Mathematica Pannonica
23/2 (2012), 277–289

SOME GLOBAL PROPERTIES OF MIXED QUASI-EINSTEIN MANIFOLDS

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Received: May 2012

MSC 2000: 53 C 25

Keywords: Quasi-Einstein manifolds, mixed quasi-Einstein manifolds, generalized quasi-Einstein manifolds, quasi-conformal curvature tensor, Killing vector field, harmonic vector field.

Abstract: The object of the present paper is to study a new type of Riemannian manifold called mixed quasi-Einstein manifolds $M(QE)_n$. Some geometric properties of mixed quasi-Einstein manifolds have been studied. Also we study some global properties of mixed quasi-Einstein manifolds. The existence of a mixed quasi-Einstein manifold have been proved by two non-trivial examples.

1. Introduction

A Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

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holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to ([1], p. 432), (1.1) is called the Einstein metric condition. Every Einstein manifold belongs to the class of Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1.2}$$

where $a, b \in \mathbb{R}$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \tag{1.3}$$

for all vector fields X.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition (1.2).

It is to be noted that Chaki and Maity [2] also introduced the notion of quasi-Einstein manifolds in a different way. They have taken a, b are scalars and the vector field U metrically equivalent to the 1-form A as a unit vector field. Such an n-dimensional manifold is denoted by $(QE)_n$. Quasi-Einstein manifolds have been studied by several authors such as De and Ghosh [6], De and De [7] and De, Ghosh and Binh [11] and many others.

Quasi-Einstein manifolds have been generalized by several authors in several ways such as generalized quasi-Einstein manifolds ([3], [8], [9], [13], [17]), super quasi-Einstein manifolds ([4], [10], [15]), pseudo quasi-Einstein manifolds [18], $N(k)$-quasi-Einstein manifolds ([14], [19]) and many others.

In a recent paper [16] Nagaraja generalizes the quasi-Einstein manifold as follows:

A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called mixed quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)B(Y) + cB(X)A(Y), \tag{1.4}$$

where a, b and c are smooth functions and A and B are non-zero 1-forms such that $g(X,U) = A(X)$ and $g(X,V) = B(X)$ for all vector fields X and U and V being the orthogonal unit vector fields called the generator of the manifold.

From (1.4), it follows that

$$S(Y,X) = ag(Y,X) + bA(Y)B(X) + cB(Y)A(X). \tag{1.5}$$

From (1.4) and (1.5), it follows that

$$(b-c)[A(X)B(Y) - A(Y)B(X)] = 0.$$

This shows that either $b = c$ or $A(X)B(Y) = A(Y)B(X)$. Motivated by this result we give the following definition:

A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called mixed quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition:

$$S(X,Y) = ag(X,Y) + b[A(X)B(Y) + A(Y)B(X)], \tag{1.6}$$

where a, b are scalars of which $b \neq 0$ and A and B are non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X), \quad g(U,V) = 0,$$

where U, V are unit vector fields. In such a case A, B are called associated 1-forms and U, V are called the generators of the manifold. Such an n-dimensional manifold is denoted by the symbol $M(QE)_n$.

If $b = 0$, then the manifold becomes an Einstein manifold. If $A = B$, then the manifold reduces to a quasi-Einstein manifold. This justifies the name mixed quasi-Einstein manifold.

A generalization of a manifold of quasi-constant curvature, called a manifold of mixed quasi-constant curvature is needed for the study of a $M(QE)_n$. Such a manifold is denoted by the symbol $M(QC)_n$ and is defined as follows:

A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is called a manifold of mixed quasi-constant curvature if its curvature tensor $\tilde{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \\ & + q[g(Y,Z)\{A(X)B(W) + B(X)A(W)\} + \\ & + g(X,W)\{A(Y)B(Z) + B(Y)A(Z)\} - \\ & - g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} - \\ & - g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}], \end{aligned} \tag{1.7}$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$ and p, q are scalars and A, B are non-zero 1-forms. If the 1-forms A and B are equal, then the manifold reduces to a manifold of quasi-constant curvature introduced by Chen and Yano [5].

