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NEW SEPARATION AXIOMS IN m-SPACES

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Abstract: In this paper, m-preopen sets are used in order to define some special type of separation axioms. Here we characterize the m-pre-R_0 space, m-pre-R_1 space and weakly m-pre-R_0 space. Some properties of such spaces are studied and its relations. Also we study its relations with the m-pre-T_i for $i = 0, 1, 2$.

1. Introduction

In [4], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. In

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[1], Carpintero et al. introduced and studied the m-pre-T_i spaces for $i = = 0, 1, 2$ and new class of functions using m-preopen sets and m-preclosed sets. Let X be a topological space and $A \subset X$. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subfamily m of the power set $P(X)$ of a nonempty set X is called a minimal structure [4] on X if $\varnothing$ and X belong to m. By (X, m), we denote a nonempty set X with a minimal structure m on X. The members of the minimal structure m are called m-open sets [4], and the pair (X, m) is called an m-space. The complement of m-open set is said to be m-closed [4]. In this paper a new classes of separation axioms are introduced and studied by making use of m-preopen sets and study its relations with the m-pre-T_i spaces for $i = 0, 1, 2$. In the second section, some preliminaries required for further proceeding is introduced. In the third section, the m-pre-R_0 spaces are introduced and characterized. In the fourth section, the m-pre-R_1 space is introduced and characterized. Also some connections with the m-pre-T_i spaces for $i = 0, 1, 2$ are studied. In the fifth section, the weakly m-pre-R_0 spaces are introduced and characterized and its relation with the m-pre-T_i spaces for $i = 0, 1, 2$ are studied.

2. Preliminaries

Definition 2.1 ([4]). Given $A \subseteq X$, the m-interior of A and the m-closure of A are defined by $m\,\mathrm{Int}(A) = \cup\{W/W \in m, W \subseteq A\}$ and $m\,\mathrm{Cl}(A) = = \cap\{F/A \subseteq F, X \setminus F \in m\}$, respectively.

Definition 2.2. Let (X, m) be an $m-$space and $A \subset X$. Then a set A is called an m-preopen [2] set in X if $A \subset m\,\mathrm{Int}(m\,\mathrm{Cl}(A))$.

A set A is called an m-preclosed set if the complement of A is m-preopen. The family of all m-preopen (resp. m-preclosed) subsets of (X, m) is denoted by $mPO(X)$ (resp. $mPC(X)$).

Definition 2.3 ([2]). Let (X, m) be an $m-$space. For $A \subset X$, the m-preclosure and the m-preinterior of A, denoted by $mp\,\mathrm{Cl}(A)$ and $mp\,\mathrm{Int}(A)$, respectively, are defined as follows: $mp\,\mathrm{Cl}(A) = \cap\{U \subset X : A \subseteq U, U \in mPC(X)\}$ and $mp\,\mathrm{Int}(A) = \cup\{F \subset X : F \subseteq A, F \in mPO(X)\}$.

Theorem 2.4 ([2]). *Let (X, m) be an m-space, and A, B be subsets of X. Then we have the following:*

(i) *$x \in mp\,\mathrm{Cl}(A)$ if and only if $U \cap A \neq \varnothing$ for every $U \in mPO(X)$*

containing x.

(ii) $mp\,\mathrm{Cl}(mp\,\mathrm{Cl}(A)) = mp\,\mathrm{Cl}(A)$.

(iii) $mp\,\mathrm{Int}(mp\,\mathrm{Int}(A)) = mp\,\mathrm{Int}(A)$.

(iv) $mp\,\mathrm{Int}(X \setminus A) = X \setminus mp\,\mathrm{Cl}(A)$.

(v) $mp\,\mathrm{Cl}(X \setminus A) = X \setminus mp\,\mathrm{Int}(A)$.

(vi) *If* $A \subseteq B$, *then* $mp\,\mathrm{Cl}(A) \subseteq mp\,\mathrm{Cl}(B)$.

(vii) $mp\,\mathrm{Cl}(A) \cup mp\,\mathrm{Cl}(B) \subseteq mp\,\mathrm{Cl}(A \cup B)$.

(viii) $A \subseteq mp\,\mathrm{Cl}(A)$ *and* $mp\,\mathrm{Int}(A) \subseteq A$.

(ix) $A \in mPO(X)$ *if and only if* $mp\,\mathrm{Int}(A) = A$.

(x) $A \in mPC(X)$ *if and only if* $mp\,\mathrm{Cl}(A) = A$.

(xi) $mp\,\mathrm{Int}(A) \in mPO(X)$ *and* $mp\,\mathrm{Cl}(A) \in mPC(X)$.

Definition 2.5 ([3])**.** Let $f : (X, m) \to (Y, \sigma)$ be a function between (X, m) and a topological space Y. Then f is said to be minimal precontinuous (briefly m-precontinuous) if for each x and each open set V containing $f(x)$, there exists an m-preopen set U containing x such that $f(U) \subset V$.

Definition 2.6. An m-space (X, m) is said to be:

(i) m-pre-T_0 [1] if for any distinct pair of points in X, there is an m-preopen sets containing one of the points but not the other.

(ii) m-pre-T_1 [1] if for each pair of distinct points x and y of X, there exist m-preopen sets U and V of X such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

(iii) m-pre-T_2 [1] if for each pair of distinct points x and y in X, there exists disjoint m-preopen sets U and V in X such that $x \in U$ and $y \in V$.

Remark 2.7. *m-pre-T_2 $\to$ m-pre-T_1 $\to$* m-*pre-T_0*.

3. m-pre-R_0 spaces

Definition 3.1. A subset S of an m-space (X, m) is said to be m-preregular if it is m-preopen and m-preclosed. The family of all m-preregular sets of (X, m) is denoted by $mPR(X)$.

The family of all m-preregular (resp. m-preopen, m-preclosed) sets of (X, m) containing a point $x \in X$ is denoted by $mPR(X, x)$ (resp. $mPO(X, x)$, $mPC(X, x)$).

Definition 3.2. A point $x \in X$ is called the m-θ-precluster point of S if $mp\,\mathrm{Cl}(U) \cap S \neq \varnothing$ for every m-preopen set U of (X, m) containing x. The set of all m-θ-precluster points of S is called the m-θ-preclosure of S and is denoted by $mp\,\mathrm{Cl}_\theta(S)$. A subset S is said to be m-θ-preclosed set is said to be m-θ-preopen.

