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ON COMPLETELY *-PRIME *-IDEALS IN INVOLUTION RINGS

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Abstract: The notion of completely *-prime *-ideal of an involution ring is introduced and investigated. In particular, it is shown that involution rings in which the zero *-ideal is completely *-prime (called *-domains) are precisely those in which there are neither symmetric nor skew-symmetric zero divisors. It is shown that such involution rings may be embedded in involution rings with unity having the same property. Several characterizations of *-domains and, in particular, of Goldie *-domains are provided. Finally, this new concept is used to describe the generalized nil radical of an involution ring.

1. Introduction

It is well-known that the symmetric and skew-symmetric elements play an important role in the determination of the algebraic structure of an involution ring. Lanski [9] and Herstein and Montgomery [8] established rather complete structure theorems for rings with involution whose symmetric and skew-symmetric elements, respectively, are not zero divisors. In this paper we investigate, in particular, involution rings in which neither symmetric nor skew-symmetric elements are zero divisors. We introduce the notion of completely *-prime *-ideal of an involution ring

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(which is a generalization of the concept of completely prime *-ideal) and show that involution rings in which there are no symmetric or skew-symmetric zero divisors are precisely those in which the zero *-ideal is completely *-prime. We show that such rings may be embedded in involution rings with identity having the same property and we characterize Goldie involution rings without either symmetric or skew-symmetric elements. Finally, we describe the generalized nil radical of an involution ring in terms of its completely *-prime *-ideals.

Throughout this paper, all rings are assumed to be associative but do not necessarily have identity. Let us recall that an involution ring R is a ring with an additional unary operation *, called involution, such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$ and $(a^*)^* = a$, for all $a, b \in R$. An element of an involution ring R which is either symmetric or skew-symmetric shall be called a *-element and a *-ideal I of R is an ideal which is closed under involution; that is, $I^* = \{a^* \in R : a \in I\} \subseteq I$. A subring B of R such that $BRB \subseteq B$ and which is closed under involution, is called a *-biideal of R.

2. Completely *-prime *-ideals and *-domains

The concepts of completely prime ideal, completely semiprime ideal, domain and reduced ring are well-known in the category of rings and have been extensively studied. We now study generalizations of these concepts in the category of involution rings and use them to prove similar results to some well-known ones for rings (without involution).

Definition 1. An involution ring R is called a *-domain if, for any nonzero $a, b \in R$, $ab \neq 0$ or $ab^* \neq 0$. If, for any nonzero $a \in R$, $a^2 \neq 0$ or $aa^* \neq 0$, then R is said to be *-reduced.

Definition 2. Let P be a proper *-ideal of an involution ring R. Then P is called completely *-prime if R/P is a *-domain. If R/P is *-reduced, then P is said to be completely *-semiprime.

Example 3. Let D be a domain and $R = D \oplus D^{op}$ where D^{op} is the opposite ring of D. Then R is a ring with involution defined by $(a, b)^* = (b, a)$ for all $a, b \in D$. This involution is known as the exchange involution. The involution ring R is a *-domain which is not prime.

Example 4. The ring $M_2(\mathbb{Z}_3)$ of 2×2 matrices over the field $\mathbb{Z}_3$, endowed with the usual transposition of matrices, is an example of a *-reduced involution ring which is not reduced.

Proposition 5. *Let R be an involution ring and P a proper *-ideal of R. Then the following statements are equivalent:*

(i) *P is a completely *-prime *-ideal of R;*

(ii) *P is a completely semiprime *-prime *-ideal of R.*

Proof. (i) implies (ii). Let P be a completely *-prime *-ideal of $R, a \in R$ and $a^2 \in P$. If a is a *-element, then $a \in P$. If a is not a *-element, then $(a^*a)(aa^*) \in P$ implies that $aa^* \in P$ or $a^*a \in P$. From this and $a^2 \in P$ we get that $a \in P$. Hence P is a completely semiprime *-ideal. To show that P is *-prime, let I and J be *-ideals of R such that $P \subset I$, $P \subset J$ and $IJ \subseteq P$. If $I \nsubseteq P$, then there exists a nonzero element $a \in I$ such that $a \notin P$. Then $(a+P)(J/P) = 0$ and $(a^*+P)(J/P) = 0$ and, since R/P is a *-domain, it follows that $J \subseteq P$.

