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# ON GENERALIZED SEMI-PSEUDO RICCI SYMMETRIC MANIFOLDS

Hülya Bağdatlı **Yılmaz**

*Marmara University, Faculty of Sciences and Letters, Department of Mathematics, Istanbul, Turkey*

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**Abstract:** The object of the present paper is to investigate generalized semi-pseudo Ricci symmetric manifolds admitting semi-symmetric non-metric connection denoted by $\left[G(SPRS)_n, \overline{\nabla}\right]$.

## 1. Introduction

It is well known that spaces admitting some sense of symmetry are important to solve the Einstein's field equations of gravitation in general relativity. Cartan [2] initiated the study of Riemannian symmetric spaces in the late twenties and, in particular, obtained a classification of those spaces.

Let $(M^n, g)$ be a Riemannian manifold of dimension $n$ and let $\nabla$ be its Levi-Civita connection. If a curvature tensor $R$ of $(M^n, g)$ is parallel with respect to its Levi-Civita connection $\nabla$, namely $\nabla R = 0$ then this manifold is called *locally symmetric* due to Cartan. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature.

The notion of locally symmetric manifolds has been weakened by

*E-mail address:* hbagdatli@marmara.edu.tr

many authors in several ways to different extent such as recurrent manifolds by Walker [20], projective symmetric manifolds by Soós [16], Ricci semi-symmetric manifolds by Szabó [17], pseudo symmetric manifolds by Chaki [3], pseudo Ricci symmetric manifolds by Chaki [4], generalized pseudo Ricci symmetric manifolds by Chaki and Koley [5], generalized pseudo symmetric manifolds by Chaki [6], weakly symmetric manifolds by Selberg [14], and weakly symmetric manifolds by Tamássy and Binh [18], almost pseudo symmetric manifolds by De and Gazi [8], pseudo cyclic Ricci symmetric manifold extended the notion of pseudo Ricci symmetric manifolds by Shaikh and Hui [15], ete. It may be noted that the notion of weakly symmetric Riemannian manifolds by Selberg is different and is not equivalent to that of Tamássy and Binh.

In 1993, the notion of semi-pseudo Ricci symmetric manifolds was introduced by Tarafdar and Jawarneh [19]. A Riemannian manifold $(M^n, g)$ $(n > 3)$ is said to be a *semi-pseudo Ricci symmetric manifold* if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the relation

$$(\nabla_X S)(Y, Z) = A(Y)S(X, Z) + A(Z)S(X, Y) \tag{1.1}$$

for all vector fields $X, Y, Z \in \chi(M^n)$, where $A$ is a non-zero 1-form such that $g(X, \rho) = A(X)$ for every vector field $X$ and $\nabla$ denotes the Levi-Civita connection on $(M^n, g)$. Such an $n-$dimensional manifold is denoted by $(SPRS)_n$.

Jawarneh and Tashtoush [11] introduced the notion of generalized semi-pseudo Ricci symmetric manifolds. A Riemannian manifold $(M^n, g)$ $(n > 3)$ is called a *generalized semi-pseudo Ricci symmetric manifold* if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(Y)S(X, Z) + B(Z)S(X, Y), \tag{1.2}$$

where $A$ and $B$ are non-zero 1-forms, $A$ and $\nabla$ have the meaning already stated, $B(X) = g(X, \mu)$ for every vector field $X$. The 1-forms are called *associated* 1*-forms* of the manifold and such an $n-$dimensional manifold is denoted by $G(SPRS)_n$. If $A = B$, then from the definitions it follows that $G(SPRS)_n$ reduces to $(SPRS)_n$.