Definition 3.3. A point $x \in X$ is called m-θ-preinterior point of S if there exists an m-preregular set U of X containing x such that $x \in U \subset S$. The set of all m-θ-interior points of S and is denoted by $mp\,\mathrm{Int}_\theta(\mathrm{S})$.

Definition 3.4. A subset A of an m-space (X, m) is said to be m-θ-preopen if $A = mp\,\mathrm{Int}_\theta(A)$. Equivalently, the complement of m-θ-preclosed set is m-θ-preopen.

Definition 3.5. A subset U of an m-space (X, m) is called an m-pre-neighborhood of a point $x \in X$ if there exists an m-preopen set V of (X, m) such that $x \in V \subset U$.

Definition 3.6. Let (X, m) be an m-space and $A \subset X$. Then the m-pre-kernel of A, denoted by $mp\,\mathrm{Ker}(A)$ is defined to be the set $mp\,\mathrm{Ker}(A) = \cap\{G \in mPO(X) \mid A \subset G\}$.

Lemma 3.7. *Let (X, m) be an m-space and $x \in X$. Then, $y \in mp\,\mathrm{Ker}(\{x\})$ if and only if $x \in mp\,\mathrm{Cl}(\{y\})$.*

Proof. Suppose that $y \notin mp\,\mathrm{Ker}(\{x\})$. Then there exists $U \in mPO(X, x)$ such that $y \notin U$. Therefore, we have $x \notin mp\,\mathrm{Cl}(\{y\})$. The proof of the converse case can be done similarly. $\Diamond$

Lemma 3.8. *Let (X, m) be an m-space and A a subset of X. Then, $mp\,\mathrm{Ker}(A) = \big\{x \in X | mp\,\mathrm{Cl}(\{x\}) \cap A \neq \varnothing\big\}$.*

Proof. Let $x \in mp\,\mathrm{Ker}(A)$ and $mp\,\mathrm{Cl}(\{x\}) \cap A = \varnothing$. Hence $x \notin X \backslash mp\,\mathrm{Cl}(\{x\})$ which is an m-preopen set containing A. This is impossible, since $x \in mp\,\mathrm{Ker}(A)$. Consequently, $mp\,\mathrm{Cl}(\{x\}) \cap A \neq \varnothing$. Next, let

$x \in X$ such that $mp\,\mathrm{Cl}(\{x\}) \cap A \neq \varnothing$ and suppose that $x \notin mp\,\mathrm{Ker}(A)$. Then, there exists an m-preopen set U containing A and $x \notin U$. Let $y \in \in mp\,\mathrm{Cl}(\{x\}) \cap A$. Hence, U is an m-preneighborhood of y which does not contain x. By this contradiction $x \in mp\,\mathrm{Ker}(A)$ and hence the claim. $\diamondsuit$

Definition 3.9. An m-space (X, m) is said to be an m-pre-R_0 space if every m-preopen set contains the m-preclosure of each of its singletons.

Example 3.10. Let $X = \{a, b, c\}$ and $m = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, c\}\}$. Then the m-space (X, m) is m-pre-R_0

Remark 3.11. Since an m-space (X, m) is m-pre-T_1 if and only if the singletons are m-preclosed [1], it is clear that every m-pre-T_1 space is m-pre-R_0. At this point there are a Question there exists an m-space (X, m) that is m-pre-R_o but not m-pre-T_1.

Proposition 3.12. *For an m-space (X, m), the following properties are equivalent:*

(i) *(X, m) is m-pre-R_0 space;*

(ii) *For any $F \in mPC(X)$, $x \notin F$ implies $F \subset U$ and $x \notin U$ for some $U \in mPO(X)$;*

(iii) *For any $F \in mPC(X)$, $x \notin F$ implies $F \cap mp\,\mathrm{Cl}(\{x\}) = \varnothing$;*

(iv) *For any distinct points x and y of X, either $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$ or $mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\}) = \varnothing$.*

Proof. (i)→(ii): Let $F \in mPC(X)$ and $x \notin F$. Then by (i) $mp\,\mathrm{Cl}(\{x\}) \subset \subset X \backslash F$. Set $U = X \backslash mp\,\mathrm{Cl}(\{x\})$, then $U \in mPO(X)$, $F \subset U$ and $x \notin U$.

(ii)→(iii): Let $F \in mPC(X)$ and $x \notin F$. There exists $U \in mPC(X)$ such that $F \subset U$ and $x \notin U$. Since $U \in mPO(X)$, $U \cap mp\,\mathrm{Cl}(\{x\}) = \varnothing$ and $F \cap mp\,\mathrm{Cl}(\{x\}) = \varnothing$.

(iii)→(iv): Suppose that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$ for distinct points $x, y \in X$. There exists $z \in mp\,\mathrm{Cl}(\{x\})$ such that $z \notin mp\,\mathrm{Cl}(\{y\})$ (or $z \in mp\,\mathrm{Cl}(\{y\})$ such that $z \notin mp\,\mathrm{Cl}(\{x\})$). There exists $V \in mPC(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \notin mp\,\mathrm{Cl}(\{y\})$. By (iii), we obtain $mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\}) = \varnothing$. The proof for the other case is similar.

(iv)→(i): Let $V \in mPC(X, x)$. For each $y \notin V$, $x \neq y$ and $x \notin mp\,\mathrm{Cl}(\{y\})$. This shows that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. By (iv),

$mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\}) = \varnothing$ for each $y \in X \backslash V$ and hence $mp\,\mathrm{Cl}(\{x\}) \cap$ $\cap \Big(\bigcup_{y \in X \backslash V} mp\,\mathrm{Cl}(\{y\}) \Big) = \varnothing$. On the other hand, since $V \in mPC(X)$ and $y \in X \backslash V$, we have $mp\,\mathrm{Cl}(\{y\}) \subset X \backslash V$ and hence $X \backslash V = \bigcup_{y \in X \backslash V} mp\,\mathrm{Cl}(\{y\})$. Therefore, we obtain $(X \backslash V) \cap mp\,\mathrm{Cl}(\{x\}) = \varnothing$ and $mp\,\mathrm{Cl}(\{x\}) \subset V$. This shows that (X, m) is an m-pre-R_0 space. $\Diamond$