(ii) implies (i). Suppose that P is a completely semiprime *-prime *-ideal of R. Let $a, b \in R$ such that $ab \in P$ and $ab^* \in P$. Then $(a^*Rb)^2 \subseteq P$ and $(a^*Rb^*)^2 \subseteq P$, whence $a^*Rb \subseteq P$ and $a^*Rb^* \subseteq P$. The *-primeness of P now implies that $a \in P$ or $b \in P$ (see [3]). $\Diamond$

It is well-known that a ring R is a domain (that is, has no zero divisors) if and only if R is a prime reduced ring. Now we have the following for rings with involution:

Corollary 6. *The following conditions are equivalent for an involution ring R:*

(i) *R is a *-domain;*

(ii) *R is reduced and *-prime.*

As for rings without involution [15], we have the following result concerning involution rings in which every left and every right ideal is two-sided, called duo involution rings:

Proposition 7. *If R is a duo involution ring and P is a proper *-ideal of R, then the following conditions are equivalent:*

(i) *P is a *-prime *-ideal;*

(ii) *P is a completely *-prime *-ideal.*

Proof. Let $a, b \in R$ with $ab \in P$ and $ab^* \in P$. The set $T = \{t \in R : at \in P\}$ is an ideal of R and $b, b^* \in T$; hence $Rb \subseteq T$ and $Rb^* \subseteq T$. Therefore $aRb \subseteq P$ and $aRb^* \subseteq P$, whence $a \in P$ or $b \in P$. $\Diamond$

The proof of the next result is immediate.

Proposition 8. *For an involution ring R, the following statements are equivalent:*

(i) *R is *-reduced;*

(ii) *R has no nonzero nilpotent *-elements.*

The next theorem gives several characterizations of *-domains. Taking into account the next lemma, the proof of the equivalence of statements (v), (vi) and (vii) is a straightforward adaptation of the proof of ([3], Th. 4.2). The concepts of "essential product", "essential ideal" and "*-essential *-ideal" are as defined in [3].

Lemma 9. *If I is a nonzero ideal of a ring R such that I is a domain, then $ann_R(I) = \{x \in R : xI = Ix = 0\}$ is a completely prime ideal of R and it is a minimal prime ideal of R.*

Proof. Let $a, b \in R$, $ab \in ann_R(I)$. We have $(bIa)^2 = 0$, which implies that $bIaI = 0$. Since I is a prime ring, $aI = 0$ or $bI = 0$. As I is a domain, its left annihilator coincides with its right annihilator and so $a \in ann_R(I)$ or $b \in ann_R(I)$. By ([3], Lemma 3.2), $ann_R(I)$ is a minimal prime ideal of R. $\Diamond$

Theorem 10. *The following conditions are equivalent for an involution ring R:*

(i) *R is a *-domain;*

(ii) *the product of any two nonzero *-elements in R is nonzero;*

(iii) *the *-elements in R are not zero divisors;*

(iv) *if $a, b \in R$ and b is a *-element such that $RabR = 0$, then $a = 0$ or $b = 0$;*

(v) *R is a domain or R has a nonzero completely prime ideal P such that $P \cap P^* = 0$;*

(vi) *R is a domain or R has a nonzero ideal P such that*

(a) *$P \cap P^* = 0$,*

(b) *$P \oplus P^*$ is a *-essential *-ideal of R,*

(c) *P and P^* are domains;*

(vii) *R is a domain or R has a nonzero completely prime ideal I such that*

(a) *$I \cap I^* = 0$,*

(b) *$I \oplus I^*$ is an essential ideal of R,*

(c) *I and I^* are domains,*

(d) *$\overline{I} \oplus \overline{I}^*$ is an essential ideal of $R/I^* \oplus R/I$ where $\overline{I} = (I \oplus I^*)/I^*$ and $\overline{I}^* = (I \oplus I^*)/I$,*

(e) *R is an essential product of R/I^* and R/I.*

Proof. Clearly, (i) implies (ii).

