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SOME DIFFERENCE SEQUENCE SPACES DEFINED BY MUSIELAK–ORLICZ FUNCTION

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Abstract: In the present paper we introduce the sequence spaces $c_0\{\mathcal{M}, \Delta^n_m, p, q\}$, $c\{\mathcal{M}, \Delta^n_m, p, q\}$ and $l_\infty\{\mathcal{M}, \Delta^n_m, p, q\}$ defined by Musielak–Orlicz function $\mathcal{M} = (M_k)$. We study some topological properties and prove some inclusion relations between these spaces.

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1. Introduction and preliminaries

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [5] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq$ $\leq kLM(x)$ for all values of $x \geq 0$, $k > 0$ and for $L > 1$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak–Orlicz function see ([6], [7]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots$$

is called the complementary function of Musielak–Orlicz function $\mathcal{M}$. For a given Musielak–Orlicz function $\mathcal{M}$, the Musielak–Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\big\},$$
$$h_{\mathcal{M}} = \big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let l_∞, c and c_0 denote the spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms respectively. The zero sequence (0,0,...) is denoted by θ and $p = (p_k)$ is a sequence of strictly positive real numbers. Further the sequence (p_k^{-1}) will be represented by (t_k).

The notion of difference sequence spaces was introduced by Kizmaz [4], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. The notion was further generalized by Et and Colak [1] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$.

Let m, n be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_m^n) = \left\{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\right\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$ studied by Et and Colak [1]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$ introduced and studied by Kizmaz [4].

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,

(2) $p(-x) = p(x)$ for all $x \in X$,

(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,

(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [12], Th. 10.4.2, pp. 183). For more details about sequence spaces see ([8], [9], [10], [11]) and references therein.

In [2] Hazarika and in [3] Hazarika and Tripathy studied some difference sequence spaces. By making the use of these spaces we have also studied some sequence spaces in this paper.

Let $\mathcal{M} = (M_k)$ be a Musielak–Orlicz function and $p = (p_k)$ be any bounded sequence of positive real numbers and let (X, q) be a seminormed space seminormed by q. Now we define the following sequence spaces:

$$c_0\left\{\mathcal{M}, \Delta_m^n, p, q\right\} = \left\{x = (x_k) \in X : \left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right]^{p_k} t_k \to 0, \right.$$
$$\left. \text{as } k \to \infty, \text{ for some } \rho > 0\right\},$$
$$c\left\{\mathcal{M}, \Delta_m^n, p, q\right\} = \left\{x = (x_k) \in X : \left[M_k\left(\frac{q(\Delta_m^n x_k - L)}{\rho}\right)\right]^{p_k} t_k \to 0, \right.$$
$$\left. \text{as } k \to \infty, \text{ for some } L \in X \text{ and for some } \rho > 0\right\}$$

and

$$l_\infty\left\{\mathcal{M}, \Delta_m^n, p, q\right\} = \left\{x = (x_k) \in X : \sup_k \left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right]^{p_k} t_k < \infty, \right.$$
$$\left. \text{for some } \rho > 0\right\}.$$

If we take $p_k = 1$, we have

$$c_0\left\{\mathcal{M}, \Delta_m^n, q\right\} = \left\{x = (x_k) \in X : \left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right] \to 0, \right.$$
$$\left. \text{as } k \to \infty, \text{ for some } \rho > 0\right\},$$
$$c\left\{\mathcal{M}, \Delta_m^n, q\right\} = \left\{x = (x_k) \in X : \left[M_k\left(\frac{q(\Delta_m^n x_k - L)}{\rho}\right)\right] \to 0, \right.$$
$$\left. \text{as } k \to \infty, \text{ for some } L \in X \text{ and for some } \rho > 0\right\}$$

and

$$l_\infty\left\{\mathcal{M}, \Delta_m^n, q\right\} = \left\{x = (x_k) \in X : \sup_k \left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right] < \infty, \right.$$
$$\left. \text{for some } \rho > 0\right\}.$$

The following inequality will be used throughout the paper. If $0 \le p_k \le$ $\le \sup p_k = K$, $D = \max(1, 2^{K-1})$, then

$$|a_k + b_k|^{p_k} \le D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^K)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some difference sequence spaces defined by Musielak–Orlicz function. We also make an effort to study some topological properties and inclusion relations between these spaces.