Theorem 3.13. *An m-space (X, m) is an m-pre-R_0 space if and only if for any x and y in X, $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$ implies $mp\,\mathrm{Cl}(\{x\}) \cap$ $\cap\, mp\,\mathrm{Cl}(\{y\}) = \varnothing$.*

Proof. Necessity. Suppose that (X, m) is m-pre-R_0 and x, $y \in X$ such that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. Then, there exists $z \in mp\,\mathrm{Cl}(\{x\})$ such that $z \in mp\,\mathrm{Cl}(\{y\})$ (or $z \notin mp\,\mathrm{Cl}(\{y\})$) such that $z \notin mp\,\mathrm{Cl}(\{x\})$. There exists $V \in mPO(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin mp\,\mathrm{Cl}(\{y\})$. Thus $x \in X \backslash mp\,\mathrm{Cl}(\{y\}) \in mPO(X)$, which implies $mp\,\mathrm{Cl}(\{x\}) \subset X \backslash mp\,\mathrm{Cl}(\{y\})$ and $mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\}) = \varnothing$. The proof for the other case is similar.

Sufficiency. Let $V \in mPO(X, x)$. We will show that $mp\,\mathrm{Cl}(\{x\}) \subset V$. Let $y \in V$, i.e., $y \in X \backslash V$. Then $x \neq y$ and $x \notin mp\,\mathrm{Cl}(\{y\})$. This shows that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. By assumption, $mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\}) =$ $= \varnothing$. Hence $y \notin mp\,\mathrm{Cl}(\{x\})$ and therefore $mp\,\mathrm{Cl}(\{x\}) \subset V$. $\Diamond$

Lemma 3.14. *The following statements are equivalent for any points x and y in an m-space (X, m):*

(i) $mp\,\mathrm{Ker}(\{x\}) \neq mp\,\mathrm{Ker}(\{y\})$;

(ii) $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$.

Proof. (i)$\rightarrow$(ii): Suppose that $mp\,\mathrm{Ker}(\{x\}) \neq mp\,\mathrm{Ker}(\{y\})$, then there exists a point z in X such that $z \in mp\,\mathrm{Ker}(\{x\})$ and $z \notin mp\,\mathrm{Ker}(\{y\})$. It follows from $z \in mp\,\mathrm{Ker}(\{x\})$ that $\{x\} \cap mp\,\mathrm{Cl}(\{z\}) \neq \varnothing$. This implies that $x \in mp\,\mathrm{Cl}(\{z\})$. By $z \notin mp\,\mathrm{Ker}(\{y\})$, we have $\{y\} \cap mp\,\mathrm{Cl}(\{z\}) = \varnothing$. Since $x \in mp\,\mathrm{Cl}(\{z\})$, $mp\,\mathrm{Cl}(\{x\}) \subset mp\,\mathrm{Cl}(\{z\})$ and $\{y\} \cap mp\,\mathrm{Cl}(\{x\}) = \varnothing$. Therefore, it follows that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. Now $mp\,\mathrm{Ker}(\{x\}) \neq$ $\neq mp\,\mathrm{Ker}(\{y\})$ implies that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$.

(ii)$\rightarrow$(i): Suppose that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. Then there exists a point z in X such that $z \in mp\,\mathrm{Cl}(\{x\})$ and $z \notin mp\,\mathrm{Cl}(\{y\})$. Then, there exists an m-preopen set containing z and therefore x but not y, namely, $y \notin mp\,\mathrm{Ker}(\{x\})$ and thus $mp\,\mathrm{Ker}(\{x\}) \neq mp\,\mathrm{Ker}(\{y\})$. $\Diamond$

Theorem 3.15. *An m-space (X, m) is an m-pre-R_0 space if and only if for any pair of points x and y in X, $mp\,\mathrm{Ker}(\{x\}) \neq mp\,\mathrm{Ker}(\{y\})$ implies $mp\,\mathrm{Ker}(\{x\}) \cap mp\,\mathrm{Ker}(\{y\}) = \varnothing$.*

Proof. Suppose that (X, m) is an m-pre-R_0 space. Thus by Lemma 3.14, for any points x and y in X if $mp\,\mathrm{Ker}(\{x\}) \neq mp\,\mathrm{Ker}(\{y\})$, then $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. Now we prove that $mp\,\mathrm{Ker}(\{x\}) \cap mp\,\mathrm{Ker}(\{y\}) = \varnothing$.

Assume that $z \in mp\,\mathrm{Ker}(\{x\}) \cap mp\,\mathrm{Ker}(\{y\})$. By $z \in mp\,\mathrm{Ker}(\{x\})$ and Lemma 3.7, it follows that $x \in mp\,\mathrm{Cl}(\{z\})$. Since $x \in mp\,\mathrm{Cl}(\{x\})$, by Th. 3.13, $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{z\})$. Similarly, we have $mp\,\mathrm{Cl}(\{y\}) = mp\,\mathrm{Cl}(\{z\}) = mp\,\mathrm{Cl}(\{x\})$. This is a contradiction. Therefore, we have $mp\,\mathrm{Ker}(\{x\}) \cap mp\,\mathrm{Ker}(\{y\}) = \varnothing$. Conversely, let (X, m) be an m-space such that for any points x and y in X, $mp\,\mathrm{Ker}(\{x\}) \neq mp\,\mathrm{Ker}(\{y\})$ implies $mp\,\mathrm{Ker}(\{x\}) \cap mp\,\mathrm{Ker}(\{y\}) = \varnothing$. If $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$, then by Lemma 3.7, $mp\,\mathrm{Ker}(\{x\}) \neq mp\,\mathrm{Ker}(\{y\})$. Hence, $mp\,\mathrm{Ker}(\{x\}) \cap mp\,\mathrm{Ker}(\{y\}) = \varnothing$, which implies $mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\}) = \varnothing$. Because $z \in mp\,\mathrm{Cl}(\{x\})$ implies $x \in mp\,\mathrm{Ker}(\{z\})$ and therefore $mp\,\mathrm{Ker}(\{x\}) \cap mp\,\mathrm{Ker}(\{y\}) \neq \varnothing$. By hypothesis, we have $mp\,\mathrm{Ker}(\{x\}) = mp\,\mathrm{Ker}(\{z\})$. Then $z \in mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\})$ implies that $mp\,\mathrm{Ker}(\{x\}) = mp\,\mathrm{Ker}(\{z\}) = mp\,\mathrm{Ker}(\{y\})$. This is a contradiction. Therefore, $mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\}) = \varnothing$ and by Th. 3.13, (X, m) is an m-preR_0 space. $\Diamond$