(ii) implies (iii). First we observe that if (ii) holds, then R is reduced. Indeed, let $a^2 = 0$ for some $a \in R$. Then $(a^*a)(aa^*) = 0$ implies that $a^*a = 0$ or $aa^* = 0$. If $a^*a = 0$, then $(aa^*)^2 = 0$ and we have $aa^* = 0$. From $(a^* + a)^2 = 0$, we obtain $a^* = -a = 0$. If $aa^* = 0$, then it follows in a similar way that $a = 0$. Next, let $bc = 0$, where $b, c \in R$ and b is a *-element. Since R is reduced, $cb = 0$ and, therefore, $b(c + c^*) = 0$. Hence $b = 0$ or $c = 0$.

(iii) implies (iv). Let $a, b \in R$ where b is a *-element such that $RabR = 0$. Then $a^2b^2 = 0$ implies that $a^2 = 0$ or $b^2 = 0$ and hence $a = 0$ or $b = 0$.

(iv) implies (i) is obvious.

(i) implies (v). Let R be a *-domain which is not a domain. Then R is *-prime but not prime. In fact, there exist nonzero elements $r, s \in R$ such that $rs = 0$ and so $(sRr)^2 = 0$. Since R is reduced, $sRr = 0$. According to [3, Th. 4.2], there exists a nonzero prime ideal P of R such that $P \cap P^* = 0$. We claim that P is a completely prime ideal of R. Indeed, suppose that $a, b \in R$ are such that $ab \in P$. Then $(abb^*)(a^*a) \in$ $\in P \cap P^* = 0$ and so $a^*a = 0$ or $abb^* = 0$. If $a^*a = 0$, then $(aRa^*)^2 = 0$. Hence $aRa^* = 0 \subseteq P$ and we get $a \in P$ or $a^* \in P$. If $a^* \in P$, then $b^*a^* \in P \cap P^* = 0$ and so $ab = 0$. Now $(bRa)^2 = 0$ implies that $bRa = 0 \subseteq P$ and we obtain $a \in P$ or $b \in P$. If $abb^* = 0$, then either $a = 0 \in P$ or $bb^* = 0$. The latter case implies, arguing as above, that $a \in P$ or $b \in P$.

(v) implies (iii). Suppose that R is not a domain. Let $a, b \in R$, where b is a *-element and $ab = 0$. From $ab \in P$, it follows that $a \in P$ or $b \in P$. If $b \in P$, then, since b is a *-element, $b \in P \cap P^* = 0$. If $b \notin P$, then $a \in P$. Now $b^*a^* = (ab)^* = 0 \in P$ yields $a^* \in P$ and thus $a \in P \cap P^* = 0$. $\Diamond$

Corollary 11. *Let R be an involution ring with descending chain condition (d.c.c.) on *-biideals of the form xRx^* ($x \in R$). Then R is a *-domain if and only if R is a division ring or R is a ring of the form $D \oplus D^{op}$, where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution.*

Proof. If R is a *-domain, then R is either a domain or R has a nonzero ideal D such that $D \cap D^* = 0$, D and D^* are domains and $D \oplus D^*$ is a *-essential *-ideal of R. If a semiprime involution ring has d.c.c. on *-biideals of the form xRx^*, then R has d.c.c. on principal right ideals, as was proved in [10, Th. 6]. So, R is a von Neumann regular ring.

Thus, in the first case, R is a division ring, as is well-known. In the second case, we claim that D (and hence also D^*) has an identity. For any $0 \neq d \in D$, there exists $x \in R$ such that $d = dxd$. Therefore, $e = xd = (xd)^2$ is a nonzero idempotent in D. Consequently, for any $y \in D$, $(y - ye)\,e = 0$ and, since D is a domain, $y = ye$. Similarly, $ey = y$. From ([12], Lemma 8), it follows that $R = D \oplus D^*$. Since $D^* \cong D^{op}$, we have $R = D \oplus D^* \cong^* D \oplus D^{op}$ with the exchange involution. Since D and D^* are domains with d.c.c on principal right ideals, they are division rings. $\Diamond$

Next, we will present some characterizations of Goldie *-domains. To this end, we start by recalling some definitions and results, due to Domokos, concerning (classical) rings of quotients and Goldie involution rings.