2. Main results

Theorem 2.1. *Suppose $\mathcal{M} = (M_k)$ is a Musielak–Orlicz function and let $p = (p_k)$ be a bounded sequence of positive real numbers for all $k \in \mathbb{N}$. Then the spaces $c_0\{\mathcal{M}, \Delta_m^n, p, q\}$, $c\{\mathcal{M}, \Delta_m^n, p, q\}$ and $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. Let $x = (x_k)$, $y = (y_k) \in c\{\mathcal{M}, \Delta_m^n, p, q\}$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\left[M_k\Big(\frac{q(\Delta_m^n x_k - L)}{\rho_1}\Big)\right]^{p_k} t_k \to 0, \quad \text{as} \quad k \to \infty,$$

and

$$\left[M_k\Big(\frac{q(\Delta_m^n y_k - L)}{\rho_2}\Big)\right]^{p_k} t_k \to 0, \quad \text{as} \quad k \to \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_k's are non-decreasing, convex function and by using inequality (1.1), we have

$$\begin{aligned}
&\left[M_k\Big(\frac{q((\alpha\Delta_m^n x_k + \beta\Delta_m^n y_k) - 2L)}{\rho_3}\Big)\right]^{p_k} t_k \le \\
&\le \left[M_k\Big(\frac{q(\alpha\Delta_m^n x_k - L)}{\rho_3} + \frac{q(\beta\Delta_m^n y_k - L)}{\rho_3}\Big)\right]^{p_k} t_k \le \\
&\le D\frac{1}{2^{p_k}}\left[M_k\Big(\frac{q(\Delta_m^n x_k - L)}{\rho_1}\Big)\right]^{p_k} t_k + D\frac{1}{2^{p_k}}\left[M_k\Big(\frac{q(\Delta_m^n y_k - L)}{\rho_2}\Big)\right]^{p_k} t_k \le \\
&\le D\left[M_k\Big(\frac{q(\Delta_m^n x_k - L)}{\rho_1}\Big)\right]^{p_k} t_k + D\left[M_k\Big(\frac{q(\Delta_m^n y_k - L)}{\rho_2}\Big)\right]^{p_k} t_k \to 0 \text{ as } k \to \infty.
\end{aligned}$$

Thus, $\alpha x + \beta y \in c\{\mathcal{M}, \Delta_m^n, p, q\}$. Hence $c\{\mathcal{M}, \Delta_m^n, p, q\}$ is a linear space. Similarly, we can prove $c_0\{\mathcal{M}, \Delta_m^n, p, q\}$ and $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$ are linear spaces over the field $\mathbb{C}$ of complex numbers.

Theorem 2.2. *For any Musielak–Orlicz function $\mathcal{M} = (M_k)$ and $p = (p_k)$ be a bounded sequence of strictly positive real numbers, then the space $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$ is a paranormed space with the paranorm defined by*

$$g(x) = q(x_1) + \inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)t_k^{\frac{1}{p_k}}\right\} \leq 1, \quad \rho > 0\right\},$$

where $H = \max(1, K)$.

Proof. (i) Clearly, $g(x) \geq 0$ for $x = (x_k) \in l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$. Since $M_k(0) = 0$, we get $g(\theta) = 0$.

(ii) $g(-x) = g(x)$.

(iii) Let $x = (x_k),\ y = (y_k) \in l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$, then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n x_k)}{\rho_1}\right)t_k^{\frac{1}{p_k}}\right\} \leq 1$$

and

$$\sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n y_k)}{\rho_2}\right)t_k^{\frac{1}{p_k}}\right\} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by Minkowski's inequality, we have

$$\begin{aligned}
&\sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n(x_k + y_k))}{\rho}\right)t_k^{\frac{1}{p_k}}\right\} = \\
&= \sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n(x_k + y_k))}{\rho_1 + \rho_2}\right)t_k^{\frac{1}{p_k}}\right\} \leq \\
&\leq \frac{\rho_1}{\rho_1 + \rho_2}\sup_{k\geq 1}\left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho_1}\right)t_k^{\frac{1}{p_k}}\right] + \\
&\quad + \frac{\rho_2}{\rho_1 + \rho_2}\sup_{k\geq 1}\left[M_k\left(\frac{q(\Delta_m^n y_k)}{\rho_2}\right)t_k^{\frac{1}{p_k}}\right] \leq 1
\end{aligned}$$