Theorem 3.16. *For an m-space (X, m), the following properties are equivalent:*

(i) *(X, m) is an m-pre-R_0 space;*

(ii) *For any nonempty sets A, $G \in mPO(X)$ such that $A \cap G \neq \varnothing$, there exists $F \in mPC(X)$ such that $A \cap F \neq \varnothing$ and $F \subset G$;*

(iii) *Any $G \in mPO(X)$, $G = \bigcup\{F \in mPC(X) \mid F \subset G\}$;*

(iv) *Any $F \in mPC(X)$, $F = \cap\{G \in mPO(X) \mid F \subset G\}$;*

(v) *For any $x \in X$, $mp\,\mathrm{Cl}(\{x\}) \subset mp\,\mathrm{Ker}(\{x\})$.*

Proof. (i)$\rightarrow$(ii): Let A be a nonempty set of X and $G \in mPO(X)$ such that $A \cap G \neq \varnothing$. There exists $x \in A \cap G$. Since $x \in G \in mPO(X)$, $mp\,\mathrm{Cl}(\{x\}) \subset G$. Set $F = mp\,\mathrm{Cl}(\{x\})$, then $F \in mPC(X)$, $F \subset G$ and $A \cap F \neq \varnothing$.

(ii)→(iii): Let $G \in mPO(X)$, then $G \supset \bigcup\{F \in mPC(X) \mid F \subset G\}$. Let x be any point of G. There exists $F \in mPC(X)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \bigcup\{F \in mPC(X) \mid F \subset G\}$ and hence $G = \bigcup\{F \in mPC(X) \mid F \subset G\}$.

(iii)→(iv): This is obvious.

(iv)→(v): Let x be any point of X and $y \notin mp\,\mathrm{Ker}(\{x\})$. There exists $V \in mPO(X, x)$ $y \notin V$; hence $mp\,\mathrm{Cl}(\{y\}) \cap V = \varnothing$.

By (iv) $(\cap\{G \in mPO(X)|mp\,\mathrm{Cl}(\{y\}) \subset G\}) \cap V = \varnothing$ and there exists $G \in mPO(X)$ such that $x \notin G$ and $mp\,\mathrm{Cl}(\{y\}) \subset G$. Therefore, $mp\,\mathrm{Cl}(\{x\}) \cap G = \varnothing$ and $y \notin mp\,\mathrm{Cl}(\{x\})$. Consequently, we obtain $mp\,\mathrm{Cl}(\{x\}) \subset mp\,\mathrm{Ker}(\{x\})$.

(v)→(i): Let $G \in mPO(X, x)$. Let $y \in mp\,\mathrm{Ker}(\{x\})$, then $x \in$ $\in mp\,\mathrm{Cl}(\{y\})$ and $y \in G$. This implies that $mp\,\mathrm{Ker}(\{x\}) \subset G$. Therefore, we obtain $x \in mp\,\mathrm{Cl}(\{x\}) \subset mp\,\mathrm{Ker}(\{x\}) \subset G$. This shows that (X, m) is an m-pre-R_0 space. ♢

Corollary 3.17. *For an m-space (X, m), the following properties are equivalent:*

(i) *(X, m) is an m-pre-R_0 space;*

(ii) *$mp\,\mathrm{Cl}(\{x\})$ $=$$mp\,\mathrm{Ker}(\{x\})$ for all $x \in X$.*

Proof. (i)→(ii): Suppose that (X, m) is an m-pre-R_0 space. By Th. 3.16, $mp\,\mathrm{Cl}(\{x\}) \subset mp\,\mathrm{Ker}(\{x\})$ for each $x \in X$. Let $y \in mp\,\mathrm{Ker}(\{x\})$, then $x \in mp\,\mathrm{Cl}(\{y\})$ and by Th. 3.13, $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$. Therefore, $y \in mp\,\mathrm{Cl}(\{x\})$ and hence $mp\,\mathrm{Ker}(\{x\}) \subset mp\,\mathrm{Cl}(\{x\})$. This shows that $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Ker}(\{x\})$.

(ii)→(i): This is obvious by Th. 3.16. ♢

Corollary 3.18. *If for any point x of an m-pre-R_0 space (X, m), $mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Ker}(\{x\}) = \{x\}$, then $mp\,\mathrm{Ker}(\{x\}) = \{x\}$.*

Proof. The proof follows from Th. 3.16(v). ♢

Theorem 3.19. *For an m-space (X, m), the following properties are equivalent:*

(i) *(X, m) is an m-pre-R_0 space;*

(ii) *$x \in mp\,\mathrm{Cl}(\{y\})$ if and only if $y \in mp\,\mathrm{Cl}(\{x\})$ for any points x and y in X.*

Proof. (i)→(ii): Assume that (X, m) is m-pre-R_0. Let $x \in mp\,\mathrm{Cl}(\{y\})$ and $A \in mPO(X, y)$. Now by hypothesis, $x \in A$. Therefore, every m-preopen set containing y contains x. Hence $y \in mp\,\mathrm{Cl}(\{x\})$.

(ii)→(i): Let $U \in mPO(X, x)$. If $y \notin U$, then $x \notin mp\,\mathrm{Cl}(\{y\})$ and hence $y \notin mp\,\mathrm{Cl}(\{x\})$. This implies that $mp\,\mathrm{Cl}(\{x\}) \subset U$. Hence (X, m) is m-pre-R_0. $\Diamond$

Theorem 3.20. *For an m-space (X, m), the following properties are equivalent:*

(i) *(X, m) is an m-pre-R_0 space;*

(ii) *If F is an m-preclosed subset of X, then $F = mp\,\mathrm{Ker}(F)$;*

(iii) *If F is an m-preclosed subset of X and $x \in F$, then $mp\,\mathrm{Ker}(\{x\}) \subset F$;*

(iv) *If $x \in X$, then $mp\,\mathrm{Ker}(\{x\}) \subset mp\,\mathrm{Cl}(\{x\})$.*

Proof. (i)→(ii): Let F be m-preclosed subset of X and $x \notin F$. Thus $X \backslash F \in mPO(X, x)$. Since (X, m) is m-pre-R_0, $mp\,\mathrm{Cl}(\{x\}) \subset X \backslash F$. Thus $mp\,\mathrm{Cl}(\{x\}) \cap F = \varnothing$ and Lemma 3.8, $x \notin mp\,\mathrm{Ker}(F)$. Therefore, $mp\,\mathrm{Ker}(F) = F$.