If S is a nonempty subset of an involution ring R, let $\langle S\rangle^*$ denote the *-biideal of R generated by S and denote by $r(S)$ the right annihilator of S in R. Then $r(S) \cap r(S)^*$ is a *-biideal of R, called an annihilator *-biideal of R.

Definition 12 [6]**.** An involution ring R is called a Goldie involution ring, if the following two conditions are satisfied:

(i) there is no infinite sequence $B_1, ..., B_n, ...$ of nonzero *-biideals of R such that $\langle B_1 + ... + B_n\rangle^* \cap B_{n+1} = 0$ for all $n = 1, 2, ...$; that is, the maximum condition on *-biideal direct sums is satisfied;

(ii) there is no infinite strictly ascending chain

$$r(S_1) \cap r(S_1)^* \subset ... \subset r(S_n) \cap r(S_n)^* \subset ...$$

where $S_1, ..., S_n, ...$ are subsets of R; that is, R satisfies the ascending chain condition on annihilator *-biideals.

Definition 13 [6]**.** An involution ring Q is a *-ring of quotients of its *-subring R, if Q is a ring of quotients of R.

Proposition 14 [6]**.** *An involution ring R has a *-ring of quotients if and only if R has a ring of quotients and, furthermore, the *-ring of quotients is uniquely determined up to *-isomorphism over R.*

Proposition 15 [6]**.** *For a semiprime involution ring R, the following two statements are equivalent:*

(i) *R is a Goldie ring;*

(ii) *R is a Goldie involution ring.*

Proposition 16 [6]**.** *If R is a *-prime involution ring, then the following two statements are equivalent:*

(i) *R is a Goldie involution ring;*

(ii) *R has a *-ring of quotients Q and*

$$Q \cong M_n(D) \ \text{or}\ Q \cong^* M_n(D) \oplus M_n^{op}(D)$$

where $M_n(D)$ denotes the full $n \times n$ matrix ring over a division ring D and $M_n^{op}(D)$ denotes its opposite ring and $M_n(D) \oplus M_n^{op}(D)$ is endowed with the exchange involution.

Proposition 17 ([13], Prop. 1.5)**.** *If R is a reduced ring having a left ring of quotients Q, then Q is reduced.*

From [11, Lemma 3.1.6], we have:

Proposition 18. *Let Q be a ring of quotients of a ring R and let S be a subring of Q. Suppose further that there are units $a, b \in Q$ such that $aRb \subseteq S$. Then S also has Q as a ring of quotients.*

Clearly, if R is a *-domain, then $r(S) \cap r(S)^* = 0$ for any nonzero subset S of R; hence (ii) of Def. 12 automatically holds in a *-domain. The equivalence of (iv) and (v) below is the involutive version of [7, Lemma 4.2].

Theorem 19. *The following conditions are equivalent for an involution ring R:*

(i) *R is a Goldie *-domain;*

(ii) *R has a *-ring of quotients which is either a division ring or a ring of the form $D \oplus D^{op}$ where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution;*

(iii) *R is a Goldie domain or R has nonzero ideal P such that P and P^* are Goldie domains, $P \cap P^* = 0$ and $P \oplus P^*$ is *-essential in R;*

(iv) *R is a *-domain which satisfies the maximum condition on *-biideal direct sums;*

(v) *R is a *-domain in which the intersection of any two nonzero *-biideals of R is nonzero.*

Proof. (i) is equivalent to (ii). Follows readily from Cor. 6, Prop. 16 and Prop. 17.

(ii) implies (iii). If R has a *-ring of quotients which is a division ring, then R is a Goldie domain. Suppose now that the *-ring of quotients of R is of the form $D \oplus D^{op}$ where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution. Then R is a *-domain and, according to Th. 10, there exists a nonzero ideal P of R such that $P \cap P^* = 0$, P and P^* are domains, and $P \oplus P^*$ is *-essential in R. By Prop. 18, $P \oplus P^*$ also has $D \oplus D^{op}$ as its *-ring of quotients. Therefore $P \oplus P^*$ is a Goldie ring and hence satisfies the maximum condition on direct sums of left (right) ideals. Now it is clear that P and P^* satisfy the maximum condition on direct sums of left (right) ideals, since any left (right) ideal of P or P^* is also a left (right) ideal of $P \oplus P^*$. Therefore P and P^* are Goldie domains.