The present paper deals with $G(SPRS)_n$ admitting a semi-symmetric non-metric connection. A $G(SPRS)_n$ admitting semi-symmetric non-metric connection is denoted by $\left[G(SPRS)_n, \overline{\nabla}\right]$. The paper is organized as follows: In Sec. 2, it is given a brief introduction to the semi-symmetric non-metric connection. In Sec. 3, it is introduced generalized semi-pseudo Ricci symmetric manifolds endowed with a semi-symmetric non-metric connection $\left[G(SPRS)_n, \overline{\nabla}\right]$ and obtained nature of its the scalar curvature $\overline{r}$. The last section deals with generalized semi-pseudo Ricci symmetric manifolds endowed with a special type of semi-symmetric non-metric connection and it is shown that such a manifold admitting a parallel vector field $\rho$ with respect to the Levi-Civita connection $\nabla$ is a quasi-Einstein manifold [7]. Moreover, it is found that $\rho$ is orthogonal to the vector field $\mu$ corresponding to the associated 1-form $B$ and the scalar curvature $r$ with respect to the connection $\nabla$ is a non-zero constant.

We assume the condition $n > 3$ throughout the paper.

## 2. Semi-symmetric non-metric connection

Let $\overline{\nabla}$ be defined a linear connection on a Riemannian manifold $(M^n, g)$ by [1]

$$\overline{\nabla}_X Y = \nabla_X Y + A(Y)X \tag{2.1}$$

for all vector fields $X, Y$. Using (2.1), the torsion tensor $\overline{T}$ of $(M^n, g)$ with respect to the connection $\overline{\nabla}$ is given by

$$\overline{T}(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y],$$

and satisfies

$$\overline{T}(X,Y) = A(Y)X - A(X)Y. \tag{2.2}$$

A linear connection satisfying (2.2) is called a *semi-symmetric connection* [10, 12, 13]. $\overline{\nabla}$ is called a *metric connection* if

$$\overline{\nabla} g = 0.$$

If $\overline{\nabla} g \neq 0$, then $\overline{\nabla}$ is said to be a *non-metric connection.* From (2.1), it follows that

$$\left(\overline{\nabla}_X g\right)(Y,Z) = -A(Y)g(X,Z) - A(Z)g(X,Y) \tag{2.3}$$

for vector fields $X, Y, Z$ on $(M^n, g)$.

Therefore, due to (2.2) and (2.3), the connection $\overline{\nabla}$ is a semi-symmetric non-metric connection.

Let us denote the curvature tensor with respect to the connections $\overline{\nabla}$ and $\nabla$ by $\overline{R}$ and $R$, respectively. Then, due to (2.1), we obtain [1]

$$\overline{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{2.4}$$

where $\alpha$ is a tensor field of type $(0,2)$ defined by

$$\alpha(X,Y) = (\nabla_X A)(Y) - A(X)A(Y). \tag{2.5}$$

By virtue of (2.1), we have

$$\left(\overline{\nabla}_X A\right)(Y) = (\nabla_X A)(Y) - A(X)A(Y). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\alpha(X,Y) = \left(\overline{\nabla}_X A\right)(Y). \tag{2.7}$$

Contracting (2.4), we get

$$\overline{S}(Y,Z) = S(Y,Z) + (1-n)\alpha(Y,Z), \tag{2.8}$$

where $\overline{S}$ and $S$ denote the Ricci tensors of the semi-symmetric non-metric connection and the Levi-Civita connection, respectively. The tensor $\alpha$ of type $(0,2)$ given in (2.5) is not symmetric in general and hence, from (2.8), it follows that the Ricci tensor $\overline{S}$ of the connection $\overline{\nabla}$ is not so either. But, using (2.7), we get

$$\begin{aligned} \alpha(Y,Z) - \alpha(Z,Y) &= \left(\overline{\nabla}_Y A\right)(Z) - \left(\overline{\nabla}_Z A\right)(Y) = \\ &= dA(Y,Z). \end{aligned}$$

Thus a tensor $\alpha$ is symmetric if and only if the 1-form $A$ is closed [1].

Again contracting (2.8), we obtain

$$\overline{r} = r + (1-n)\mathrm{trace}\ \alpha, \tag{2.9}$$

where $\overline{r}$ and $r$ denote the scalar curvatures of the connections $\overline{\nabla}$ and $\nabla$, respectively.