(ii)→(iii): In general, $A \subset B$ implies $mp\,\mathrm{Ker}(A) \subset mp\,\mathrm{Ker}(B)$. Therefore, it follows from (ii) that $mp\,\mathrm{Ker}(\{x\}) \subset mp\,\mathrm{Ker}(F) = F$.

(iii)→(iv): Since $x \in mp\,\mathrm{Cl}(\{x\})$ and $mp\,\mathrm{Cl}(\{x\})$ is m-preclosed, by (iii) $mp\,\mathrm{Ker}(\{x\}) \subset mp\,\mathrm{Cl}(\{x\})$.

(iv)→(i): We show the implication by using Th. 3.19. Let $x \in$ $\in mp\,\mathrm{Cl}(\{y\})$. Then by Lemma 3.7, $y \in mp\,\mathrm{Ker}(\{x\})$. Since $x \in mp\,\mathrm{Cl}(\{x\})$ and $mp\,\mathrm{Cl}(\{x\})$ is m-preclosed, by (iv) we obtain $y \in mp\,\mathrm{Ker}(\{x\}) \subset$ $\subset mp\,\mathrm{Cl}(\{x\})$. Therefore, $x \in mp\,\mathrm{Cl}(\{x\})$ implies $y \in mp\,\mathrm{Cl}(\{x\})$. The converse is obvious and (X, m) is m-pre-R_0. $\Diamond$

Definition 3.21. A filterbase $\mathcal{F}$ is called m-preconvergent to a point x in X, if for any $U \in mPO(X, x)$, there exists $B \in \mathcal{F}$ such that B is a subset of U.

Lemma 3.22. *Let (X, m) be an m-space and let x and y be any two points in X such that every net in X m-preconverging to y m-preconverges to x. Then $x \in mp\,\mathrm{Cl}(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $mp\,\mathrm{Cl}(\{y\})$. Since $\{x_n\}_{n \in N}$ m-preconverges to y, then $\{x_n\}_{n \in N}$ m-preconverges to x and this implies that $x \in mp\,\mathrm{Cl}(\{y\})$. $\Diamond$

Theorem 3.23. *For an m-space (X, m), the following statements are equivalent:*

(i) *(X, m) is an m-pre-R_0 space;*

(ii) *If x, $y \in X$, then $y \in mp\,\mathrm{Cl}(\{x\})$ if and only if every net in X m-preconverging to y, m-preconverges to x.*

Proof. (i)$\rightarrow$(ii): Let x, $y \in X$ such that $y \in mp\,\mathrm{Cl}(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha\in N}$ be a net in X such that $\{x_\alpha\}_{\alpha\in N}$ m-preconverges to y. Since $y \in mp\,\mathrm{Cl}(\{x\})$, by Th. 3.13, we have $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$. Therefore $x \in mp\,\mathrm{Cl}(\{y\})$. This means that $\{x_\alpha\}_{\alpha\in N}$ m-preconverges to x. Conversely, let $x, y \in X$ such that every net in X m-preconverging to y m-preconverges to x. Then $x \in mp\,\mathrm{Cl}(\{y\})$ by Lemma 3.8. By Th. 3.13, we have $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$. Therefore $y \in mp\,\mathrm{Cl}(\{x\})$.

(ii)$\rightarrow$(i): Assume that x and y are any two points of X such that $mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\}) \neq \varnothing$. Let $z \in mp\,\mathrm{Cl}(\{x\}) \cap mp\,\mathrm{Cl}(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha\in N}$ in $mp\,\mathrm{Cl}(\{x\})$ such that $\{x_\alpha\}_{\alpha\in N}$ m-preconverges to z. Since $z \in mp\,\mathrm{Cl}(\{y\})$, then $\{x_\alpha\}_{\alpha\in N}$ m-preconverges to y. It follows that $y \in mp\,\mathrm{Cl}(\{x\})$. By the same token we obtain $x \in mp\,\mathrm{Cl}(\{y\})$. Therefore $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$ and by Th. 3.13, (X, m) is m-pre-R_0. $\Diamond$

Example 3.24. Let $X = \{a, b, c\}$ and $m = \{\emptyset, X, \{b\}, \{c\}, \{a, c\}\}$. Then the m-space (X, m) is m-pre-T_0 but not m-pre-R_0

4. m-pre-R_1 spaces

Definition 4.1. An m-space (X, m) is said to be m-pre-R_1 if for x, y in X with $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$, there exist disjoint m-preopen sets U and V such that $mp\,\mathrm{Cl}(\{x\}) \subset U$ and $mp\,\mathrm{Cl}(\{y\}) \subset V$.

Example 4.2. Let $X = \{a, b, c\}$ and $m = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}$. Then the m-space (X, m) is m-pre-R_1.

Proposition 4.3. *If (X, m) is m-pre-R_1, then it is m-pre-R_0.*

Proof. Let $U \in mPO(X, x)$. If $y \notin U$, then since $x \notin mp\,\mathrm{Cl}(\{y\})$, $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. Hence there exists an m-preopen V_y such that $mp\,\mathrm{Cl}(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin mp\,\mathrm{Cl}(\{x\})$. Thus $mp\,\mathrm{Cl}(\{x\}) \subset U$. Therefore (X, m) is m-pre-R_0. $\Diamond$

Example 4.4. Let $X = \{a, b, c\}$ and $m = \{\emptyset, X, \{a, b\}, \{b, c\}\}$. Then the m-space (X, m) is m-pre-R_0 but is not m-pre-R_1

Theorem 4.5. *An m-space (X, m) is m-pre-R_1 if and only if for x, $y \in X$, $mp\,\mathrm{Ker}(\{x\}) \neq mp\,\mathrm{Ker}(\{y\})$, there exist disjoint m-preopen sets U and V such that $mp\,\mathrm{Cl}(\{x\}) \subset U$ and $mp\,\mathrm{Cl}(\{y\}) \subset V$.*