(iii) implies (i). Suppose that R is not a domain. By assumption, there exists a nonzero ideal P of R such that P and P^* are Goldie domains, $P \cap P^* = 0$ and $P \oplus P^*$ is *-essential in R. By Th. 10, R is a *-domain and so is $P \oplus P^*$. Moreover, since P and P^* are Goldie rings, $P \oplus P^*$ is also a Goldie ring. Since $P \oplus P^*$ is an essential ideal of R (see [3], Cor. 3.4), R is also a Goldie involution ring, as is well-known.

Clearly (i) is equivalent to (iv) and (v) implies (i).

(i) implies (v). Suppose that B_1 and B_2 are nonzero *-biideals of R such that $B_1 \cap B_2 = 0$. First, we consider the case when R is a domain. We have $(B_1R \cap B_2R)\,(RB_1 \cap RB_2) = 0$, implying that $B_1R \cap B_2R = 0$ or $RB_1 \cap RB_2 = 0$. However, R is a Goldie domain and so the intersection of any two nonzero left (right) ideals of R is nonzero (see [7], Lemma 4.2). We have thus arrived at a contradiction. Next we assume that R is not a domain. From Th. 10, R contains a nonzero proper ideal P such that P and P^* are domains, $P \cap P^* = 0$ and $P \oplus P^*$ is *-essential in R. Now B_1PB_1 and B_2PB_2 are nonzero biideals of P and $B_1PB_1 \cap B_2PB_2 = 0$. Thus $(B_1PB_1P \cap B_2PB_2P)\,(PB_1PB_1 \cap PB_2PB_2) = 0$, implying that $B_1PB_1P \cap B_2PB_2P = 0$ or $PB_1PB_1 \cap PB_2PB_2 = 0$. This contradicts the fact that P is a Goldie domain. $\Diamond$

Szendrei [14] showed that every domain can be embedded in a domain with identity. We terminate this section showing that every *-domain is embeddable in a *-domain with identity. To this end, we need the following lemma:

Lemma 20. *Let R be an involution ring and I a *-ideal of R which, considered as an involution ring, is a *-domain. If $ra = 0$ for some nonzero *-element $a \in I$ and $r \in R$, then $r \in ann_R(I)$.*

Proof. If $r = 0$, the result is clearly true. Suppose now that $r \neq 0$ and let $b \in I$. Then $0 = (br)\,a = (b^*r)\,a$. Hence $br = b^*r = 0$ and $(b - b^*)\,rb = 0$. This implies that $rb = 0$ or $b^* = b$. In the latter case, $b\,(rb) = 0$ implies that $rb = 0$. Hence the result follows. $\Diamond$

Proposition 21. *Let I be a nonzero *-ideal of an involution ring R such that I, as an involution ring, is a *-domain and let $A = ann_R(I)$. Then R/A is a *-domain and $(I + A)/A \cong^* I$.*

Proof. Let $r, s \in R$ such that $rs \in A$ and $s - s^* \in A$ or $s + s^* \in A$. For any nonzero *-element $a \in I$, we have $r(sas^*) = 0$, Clearly, sas^* is a *-element in I. If $sas^* \neq 0$, then, from the previous lemma, $r \in A$. If, on the other hand, $sas^* = 0$, then we get $0 = s(as^*a)$. If $s - s^* \in A$, then $(s - s^*)a = 0$; that is, $sa = s^*a$ and so as^*a is a *-element in I. Therefore, if $as^*a \neq 0$, the previous lemma implies that $s \in A$; otherwise, $as^*a = a(s^*a) = 0$ implies that $s^*a = 0$ and we conclude that $s \in A$. When $s + s^* \in A$, it follows in a similar way that $s \in A$. Finally, noting that $I \cap A = 0$ since I is semiprime, we have $(I+A)/A \cong^* I/(I \cap A) \cong^* I$. $\Diamond$