## 3. Generalized semi-pseudo Ricci symmetric manifold admitting a semi symmetric non-metric connection

**Definition 1.** A Riemannian manifold $(M^n, g)$ $(n > 3)$ is called a generalized semi-pseudo Ricci symmetric manifold admitting a semi-symmetric non-metric connection $\overline{\nabla}$ if its Ricci tensor $\overline{S}$ of type $(0, 2)$ is not identically zero and satisfies the relation

$$\left(\overline{\nabla}_X \overline{S}\right)(Y, Z) = \overline{A}(Y)\overline{S}(X, Z) + \overline{B}(Z)\overline{S}(X, Y), \tag{3.1}$$

where $\overline{A}$ and $\overline{B}$ are distinct non-zero 1-forms.

The 1-forms $\overline{A}$ and $\overline{B}$ are called *associated* 1*-forms* of the manifold and such an $n-$dimensional manifold is denoted by $\left[G(SPRS)_n, \overline{\nabla}\right]$.

**Theorem 1.** *If, in a* $\left[G(SPRS)_n, \overline{\nabla}\right]$ *the* 1*-form* $A$*, associated with the torsion tensor* $T$*, is closed, then its Ricci tensor* $\overline{S}$ *is of the form:*

$$\overline{S}(X, Y) = \overline{r}H(X)H(Y), \tag{3.2}$$

*where* $H$ *is a non-zero* 1*-form defined by* $H(X) = g(X, h), h$ *being a unit vector field.*

**Proof.** Let us interchange $Y$ and $Z$ in (3.1). Then we get

$$\left(\overline{\nabla}_X \overline{S}\right)(Z, Y) = \overline{A}(Z)\overline{S}(X, Y) + \overline{B}(Y)\overline{S}(X, Z). \tag{3.3}$$

Since 1-form $A$ is closed, the Ricci tensor $\overline{S}$ is symmetric. Substracting (3.3) from (3.1), we obtain

$$\begin{aligned} &\left(\overline{\nabla}_X \overline{S}\right)(Y, Z) - \left(\overline{\nabla}_X \overline{S}\right)(Z, Y) = \\ &= \left\{\overline{B}(Z) - \overline{A}(Z)\right\}\overline{S}(X, Y) + \left\{\overline{A}(Y) - \overline{B}(Y)\right\}\overline{S}(X, Z). \end{aligned} \tag{3.4}$$

If the symmetry property of $\overline{S}$ is used, then it follows from (3.4) that

$$\left\{\overline{A}(Y) - \overline{B}(Y)\right\}\overline{S}(X, Z) = \left\{\overline{A}(Z) - \overline{B}(Z)\right\}\overline{S}(X, Y). \tag{3.5}$$

Let us now consider $\overline{D}(X) = \overline{A}(X) - \overline{B}(X)$ such that $\overline{D}(X) = g(X, \nu)$ for all vector fields $X$ and $\nu$ is a vector field associated with the 1-form $\overline{D}$. In virtue of Definition 1, $\overline{D}(X) \neq 0$. Thus (3.5) reduces to the following form

$$\overline{D}(Y)\overline{S}(X,Z) = \overline{D}(Z)\overline{S}(X,Y). \tag{3.6}$$

Contracting (3.6) with respect to $X$ and $Z$, we have

$$\overline{r}\overline{D}(Y) = \overline{D}(LY), \tag{3.7}$$

where $\overline{L}$ is the Ricci operator associated with the Ricci tensor such that $\overline{S}(X,Y) = g(\overline{L}X,Y)$ for all vector fields $X, Y$.