Proof. It follows from Lemma 3.14. $\Diamond$

Theorem 4.6. *The following properties are equivalent:*

(i) *(X, m) is m-pre-T_2,*

(ii) *(X, m) is m-pre-R_1 and m-pre-T_1, and*

(iii) *(X, m) is m-pre-R_1 and m-pre-T_0.*

Proof. (i)$\rightarrow$(ii): Since (X, m) is m-pre-T_2, then it is m-pre-T_1. If x, $y \in X$ such that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$, then $x \neq y$ and there exist disjoint m-preopen sets U and V such that $x \in U$ and $y \in V$ and $mp\,\mathrm{Cl}(\{x\}) = \{x\} \subset U$ and $mp\,\mathrm{Cl}(\{y\}) = \{y\} \subset V$. Hence (X, m) is m-pre-R_1.

(ii)$\rightarrow$(iii): Since (X, m) is m-pre-T_1, then (X, m) is m-pre-T_0.

(iii)$\rightarrow$(i): Since (X, m) is m-pre-R_1, and m-pre-T_1, then (X, m) is m-pre-R_0 and m-pre-T_0, which implies (X, m) is m-pre-T_1. Let x, $y \in X$ such that $x \neq y$. Since $mp\,\mathrm{Cl}(\{x\}) = \{x\} \neq \{y\} = mp\,\mathrm{Cl}(\{y\})$, then there exist disjoint m-preopen sets U and V such that $x \in U$ and $y \in V$. Hence, (X, m) is m-pre-T_2. $\Diamond$

Lemma 4.7. *For any subset A of an m-space (X, m), $mp\,\mathrm{Cl}(A) \subset$ $\subset mp\,\mathrm{Cl}_\theta(A)$.*

Lemma 4.8. *Let x and y be any points in an m-space (X, m). Then $y \in mp\,\mathrm{Cl}_\theta(\{x\})$ if and only if $x \in mp\,\mathrm{Cl}_\theta(\{y\})$.*

Proof. Let $y \notin mp\,\mathrm{Cl}_\theta(\{x\})$. This implies that there exists $V \in mPO(X, y)$ such that $mp\,\mathrm{Cl}(V) \cap \{x\} = \emptyset$ and $X \backslash mp\,\mathrm{Cl}(V) \in mPR(X, x)$, which means that $x \notin mp\,\mathrm{Cl}_\theta(\{y\})$. $\Diamond$

Theorem 4.9. *An m-space (X, m) is m-pre-R_1 if and only if*

$$mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}_\theta(\{x\}) \quad \textit{for each} \quad x \in X.$$

Proof. Necessity. Assume that (X, m) is m-pre-R_1 and

$$y \in mp\,\mathrm{Cl}_\theta(\{x\}) \backslash mp\,\mathrm{Cl}(\{x\}).$$

Then $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$, therefore, there exist disjoint m-preopen sets U_1 and U_2 such that $mp\,\mathrm{Cl}(\{x\}) \subset U_1$ and $mp\,\mathrm{Cl}(\{y\}) \subset U_2$. In consequence, $mp\,\mathrm{Cl}(U_2) \cap \{x\} \subseteq mp\,\mathrm{Cl}(U_2) \cap U_1 = \emptyset$ and hence $y \notin mp\,\mathrm{Cl}_\theta(\{x\})$. This is a contradiction, therefore $mp\,\mathrm{Cl}_\theta(\{x\}) \subseteq mp\,\mathrm{Cl}(\{x\})$. Now using Lemma 4.8, we obtain $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}_\theta(\{x\})$.

Sufficiency. Suppose that $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}_\theta(\{x\})$ for each $x \in X$. We first prove that (X,m) is m-pre-R_0. Let $U \in mPO(X,x)$ and $y \notin U$. Since $mp\,\mathrm{Cl}_\theta(\{y\}) = mp\,\mathrm{Cl}(\{y\}) \subset X \backslash U$, we have $x \notin mp\,\mathrm{Cl}_\theta(\{x\})$ and by Lemma 4.8, $y \notin mp\,\mathrm{Cl}_\theta(\{x\}) = mp\,\mathrm{Cl}(\{x\})$. It follows that $mp\,\mathrm{Cl}(\{x\}) \subset U$. Therefore, (X,m) is m-pre-R_0. Now, let $a, b \in X$ with $mp\,\mathrm{Cl}(\{a\}) \neq mp\,\mathrm{Cl}(\{b\})$. It follows that, (X,m) is m-pre-T_1 and $a \neq b$. Since $mp\,\mathrm{Cl}(\{a\}) = mp\,\mathrm{Cl}_\theta(\{a\})$ for each $a \in X$, $b \notin mp\,\mathrm{Cl}_\theta(\{a\})$ and hence there exists $U \in mPO(X,b)$ such that $a \notin mp\,\mathrm{Cl}(U)$. Therefore, we obtain $b \in U$, $a \in X \backslash mp\,\mathrm{Cl}(U)$ and $U \cap X \backslash mp\,\mathrm{Cl}(U) = \emptyset$. This shows that (X,m) is m-pre-R_1. $\Diamond$

Theorem 4.10. *An m-space (X,m) is m-pre-R_1 if and only if for each m-preopen set U and each $x \in U$, $mp\,\mathrm{Cl}_\theta(\{x\}) \subset U$.*

Proof. Necessity. Assume that (X,m) is m-pre-R_1. Suppose that $U \in$ $\in mPO(X,x)$. Let $y \in X \backslash U$. Since (X,m) is m-pre-R_1, by Th. 4.9, $mp\,\mathrm{Cl}_\theta(\{y\}) = mp\,\mathrm{Cl}(\{y\}) \subset X \backslash U$. Hence we have that $x \notin mp\,\mathrm{Cl}_\theta(\{y\})$ and by Lemma 4.8, $y \notin mp\,\mathrm{Cl}_\theta(\{x\})$. It follows that $mp\,\mathrm{Cl}_\theta(\{x\}) \subset U$.