The notion of quasi-conformal curvature tensor was given by Yano and Sawaki [21] and is defined as follows:

$$C^\star(X,Y)Z = a_1R(X,Y)Z + b_1[S(Y,Z)X - S(X,Z)Y + + g(Y,Z)QX - g(X,Z)QY] - - \frac{r}{n}\left[\frac{a_1}{n-1} + 2b_1\right][g(Y,Z)X - g(X,Z)Y]. \tag{1.8}$$

Here a_1 and b_1 are constant, R is the Riemannian curvature tensor of type $(1,3)$, S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator and r is the scalar curvature of the manifold.

If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (1.8) reduces to the conformal curvature tensor. A Riemannian manifold is called quasi-conformally flat if $C^\star = 0$ for $n > 3$.

The present paper is organised as follows:

After preliminaries in Sec. 3, we prove that a quasi-conformally flat $M(QE)_n$ is a $M(QC)_n$. In the next section we look for a sufficient condition in order that a $M(QE)_n$ may be quasi-conformally conservative. In Sec. 5, we study compact orientable $M(QE)_n$. In the next section we study Killing vector field in a compact orientable $M(QE)_n$. Sec. 7 is devoted to study Harmonic vector field in a $M(QE)_n$. Finally, we construct two non-trivial examples of a $M(QE)_n$.

2. Preliminaries

From (1.6), we have

$$S(X,X) = a|X|^2 + 2b|g(X,U)g(X,V)|, \quad \forall X. \tag{2.1}$$

Let θ_1 be the angle between U and any vector X; θ_2 be the angle between V and any vector X. Then $\cos\theta_1 = \frac{g(X,U)}{\sqrt{g(X,X)}}$ and $\cos\theta_2 = \frac{g(X,V)}{\sqrt{g(X,X)}}$.

If $b > 0$, we have from (2.1)

$$(a+2b)|X|^2 \geq a|X|^2 + 2b|g(X,U)g(X,V)| = S(X,X). \tag{2.2}$$

Now contracting (1.6) over X and Y we obtain

$$r = an, \tag{2.3}$$

where r is the scalar curvature of the manifold.

Putting $X = Y = U$ in (1.6), we get

$$S(U, U) = a. \tag{2.4}$$

Similarly, we have

$$S(V, V) = a \tag{2.5}$$

and

$$S(U, V) = b. \tag{2.6}$$

Let Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. Then

$$S(X, Y) = g(QX, Y), \tag{2.7}$$

for all $X, Y \in TM$.

Let l^2 denote the square of the length of the Ricci tensor S. Then

$$l^2 = S(Qe_i, e_i), \tag{2.8}$$

where $\{e_i\}$, $(i = 1, 2, 3, \ldots\ldots, n)$ is an orthonormal basis of the tangent space at each point of the manifold.

Then from (1.6), we have

$$l^2 = na^2 + 2b^2. \tag{2.9}$$

This result will be used in the sequel.

3. Quasi-conformally flat $M(QE)_n$ $(n > 3)$

A $M(QE)_n$ $(n > 3)$ is not, in general a $M(QC)_n$. In this section we consider a conformally flat $M(QE)_n$ $(n > 3)$ and show that such a $M(QE)_n$ is a $M(QC)_n$.

From (1.8) it follows that in a quasi-conformally flat Riemannian manifold (M^n, g) $(n > 3)$ the curvature tensor $\tilde{R}$ of type $(0, 4)$ has the following form:

$$\begin{aligned} a_1\tilde{R}(X,Y,Z,W) = &-b_1[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + \\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] + \\ &+ \frac{r}{n}\left\{\frac{a_1}{n-1} + 2b_1\right\}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.1}$$

Using (1.6) in (3.1), we obtain

$$
\begin{aligned}
(3.2)\qquad \tilde{R}(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]+\\
& + q[g(Y,Z)\{A(X)B(W) + B(X)A(W)\}+\\
& + g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\}-\\
& - g(X,Z)\{A(Y)B(W) + A(W)B(Y)\}-\\
& - g(Y,W)\{A(X)B(Z) + B(X)A(Z)\}],
\end{aligned}
$$

where $p = \frac{r}{a_1 n}(\frac{a_1}{n-1} + 2b_1) - 2ab_1$ and $q = -\frac{bb_1}{a_1}$.