and thus

$$\begin{aligned}
&g(x + y) = \\
&= q(x_1 + y_1) + \\
&\quad + \inf\left\{(\rho_1 + \rho_2)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n x_k + \Delta_m^n y_k)}{\rho}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1,\ \rho > 0\right\} \leq \\
&\leq q(x_1) + \inf\left\{(\rho_1)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n x_k)}{\rho_1}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1, \quad \rho > 0\right\} + \\
&\quad + q(y_1) + \inf\left\{(\rho_2)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n y_k)}{\rho_2}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1, \quad \rho > 0\right\} \leq \\
&\leq g(x) + g(y).
\end{aligned}$$

(iv) Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = q(\lambda x_1) + \inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q(\lambda\Delta_m^n x_k)}{\rho}\right)\right\} t_k{}^{\frac{1}{p_k}} \leq 1,\ \rho > 0\right\} =$$

$$= |\lambda| q(x_1) + \inf\left\{(|\lambda| r)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n x_k)}{r}\right)\right\} t_k{}^{\frac{1}{p_k}} \leq 1,\ r > 0\right\},$$

where $r = \frac{\rho}{|\lambda|}$. Hence $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$ is a paranormed space.

Theorem 2.3. *If $\mathcal{M} = (M_k)$ is a Musielak–Orlicz function and $p = (p_k) \in l_\infty$, then the spaces $c_0\{\mathcal{M}, \Delta_m^n, p, q\}$, $c\{\mathcal{M}, \Delta_m^n, p, q\}$ and $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$ are complete paranormed spaces paranormed by g.*

Proof. Suppose (x^n) is a Cauchy sequence in $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$, where $x^n = (x_k^n)_{k=1}^\infty$ for all $n \in \mathbb{N}$. So that $g(x^i - x^j) \to 0$ as $i, j \to \infty$. Suppose $\epsilon > 0$ is given and let s and x_0 be such that $\frac{\epsilon}{sx_0} > 0$ and $M_k\left(\frac{sx_0}{2}\right) \geq \sup_{k\geq 1}(p_k)^{t_k}$. Since $g(x^i - x^j) \to 0$, as $i, j \to \infty$ which means that there exists $n_0 \in \mathbb{N}$ such that

$$g(x^i - x^j) < \frac{\epsilon}{sx_0}, \quad \text{for all } \ i, j \geq n_0.$$

This gives $g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0}$ and

(2.1)
$$\inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q(\Delta_m^n x_k^i - \Delta_m^n x_k^j)}{\rho}\right) t_k{}^{\frac{1}{p_k}}\right\} \leq 1, \quad \rho > 0\right\} < \frac{\epsilon}{sx_0}.$$

It shows that (x_1^i) is a Cauchy sequence in X. Therefore (x_1^i) is convergent in X because X is complete. Suppose $\lim_{i\to\infty} x_1^i = x_1$, then

$$\lim_{j\to\infty} g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0},$$

we get

$$g(x_1^i - x_1) < \frac{\epsilon}{sx_0}.$$

Thus, we have

$$M_k\left(\frac{q(\Delta_m^n x_k^i - \Delta_m^n x_k^j)}{g(x^i - x^j)}\right) t_k{}^{\frac{1}{p_k}} \leq 1.$$