Sufficiency. Assume now that $y \in mp\,\mathrm{Cl}_\theta(\{x\}) \backslash mp\,\mathrm{Cl}(\{x\})$ for some $x \in X$. Then there exists $U \in mPO(X,y)$ such that $mp\,\mathrm{Cl}(U) \cap \{x\} \neq \emptyset$ but $U \cap \{x\} = \emptyset$. Then $mp\,\mathrm{Cl}_\theta(\{y\}) \subset U$ and $mp\,\mathrm{Cl}_\theta(\{y\}) \cap \{x\} \neq \emptyset$. Hence, $x \notin mp\,\mathrm{Cl}_\theta(\{y\})$. Thus, $y \notin mp\,\mathrm{Cl}_\theta(\{x\})$. By this contradiction, we obtain $mp\,\mathrm{Cl}_\theta(\{x\}) = mp\,\mathrm{Cl}(\{x\})$ for each $x \in X$. Thus, by Th. 4.9, (X,m) is m-pre-R_1. $\Diamond$

Theorem 4.11. *The following properties are equivalent:*

(i) *(X,m) is m-pre-R_1.*

(ii) *For each x, $y \in X$ one of the following holds:*

  (a) *If U is m-preopen, then $x \in U$ if and only if $y \in U$.*

  (b) *There exist disjoint m-preopen sets U and V such that $x \in U$ and $y \in V$.*

(iii) *If x, $y \in X$ such that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$, then there exist m-preclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_2$, $y \in F_1$, $x \notin F_2$, and $X = F_1 \cup F_2$.*

Proof. (i)→(ii): Let x, $y \in X$. Then $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$ or $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. If $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$ and U is m-preopen, then $x \in U$ implies $y \in mp\,\mathrm{Cl}(\{x\}) \subset U$ and $y \in U$ implies $x \in mp\,\mathrm{Cl}(\{y\}) \subset U$. Thus consider the case that $mp\,\mathrm{Cl}(\{x\}) \neq$ $\neq mp\,\mathrm{Cl}(\{y\})$. Then there exist disjoint m-preopen sets U and V such that $x \in mp\,\mathrm{Cl}(\{x\}) \subset U$ and $y \in mp\,\mathrm{Cl}(\{y\}) \subset V$.

(ii)→(iii): Let x, $y \in X$ such that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$. Then $x \notin mp\,\mathrm{Cl}(\{y\})$ or $y \notin mp\,\mathrm{Cl}(\{x\})$, say $x \notin mp\,\mathrm{Cl}(\{y\})$. Then there exist an m-preopen set A such that $x \in$ A and $y \notin A$, which implies there exist disjoint m-preopen sets U and V such that $x \in U$ and $y \in V$. Then $F_1 = X\backslash V$ and $F_2 = X\backslash U$ are m-preclosed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, $X = F_1 \cup F_2$.

(iii)→(i): Let U be m-preopen and let $x \in U$. Then $mp\,\mathrm{Cl}(\{x\}) \subset U$, for suppose not. Let $y \in mp\,\mathrm{Cl}(\{x\}) \cap (X\backslash U)$. Then $mp\,\mathrm{Cl}(\{x\}) \neq$ $\neq mp\,\mathrm{Cl}(\{y\})$ and there exist m-preclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, $X = F_1 \cup F_2$. Then $y \in F_2\backslash F_1 = X\backslash F_1$, which is m-preopen, and $x \notin X\backslash F_1$, which is a contradiction. Hence, (X, m) is m-pre-R_0. Let $a, b \in X$ such that $mp\,\mathrm{Cl}(\{a\}) \neq mp\,\mathrm{Cl}(\{b\})$. Then there exist m-preclosed sets A_1 and A_2 such that $a \in A_1$, $b \notin A_1$, $a \notin A_2$, and $X = A_1 \cup A_2$. Thus $a \in A_1\backslash A_2$ and $b \in A_2\backslash A_1$, which are m-preopen, which implies $mp\,\mathrm{Cl}(\{a\}) \subset A_1\backslash A_2$ and $mp\,\mathrm{Cl}(\{b\}) \subset A_2\backslash A_1$. Hence, (X, m) is m-pre-R_1. $\Diamond$

Theorem 4.12. *An m-space (X, m) is m-pre-T_2 if and only if for x, $y \in X$ such that $x \neq y$, there exist m-preclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \notin F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.*

Proof. The straightforward proof is omitted. $\Diamond$

Remark 4.13. If $\{x_\lambda\}_{\lambda\in A}$ is a net in (X, m), let $mp\lim(\{x_\lambda\}_{\lambda\in A}) =$ $= \{x \in X : \{x_\lambda\}_{\lambda\in A}$ m-preconverges to $x\}$.

Theorem 4.14. *The following properties are equivalent:*

(i) *(X, m) is m-pre-R_1;*

(ii) *for $x, y \in X$, $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$, whenever there exists a net $\{x_\lambda\}_{\lambda\in A}$ such that $x, y \in mp\lim(\{x_\lambda\}_{\lambda\in A})$;*

(iii) *(X, m) is m-pre-R_0, and for every m-preconvergent net $\{x_\lambda\}_{\lambda\in A}$ in X, $mp\lim(\{x_\lambda\}_{\lambda\in A}) = mp\,\mathrm{Cl}(\{x\})$ for some $x \in X$.*

Proof. (i)→(ii): Let $x, y \in X$ such that there exists a net $\{x_\lambda\}_{\lambda\in A}$ in X such that $x, y \in mp\lim(\{x_\lambda\}_{\lambda\in A})$. Then (a) if U is m-preopen, then

$x \in U$ if and only if $y \in U$ or (b) there exist disjoint m-preopen sets U and V such that $x \in U$ and $y \in V$. Since $x, y \in mp\lim(\{x_\lambda\}_{\lambda\in A})$, then (i) is satisfied, which implies $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{y\})$.