Substituting $Z = \nu$ into (3.6), we obtain

$$\overline{D}(Y)\overline{S}(X,\nu) = \overline{D}(\nu)\overline{S}(X,Y) = \overline{D}(Y)\overline{D}(\overline{L}X). \tag{3.8}$$

From (3.7), it follows that

$$\overline{D}(\nu)\overline{S}(X,Y) = \overline{r}\overline{D}(Y)\overline{D}(X). \tag{3.9}$$

Hence we have

$$\overline{S}(X,Y) = \frac{\overline{r}}{\overline{D}(\nu)}\overline{D}(Y)\overline{D}(X) = \overline{r}H(X)H(Y), \tag{3.10}$$

where $H(X) = g(X,h) = \frac{\overline{D}(X)}{\sqrt{\overline{D}(\nu)}}$, $h$ being a unit vector field associated with the 1-form $H$. Thus the theorem is proved. $\Diamond$

From (3.2), it follows that if $\overline{r} = 0$, then $\overline{S}(X,Y) = 0$ which is inadmissible by the definition of a $\left[G(SPRS)_n, \overline{\nabla}\right]$.

**Theorem 2.** *In a* $\left[G(SPRS)_n, \overline{\nabla}\right]$ *with* $\overline{D}(X) \neq 0$, $\overline{r}$ *is an eigenvalue of the Ricci tensor* $\overline{S}$ *corresponding to the eigenvector* $\nu$.

**Proof.** From (3.7), it follows that

$$\overline{r}g(Y,\nu) = g(LY,\nu) = \overline{S}(Y,\nu).$$

This relation shows that $\overline{r}$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector $\nu$. Hence the theorem is proved. $\Diamond$

**Theorem 3.** *If, in a* $\left[G(SPRS)_n, \overline{\nabla}\right]$, *the length of the Ricci tensor* $\overline{S}$ *is* $l$, *then that of the scalar curvature* $\overline{r}$ *is also* $l$.

**Proof.** From (3.2), it follows that

$$\overline{S}(X,X) = \overline{r}\left[g(X,h)\right]^2 \tag{3.11}$$

for all vector fields $X$. From (3.11), since $h$ is a unit vector field, i.e. $g(h,h)=1$, we get

$$\overline{S}(h,h)=\overline{r}. \tag{3.12}$$

Let $\theta$ be the angle between $h$ and an arbitrary vector field $X$. Then remembering that $h$ is a unit vector field, we have

$$\cos\theta=\frac{g(X,h)}{\sqrt{g(h,h)}\sqrt{g(X,X)}}=\frac{g(X,h)}{\sqrt{g(X,X)}}.$$

Hence $[g(X,h)]^2\leq g(X,X)=|X|^2$ and if $\overline{r}>0$, then $\overline{r}\,[g(X,h)]^2\leq$ $\leq\overline{r}\,|X|^2$, from (3.11), we get

$$\overline{S}(X,X)\leq\overline{r}\,|X|^2\,.$$

Let $l^2$ be the square of the length of the Ricci tensor $\overline{S}$. Then we can write

$$l^2=\overline{S}(\overline{L}e_i,e_i), \tag{3.13}$$

where $\{e_i\}\,,(1\leq i\leq n)$ is an orthonormal basis of the tangent space at a point.

From (3.2), it follows that

$$\begin{aligned}\overline{S}(\overline{L}e_i,e_i) &= \overline{r}H(\overline{L}e_i)H(e_i)=\\ &= \overline{r}g(\overline{L}e_i,h)g(e_i,h)=\\ &= \overline{r}g(\overline{L}h,h)=\\ &= \overline{r}\overline{S}(h,h).\end{aligned}$$

Thus, using (3.12) and (3.13), we get

$$l^2=\overline{r}.\overline{r}=\overline{r}^2.$$

Hence we can say that the length of the scalar curvature $\overline{r}$ is also $l$. This completes the proof. $\Diamond$

## 4. $G(SPRS)_n$ admitting a special type of semi-symmetric non-metric connection

In this section we consider a generalized semi-pseudo Ricci symmetric manifold $G(SPRS)_n$ admitting a type of semi-symmetric metric connection $\overline{\nabla}$ whose curvature tensor $\overline{R}$ satisfies the condition

$$\overline{R}(X,Y)Z=0. \tag{4.1}$$

Agashe and Chafle [1] proved that if a Riemannian manifold $(M^n, g)$ $(n > 3)$ admits a semi-symmetric non-metric connection whose curvature tensor vanishes, then the manifold is of constant curvature and hence is conformally flat.