This implies that

$$M_k\left(\frac{q(\Delta_m^n x_k^i - \Delta_m^n x_k^j)}{g(x^i - x^j)}\right) \le (p_k)^{t_k} \le M_k\left(\frac{sx_0}{2}\right)$$

and thus

$$q(\Delta_m^n x_k^i - \Delta_m^n x_k^j) < \frac{sx_0}{2}.\frac{\epsilon}{sx_0} < \frac{\epsilon}{2}$$

which shows that $(\Delta_m^n x_k^i)$ is a Cauchy sequence in X for all $k \in \mathbb{N}$. Therefore, $(\Delta_m^n x_k^i)$ converges in X. Suppose $\lim\limits_{i\to\infty} \Delta_m^n x_k^i = y_k$ for all $k \in \mathbb{N}$. Also, we have $\lim\limits_{i\to\infty} \Delta_m^n x_2^i = y_1 - x_1$. On repeating the same procedure, we obtain $\lim\limits_{i\to\infty} \Delta_m^n x_{k+1}^i = y_k - x_k$ for all $k \in \mathbb{N}$. Therefore by continuity of M_k, we get

$$\lim_{j\to\infty} \sup_{k\ge1} M_k\left(\frac{q(\Delta_m^n x_k^i - \Delta_m^n x_k^j)}{\rho}\right) t_k^{\frac{1}{p_k}} \le 1,$$

so that

$$\sup_{k\ge1} M_k\left(\frac{q(\Delta_m^n x_k^i - \Delta_m^n x_k^j)}{\rho}\right) t_k^{\frac{1}{p_k}} \le 1.$$

Let $i \ge n_0$ and taking infimum of each ρ's, we have $g(x^i - x) < \epsilon$. So $(x^i - x) \in l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$. Hence $x = x^i - (x^i - x) \in l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$, since $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$ is a linear space. Hence, $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$ is a complete paranormed space. Similarly, we can prove that the spaces $c_0\{\mathcal{M}, \Delta_m^n, p, q\}$ and $c\{\mathcal{M}, \Delta_m^n, p, q\}$ are complete paranormed spaces.

Theorem 2.4. *If $0 < p_k \le r_k < \infty$ for each k, then $Z\{\mathcal{M}, \Delta_m^n, p, q\} \subseteq$ $\subseteq Z\{\mathcal{M}, \Delta_m^n, r, q\}$ for $Z = c_0$ and c.*

Proof. Let $x = (x_k) \in c\{\mathcal{M}, \Delta_m^n, p, q\}$. Then there exists some $\rho > 0$ and $L \in X$ such that

$$M_k\Big(\frac{q(\Delta_m^n x_k - L)}{\rho}\Big)^{p_k} t_k \to 0 \text{ as } k \to \infty.$$

This implies that

$$M_k\Big(\frac{q(\Delta_m^n x_k - L)}{\rho}\Big) < \epsilon, \ (0 < \epsilon < 1)$$

for sufficiently large k. Hence we get

$$M_k\Big(\frac{q(\Delta_m^n x_k - L)}{\rho}\Big)^{r_k} t_k \le M_k\Big(\frac{q(\Delta_m^n x_k - L)}{\rho}\Big)^{p_k} t_k \to 0 \text{ as } k \to \infty.$$

This implies that $x = (x_k) \in c\{\mathcal{M}, \Delta_m^n, r, q\}$. This completes the proof. $\Diamond$

Similarly, we can prove for the case $Z = c_0$.

Theorem 2.5. *Suppose $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ are Musielak–Orlicz functions satisfying the Δ_2-condition then we have the following results:*

(i) *if $(p_k) \in l_\infty$ then $Z\{\mathcal{M}', \Delta_m^n, p, q\} \subseteq Z\{\mathcal{M}'' \circ \mathcal{M}', \Delta_m^n, p, q\}$ for $Z = c, c_0$ and l_∞.*

(ii) *$Z\{\mathcal{M}', \Delta_m^n, p, q\} \cap Z\{\mathcal{M}'', \Delta_m^n, p, q\} \subseteq Z\{\mathcal{M}' + \mathcal{M}'', \Delta_m^n, p, q\}$ for $Z = c, c_0$ and l_∞.*

Proof. If $x = (x_k) \in c_0\{\mathcal{M}, \Delta_m^n, p, q\}$ then there exists some $\rho > 0$ such that

$$\left\{ M_k' \left(\frac{q(\Delta_m^n x_k)}{\rho} \right) \right\}^{p_k} t_k \to 0 \text{ as } k \to \infty.$$

Suppose

$$y_k = M_k' \left(\frac{q(\Delta_m^n x_k)}{\rho} \right) \text{ for all } k \in \mathbb{N}.$$