(ii)$\rightarrow$(iii): Let $U \in mPO(X, x)$. Let $y \notin U$. For each $n \in N$ let $x_n = x$. Then $\{x_n\}_{n\in N})$ m-preconverges to x and since $mp\,\mathrm{Cl}(\{x\}) \neq$ $\neq mp\,\mathrm{Cl}(\{y\})$, that $y \in A$ and $x \notin A$. Thus, $y \notin mp\,\mathrm{Cl}(\{x\})$ and $mp\,\mathrm{Cl}(\{y\}) \subset U$. Hence (X, m) is m-preR_0. Let $\{x_\lambda\}_{\lambda\in A}$ be an m-preconvergent net in X. Let $x \in X$ such that $\{x_\lambda\}_{\lambda\in A}$ m-preconverges to x. If $y \in mp\,\mathrm{Cl}(\{x\})$, then $\{x_\lambda\}_{\lambda\in A}$ m-preconverges to y, which implies $mp\,\mathrm{Cl}(\{x\}) \subset mp\lim(\{x_\lambda\}_{\lambda\in A})$ and if $y \in mp\lim(\{x_\lambda\}_{\lambda\in A})$, then $x, y \in$ $\in mp\lim(\{x_\lambda\}_{\lambda\in A})$, which implies $y \in mp\,\mathrm{Cl}(\{y\}) = mp\,\mathrm{Cl}(\{x\})$. Hence $mp\lim(\{x_\lambda\}_{\lambda\in A}) = mp\,\mathrm{Cl}(\{x\})$.

(iii)$\rightarrow$(i): Assume that (X, m) is not m-pre-R_1. Then there exists $x, y \in X$ such that $mp\,\mathrm{Cl}(\{x\}) \neq mp\,\mathrm{Cl}(\{y\})$ and every m-preopen set containing $mp\,\mathrm{Cl}(\{x\})$ intersects every m-preopen set containing $mp\,\mathrm{Cl}(\{y\})$. Since (X, m) is m-pre-R_0, then every m-preopen set containing x contains $mp\,\mathrm{Cl}(\{x\})$ and every m-preopen set containing y contains $mp\,\mathrm{Cl}(\{y\})$, which implies that every m-preopen set containing x intersects every m-preopen set containing y. Let $D_x = \{U \subset X : U \in$ $\in mPO(X, x)\}$. Let $\geq_x$ be the binary relation on D_x defined by $U_1 \geq_x U_2$ if and only if $U_1 \subset U_2$. Then, clearly $(D_x, \geq_x)$ is a directed set. Let $D_y = \{U \subset X : U \in mPO(X, y)\}$ and let $\geq_y$ be the binary relation on D_y defined by $U_1 \geq_y U_2$ if and only if $U_1 \subset U_2$. Then, $(D_y, \geq_y)$ is also a directed set. Let $D = \{(U_1, U_2) : U_1 \in D_x and U_2 \in D_y\}$ and let $\geq$ be the binary relation on D defined by $(U_1, U_2) \geq (V_1, V_2)$ if and only if $U_1 \geq_x V_1$ and $U_2 \geq_y V_2$. Then, $(D, \geq)$ is a directed set. For each $(U_1, U_2) \in D$, let $x_{(U_1,U_2)} \in (U_1, U_2)$. Then $\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}$ is a net in X that m-preconverges to both x and y. Thus, there exists $z \in X$ such that $mp\lim(\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}) = mp\,\mathrm{Cl}(\{z\})$, which implies $x, y \in mp\,\mathrm{Cl}(\{z\})$. Since $\{mp\,\mathrm{Cl}(\{w\}) : w \in X\}$ is a decomposition of X, then $mp\,\mathrm{Cl}(\{x\}) = mp\,\mathrm{Cl}(\{z\}) = mp\,\mathrm{Cl}(\{y\})$, which is a contradiction. Hence (X, m) is m-pre-R_1. $\Diamond$

Theorem 4.15. *An m-space (X, m) is m-pre-T_2 if and only if every m-preconvergent net in X m-preconverges to a unique point.*

Proof. The proof follows from Theorems 4.14 and 4.6. $\Diamond$

5. Weakly m-preR_0 spaces

Definition 5.1. An m-space (X, m) is said to be weakly m-pre-R_0 if and only if $\cap_{x\in X}\ mp\,\mathrm{Cl}(\{x\}) = \varnothing$.

Remark 5.2. It should be noticed that m-pre-R_0 space is weakly m-pre-R_0, but the converse is not true.

Example 5.3. Let $X = \{a, b, c\}$ and $m = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. Then the m-space (X, m) is weakly m-pre-R_0 but is not m-pre-R_0.

Theorem 5.4. *An m-space (X, m) is weakly m-pre-R_0 if and only if $mp\,\mathrm{Ker}(\{x\}) \neq X$ for every x in X.*

Proof. Suppose that (X, m) is weakly m-pre-R_0 and that there is a point $z \in X$ such that $mp\,\mathrm{Ker}(\{z\}) = X$. It follows that $z \notin U$, where U is some proper m-preopen subset of X. It means that $z \in \cap_{x\in X}\ mp\,\mathrm{Cl}(\{x\})$ which is a contradiction to our assumption. Conversely, suppose that $mp\,\mathrm{Ker}(\{x\}) \neq X$ for every $x \in X$. If there exists a point $z \in X$ such that $z \in \cap_{x\in X}\ mp\,\mathrm{Cl}(\{x\})$, then every m-preopen set containing z must contain every point of X. It follows that X is the unique m-preopen set containing z. Therefore, $mp\,\mathrm{Ker}(\{x\}) = X$ which is a contradiction and hence (X, m) is weakly m-pre-R_0. $\lozenge$

Definition 5.5. A function $f : (X, m) \to (Y, m')$ is called (m, m')-preclosed, if the image of every m-preclosed subset of (X, m) is m'-preclosed in (Y, m').

Theorem 5.6. *If $f : (X, m) \to (Y, m')$ is an injective (m, m')-preclosed function and X is weakly m-pre-R_0, then (Y, m') is weakly m'-pre-R_0.*

Proof. From the assumption follows the following relation:

$$\cap_{y\in Y}\ m'p\,\mathrm{Cl}(\{y\}) \subset \cap_{x\in X}\ m'p\,\mathrm{Cl}(\{f(x)\}) \subset$$
$$\subset f(\cap_{x\in X}\ mp\,\mathrm{Cl}(\{x\})) = f(\varnothing) = \varnothing. \qquad \lozenge$$

Remark 5.7. In the following diagram, we denote by arrows the implications between the separation axioms which we have introduced and discussed in this paper and the examples show that no other implications hold between them:

$$m\text{-}pre\text{-}R_1 \quad \to \quad m\text{-}pre\text{-}R_0 \quad \to \quad weakly\ \mathrm{m}\text{-}pre\text{-}R_0\,.$$

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