Thus we can state the theorem:

Theorem 3.1. *Every quasi-conformally flat $M(QE)_n$ is a $M(QC)_n$.*

From Th. 3.1 we can also have the following corollary:

Corollary 1. *A conformally flat $M(QE)_n$ is a $M(QC)_n$.*

4. $M(QE)_n$ ($n > 3$) with divergence free quasi-conformal curvature tensor

In this section we look for sufficient condition in order that a $M(QE)_n$ ($n>3$) may be quasi-conformally conservative. Quasi-conformal curvature tensor is said to be conservative [12] if the divergence of $C^\star$ vanishes, i.e., div $C^\star = 0$.

In a $M(QE)_n$ if both a and b are constant, then contracting (1.6) we have r =constant, where r is the scalar curvature. Then $dr = 0$.

Using $dr = 0$ we obtain from (1.8)

$$
\begin{aligned}
(4.1)\qquad (\nabla_W C^\star)(X,Y,Z) = {} & a_1(\nabla_W R)(X,Y)Z + b_1[(\nabla_W S)(Y,Z)X-\\
& - (\nabla_W S)(X,Z)Y + g(Y,Z)(\nabla_W Q)(X)-\\
& - g(X,Z)(\nabla_W Q)(Y)].
\end{aligned}
$$

We know that $(\operatorname{div} R)(X,Y,Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$, and from (1.6) we get

$$
\begin{aligned}
(4.2)\qquad (\nabla_X S)(Y,Z) = {} & b[(\nabla_X A)(Y)B(Z) + (\nabla_X B)(Z)A(Y)+\\
& + (\nabla_X B)(Y)A(Z) + (\nabla_X A)(Z)B(Y)],
\end{aligned}
$$

since both a and b are constant.

Contracting (4.1) and using (4.2) we get

$$(\mathrm{div}\, C^\star)(X,Y,Z) = (a_1+b_1)\big[(\nabla_X A)(Y)B(Z)+(\nabla_X B)(Z)A(Y)+ + (\nabla_X B)(Y)A(Z) + (\nabla_X A)(Z)B(Y)- - (\nabla_Y A)(X)B(Z) - (\nabla_Y B)(Z)A(X)- - (\nabla_Y B)(X)A(Z) - (\nabla_Y A)(Z)B(X)\big]. \tag{4.3}$$

Imposing the condition that the generators U and V of the manifold are parallel vector fields give $\nabla_X U = 0$, $\nabla_X V = 0$.

Hence

$$g(\nabla_X U, Y) = 0, \text{ i.e., } (\nabla_X A)(Y) = 0, \tag{4.4}$$

and

$$g(\nabla_X V, Y) = 0, \text{ i.e., } (\nabla_X B)(Y) = 0. \tag{4.5}$$

Therefore, from (4.3) it follows

$$(\mathrm{div}\ C^\star)(X,Y,Z) = 0.$$

Thus we can state the following:

Theorem 4.1. *If in a $M(QE)_n$, the associated scalars are constants and the generators U and V of the manifold are parallel vector fields, then the manifold is quasi-conformally conservative.*

5. Compact orientable $M(QE)_n$

In this section we consider a compact orientable $M(QE)_n$ without boundary having constant associated scalars a, b. Then from (2.3) and (2.9), it follows that the scalar curvature is constant and so also is the length of the Ricci tensor. We further suppose that $M(QE)_n$ under consideration admits a non-isometric conformal motion generated by a vector field X.

Since l^2 is constant, it follows that

$$\pounds_X(l^2) = 0, \tag{5.1}$$

where $\pounds_X$ denotes Lie differentiation with respect to X.

Now, it is known [20] that if a compact Riemannian manifold M of dimension $n \geq 3$ with constant scalar curvature admits an infinitesimal non-isometric conformal transformation X such that $\pounds_X(l^2) = 0$, then M is isometric to a sphere. But a sphere is Einstein so that b vanishes which is a contradiction.

This leads to the following:

Theorem 5.1. *A compact orientable $M(QE)_n$ $(n \geq 3)$ without boundary does not admit non-isometric conformal vector fields.*

6. Killing vector fields in compact orientable $M(QE)_n$

In this section we consider a compact orientable $M(QE)_n\,(n \geq 3)$ without boundary with a, b as associated scalars and U and V as associated generators.

It is known that for a vector field X in a Riemannian manifold M, the following relation holds

$$\int_M [S(X,X) - |\nabla X|^2 - (\operatorname{div} X)^2]dv \leq 0, \tag{6.1}$$

where dv denotes the volume element of M.