Choose $\delta > 0$ be such that $0 < \delta < 1$, then for $y_k \geq \delta$ we have $y_k < \frac{y_k}{\delta} < < 1 + \frac{y_k}{\delta}$. Now M_k'' satisfies Δ_2-condition so that there exists $J \geq 1$ such that

$$M_k''(y_k) < \frac{Jy_k}{2\delta} M_k''(2) + \frac{Jy_k}{2\delta} M_k''(2) = \frac{Jy_k}{\delta} M_k''(2).$$

We obtain

$$\left[(M_k'' \circ M_k') \left(\frac{q(\Delta_m^n x_k)}{\rho} \right) \right]^{p_k} t_k =$$

$$= \left[M_k'' \left\{ M_k' \left(\frac{q(\Delta_m^n x_k)}{\rho} \right) \right\} \right]^{p_k} t_k = [M_k''(y_k)]^{p_k} t_k \leq$$

$$\leq \max \left\{ \sup_k \left([M_k''(1)]^{p_k} \right), \sup_k \left([k M_k''(2) \delta^{-1}]^{p_k} \right) \right\} [y_k]^{p_k} t_k \to 0, \text{ as } k \to \infty.$$

Similarly, we can prove the other cases.

(ii) Suppose $x = (x_k) \in c_0\{M_k', \Delta_m^n, p, q\} \cap c_0\{M_k'', \Delta_m^n, p, q\}$, then there exist $\rho_1,\ \rho_2 > 0$ such that

$$\left\{ \left(M_k' \left(\frac{q(\Delta_m^n x_k)}{\rho_1} \right) \right)^{p_k} t_k \right\} \to 0, \text{ as } k \to \infty$$

and

$$\left\{\left(M_k''\left(\frac{q(\Delta_m^n x_k)}{\rho_2}\right)\right)^{p_k} t_k\right\} \to 0, \quad \text{as} \quad k \to \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The remaining proof follows from the inequality

$$\left\{\left[(M_k' + M_k'')\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right]^{p_k} t_k\right\} \leq$$
$$\leq D\left\{\left[M_k'\left(\frac{q(\Delta_m^n x_k)}{\rho_1}\right)\right]^{p_k} t_k + \left[M_k''\left(\frac{q(\Delta_m^n x_k)}{\rho_2}\right)\right]^{p_k} t_k\right\}.$$

Hence $c_0\{M_k', \Delta_m^n, p, q\} \cap c_0\{M_k'', \Delta_m^n, p, q\} \subseteq c_0\{M_k' + M_k'', \Delta_m^n, p, q\}$. Similarly we can prove the other cases.

Theorem 2.6. (i) *If* $0 < \inf p_k \leq p_k < 1$, *then* $l_\infty\{\mathcal{M}, \Delta_m^n, p, q\} \subset \subset l_\infty\{\mathcal{M}, \Delta_m^n, q\}$.

(ii) *If* $1 \leq p_k \leq \sup p_k < \infty$, *then* $l_\infty\{\mathcal{M}, \Delta_m^n, q\} \subset l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$.

Proof. (i) Let $x = (x_k) \in l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$. Since $0 < \inf p_k \leq 1$, we have

$$\sup_k \left\{\left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right]\right\} \leq \sup_k \left\{\left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right]^{p_k} t_k\right\}$$

and hence $x = (x_k) \in l_\infty\{\mathcal{M}, \Delta_m^n, q\}$.

(ii) Let $p_k \geq 1$ for each k and $\sup_k p_k < \infty$. Let

$$x = (x_k) \in l_\infty\{\mathcal{M}, \Delta_m^n, q\},$$

then for each ϵ, $0 < \epsilon < 1$, there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\sup_k \left\{M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right\} \leq \epsilon < 1.$$

This implies that

$$\sup_k \left\{\left[M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right]^{p_k} t_k\right\} \leq \sup_k \left\{M_k\left(\frac{q(\Delta_m^n x_k)}{\rho}\right)\right\}.$$

Thus $x = (x_k) \in l_\infty\{\mathcal{M}, \Delta_m^n, p, q\}$ and this completes the proof. $\Diamond$

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