Corollary 22. *For each *-domain R, there exists a *-domain with identity which contains R as a *-ideal.*

Proof. Every involution ring R can be embedded as a *-ideal in an involution ring $R^\sharp$ with identity [1]. By application of the previous proposition to the involution ring $R^\sharp$, we obtain the desired result. $\Diamond$

3. The generalized nil radical

Our main goal in this section is to show that a completely semiprime *-ideal of an involution ring R is an intersection of completely *-prime *-ideals of R. The proofs are a straightforward adaptation of the proofs of the corresponding results in [2], for rings without involution. First, however, we require some preliminary results.

In [3], Birkenmeier and Groenewald called a subset N of an involution ring R a *-m-system if, for all $a, b \in N$, $aRb \cap N \neq \varnothing$ or $aRb^* \cap N \neq \varnothing$. It is clear that if P is a *-prime *-ideal, then $C(P)$, the complement of P in R, is a *-m-system. In some cases, the converse is also true. In fact, as in [2], it follows that any *-m-system N, not intersecting a *-ideal $I \neq R$, is contained in some maximal *-m-system M not intersecting I. We claim that the complement $C(M)$ is a *-prime *-ideal. Indeed, by Zorn's Lemma, among the *-ideals of R containing I and not intersecting M, there is a maximal *-ideal P. Now P is a *-prime *-ideal. Indeed, if I_1 and I_2 are *-ideals in R such that $P \subset I_1$ and $P \subset I_2$, then, by the maximality of P, there exist elements $a_1, a_2 \in R$ such that

$a_1 \in I_1 \cap M$ and $a_2 \in I_2 \cap M$. Since M is a *-m-system, there exists an element $x \in R$ such that $a_1xa_2 \in M$ or $a_1xa_2^* \in M$. However, $a_1xa_2 \in \in I_1I_2$ and $a_1xa_2^* \in I_1I_2$. Since P is a *-prime *-ideal, it is clear from [3, Th. 5.4] that $C(P)$ is a *-m-system. From $I \subseteq P$ and $P \cap M = \varnothing$, we obtain, $C(P) \cap I = \varnothing$ and $M \subseteq C(P)$ whence, by the maximality of M, $M = C(P)$. Thus $C(M) = P$ and P is a minimal *-prime *-ideal containing I, for if P' is a *-prime *-ideal such that $I \subseteq P' \subset P$, then $C(P') \cap I = \varnothing$ and $C(P') \cap I \neq \varnothing$, since $M = C(P) \subset C(P')$.

Theorem 23. *A completely semiprime *-ideal is an intersection of completely *-prime *-ideals.*