We now give the definition of Weyl conformal curvature tensor, or simply the conformal curvature tensor. The conformal curvature tensor $C$ on a Riemannian manifold $(M^n, g)$ $(n > 3)$ is defined by [9]

$$\begin{aligned} C(X,Y,Z,W) &= \\ &= R(X,Y,Z,W) - \frac{1}{n-2}\left[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + \right. \\ &\quad \left. + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\right] + \\ &\quad + \frac{r}{(n-1)(n-2)}\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right]. \end{aligned} \tag{4.2}$$

**Theorem 4.** *If, in a* $\left[G(SPRS)_n, \overline{\nabla}\right]$ *whose curvature tensor* $\overline{R}$ *vanishes,* $\rho$ *is a parallel vector field with respect to the Levi-Civita connection* $\nabla$*, then the manifold reduces to a quasi-Einstein manifold.*

**Proof.** Let $\rho$ be a paralel vector field. Then we get

$$\nabla_X \rho = 0, \tag{4.3}$$

and hence we obtain

$$R(X,Y)\rho = \nabla_X \nabla_Y \rho - \nabla_Y \nabla_X \rho - \nabla_{[X,Y]}\rho = 0.$$

Thus we have

$$A(R(X,Y)Z) = g(R(X,Y)Z, \rho) = -g(R(X,Y)\rho, Z) = 0. \tag{4.4}$$

Since $C = 0$, it follows from (4.2) that

$$\begin{aligned} R(X,Y,Z,W) = &\frac{1}{n-2}\left[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + \right. \\ &\left. + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\right] - \\ &- \frac{r}{(n-1)(n-2)}\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right]. \end{aligned} \tag{4.5}$$

Substituting $W = \rho$ into (4.5) and using (4.4), we get

$$S(Y,Z)A(X) - S(X,Z)A(Y) = \frac{r}{(n-1)}\left[g(Y,Z)A(X) - g(X,Z)A(Y)\right]. \tag{4.6}$$

Taking $X = \rho$ in (4.6), we get

$$S(Y,Z) = \frac{r}{(n-1)} g(Y,Z) - \frac{1}{A(\rho)} A(Y)A(Z) \tag{4.7}$$

for $A(\rho) \neq 0$. Thus we have

$$S(Y,Z) = ag(Y,Z) + bA(Y)A(Z), \tag{4.8}$$

where $a = \frac{r}{(n-1)} \neq 0$ and $b = -\frac{1}{A(\rho)} \neq 0$. Due to (4.8), it is shown that this manifold is a quasi-Einstein manifold. ♢

**Theorem 5.** *If a $\left[G(SPRS)_n, \overline{\nabla}\right]$ whose curvature tensor $\overline{R}$ vanishes admits a parallel vector field $\rho$ with respect to the connection $\nabla$, then $\rho$ is orthogonal to the vector field $\mu$ corresponding to the associated* 1*-form $B$ of a $\left[G(SPRS)_n, \overline{\nabla}\right]$ and the scalar curvature $r$ with respect to the connection $\nabla$ is a non-zero constant.*

**Proof.** Since $\rho$ is a parallel vector field, we know that

$$\nabla_X \rho = 0.$$

From (4.4), it follows that

$$S(X,\rho) = 0. \tag{4.9}$$

Putting $Y = \rho$ into (4.7) and using (4.9), we get

$$\left(\frac{r}{n-1} - 1\right) A(X) = 0. \tag{4.10}$$

Since $A(X) \neq 0$ then by (4.10) one can get $r = n - 1$, which means that the scalar curvature of this manifold is a non-zero constant. Moreover, substituting $Z = \rho$ into (1.2) and using (4.9), it follows that

$$B(\rho)S(X,Y) = 0. \tag{4.11}$$

Since the Ricci tensor $S$ with respect to the connection $\nabla$ is not identically zero, we have

$$B(\rho) = g(\rho,\mu) = 0. \tag{4.12}$$

It follows from (4.12) that $\rho$ is orthogonal to $\mu$. Hence the proof is completed. ♢

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