Proof. We show that if Q is a completely semiprime *-ideal of an involution ring R and I is any *-ideal of R such that $Q \subset I$, then there is a completely *-prime *-ideal P such that $Q \subseteq P$ and $I \nsubseteq P$. Choose $a \in R$ such that $a \in I$ and $a \notin Q$. Then the *-m-system $N = \{a^i : i = 1, 2, ...\}$ does not intersect the *-ideal Q. Let M be a maximal *-m-system containing N and not intersecting Q. As seen above, the complement $C(M) = P$ is a minimal *-prime *-ideal containing Q. Since $a \notin C(M) = P$, we have $I \nsubseteq P$. It remains to show that P is a completely *-prime *-ideal. To this end, we consider the set M_1 of all elements of the form $a_1^{r_1}a_2^{r_2}...a_n^{r_n}$ where each r_i is a positive integer and $a_1a_2...a_n \in M$. It is clear that $M \subseteq M_1$. Moreover, M_1 is a *-m-system. Indeed, if $a, b \in M_1$, then $a = a_1^{r_1}a_2^{r_2}...a_n^{r_n}$ with $a_1a_2...a_n \in M$ and $b = b_1^{s_1}b_2^{s_2}...b_m^{s_m}$ with $b_1b_2...b_m \in M$. Since M is a *-m-system, there exists $x \in R$ such that $a_1a_2...a_nxb_1b_2...b_m \in M$ or $a_1a_2...a_nx(b_1b_2...b_m)^* = a_1a_2...a_nxb_m^*...b_2^*b_1^* \in M$. By the definition of M_1, $axb = a_1^{r_1}a_2^{r_2}...a_n^{r_n}xb_1^{s_1}b_2^{s_2}...b_m^{s_m} \in M_1$ or $axb^* = = a_1^{r_1}a_2^{r_2}...a_n^{r_n}x(b_m^{s_m})^*...(b_2^{s_2})^*(b_1^{s_1})^* = a_1^{r_1}a_2^{r_2}...a_n^{r_n}x(b_m^*)^{s_m}...(b_2^*)^{s_2}(b_1^*)^{s_1} \in \in M_1$. Now we show that $M_1 \cap Q = \varnothing$. Since Q is completely semiprime, it holds that, for any $x, y \in Q$, $xy \in Q$ if and only if $yx \in Q$. Assume that there exists $c \in M_1 \cap Q$. Then $c = c_1^{r_1}c_2^{r_2}...c_n^{r_n}$ with $c_1c_2...c_n \in M$. If $r_1 > 1$, then $c_1^{r_1-1}c_2^{r_2}...c_n^{r_n}c_1 \in Q$ and hence $\left(c_1^{r_1-1}c_2^{r_2}...c_n^{r_n}\right)^2 \in Q$. This implies that $c_1^{r_1-1}c_2^{r_2}...c_n^{r_n} \in Q$. Continuing in this way, we eventually obtain, after a finite number of steps, $c_1c_2^{r_2}...c_n^{r_n} \in Q$. But then $c_2^{r_2}...c_n^{r_n}c_1 \in Q$. Repeating these arguments for $r_2, ..., r_n$, we obtain, after a finite number of steps, $c_1c_2...c_n \in Q$ and, therefore, $c_1c_2...c_n \in Q \cap M$; a contradiction with the choice of M. From $Q \cap M = \varnothing$ and the maximality of the *-m-system M, we obtain $M_1 = M = C(P)$. Now $t \in M$ implies that $t^2 \in M_1 = M$, so $t^2 \in C(M) = P$ implies that $t \in P$. So R/P is reduced and *-prime, hence a *-domain by Cor. 6. $\Diamond$

The generalized nil radical of a ring R, denoted by $\mathcal{N}_g(R)$, was introduced simultaneously by Andrunakievic and Thierrin (see [5]). It is clear that $\mathcal{N}_g(R) = R$ implies that $\mathcal{N}_g(R^{op}) = R^{op}$, whence, by [4, Prop. 1.1], for any involution ring R, $\mathcal{N}_g(R) = \mathcal{N}_g(R)^*$. By Th. 23, $\mathcal{N}_g(R)$ is an intersection of completely *-prime *-ideals of R. If Λ denotes the family of all completely *-prime *-ideals of R, then $\cap(P : P \in \Lambda) \subseteq$ $\subseteq \mathcal{N}_g(R)$. If the inclusion is proper, then, by the Lemma in [2], there exists a completely prime ideal P' such that $\cap(P : P \in \Lambda) \subseteq P'$ and $\mathcal{N}_g(R) \nsubseteq P'$; a contradiction with the fact that $\mathcal{N}_g(R)$ coincides with the intersection of all completely prime ideals of R [2]. We may now state the following:

Corollary 24. *For any involution ring R, $\mathcal{N}_g(R)$ coincides with the intersection of all completely *-prime *-ideals of R.*

Proposition 25. *Let R be a ring with involution. Any maximal *-m-system M not intersecting a given completely semiprime *-ideal $Q \neq R$, is a *-multiplicative system; that is, if $a, b \in M$, then $ab \in M$ or $ab^* \in M$.*

Proof. In the proof of the previous theorem, it was seen that $M = C(P)$ where P is a completely *-prime *-ideal. $\Diamond$

Corollary 26. *Each minimal *-prime *-ideal of a reduced involution ring R is a completely *-prime *-ideal.*

Proof. If P is a minimal *-prime *-ideal of R, then $C(P)$ is a maximal *-m-system not containing the completely semiprime *-ideal 0. Hence P is a completely *-prime *-ideal. $\Diamond$

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