If X is a Killing vector field, then $\operatorname{div} X = 0$ [20]. Hence (6.1) takes the following form

$$\int_M [S(X,X) - |\nabla X|^2]dv = 0. \tag{6.2}$$

Let $b > 0$, then by (2.2) $(a + 2b)|X|^2 \geq S(X,X)$.

Consequently,

$$\int_M [(a + 2b)|X|^2]dv \geq \int_M [S(X,X) - |X|^2]dv.$$

Hence by (6.2), we have

$$\int_M [(a + 2b)|X|^2 - |\nabla X|^2]dv \geq 0. \tag{6.3}$$

If $a + 2b < 0$, then

$$\int_M [(a + 2b)|X|^2 - |\nabla X|^2]dv = 0.$$

Therefore, $X = 0$. This leads to the following theorem:

Theorem 6.1. *If in a compact orientable $M(QE)_n$ $(n \geq 3)$ without boundary the associated scalars and the structure tensor are such that $b > 0$ and $a + 2b < 0$, then there exists no non-zero Killing vector field in this manifold.*

7. Harmonic vector fields in a compact orientable $M(QE)_n$ $(n \geq 3)$ without boundary

Let us assume $\theta_2 \leq \theta_1$, θ_1 is the angle between U and any vector field X and θ_2 is the angle between V and any vector field X. Then we have $\cos\theta_2 \geq \cos\theta_1$ and $g(X,U) \geq g(X,V)$. Then from (2.1) we obtain

$$S(X,X) \geq (a+2b)\{g(X,U)\}^2, \tag{7.1}$$

where a, b are positive.

A vector field H in a Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be harmonic if [20] $d\tau = 0$ and $\delta\tau = 0$, where $\tau(X) = g(X,H)$ $\forall X$.

It is known that in a compact orientable Riemannian manifold (M^n, g) $(n \geq 3)$, the following relation holds

$$\int_M [S(X,X) - \frac{1}{2}|d\tau|^2 + |\nabla X|^2 - (\delta\tau)^2]dv = 0, \tag{7.2}$$

for any vector field X and dv denotes the volume element of M.

Now let $X \in \chi(M)$ be a harmonic vector field. Then from (7.2) we get

$$\int_M [S(X,X) + |\nabla X|^2]dv = 0, \tag{7.3}$$

for any vector field X.

Hence if a, b of $M(QE)_n$ are positive, then (7.1) and (7.3) together yield

$$\int_M [(a+2b)|g(X,U)|^2 + |\nabla X|^2]dv \leq 0, \tag{7.4}$$

which implies by virtue of $a+2b > 0$ that $g(X,U) = 0$ and $\nabla X = 0$, for any vector field X. This follows that X is orthogonal to U and X is a parallel vector field.

Similarly, for the case $\theta_1 \leq \theta_2$ and calculating as before it can be shown that $g(X,V) = 0$ and $\nabla X = 0$, for any vector field X.

Thus we can state the following:

Theorem 7.1. *In a compact orientable $M(QE)_n$ $(n \geq 3)$ without boundary any harmonic vector field X is parallel and orthogonal to one of the generators of the manifold which makes greatest angle with the vector field X, provided that a, b are positive scalars.*

8. Example of a $M(QE)_n$

Example 1. We consider a Riemannian manifold (M^4, g) endowed with the metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 + (dx^4)^2, \tag{8.1}$$

$i, j = 1, 2, 3, 4.$

The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = -x^1, \quad \Gamma^2_{33} = -\frac{x^2}{(x^1)^2}, \quad \Gamma^2_{12} = \frac{1}{x^1}, \quad \Gamma^3_{23} = \frac{1}{x^2},$$

$$R_{1332} = -\frac{x^2}{x^1}, \qquad S_{12} = -\frac{1}{x^1x^2}.$$

It can be easily shown that the scalar curvature of the manifold is zero. Therefore R^4 with the considered metric is a Riemannian manifold (M^4, g) of vanishing scalar curvature. We shall now show that this M^4 is a $M(QE)_4$ i.e., it satisfies the defining relation (1.6).

We take the associated scalars as follows:

$$a = \frac{1}{x^1(x^2)^2}, \quad b = -\frac{2}{(x^1)^2x^2}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} x^1, & \text{for } i = 2, \\ 0, & \text{for } i = 1, 3, 4, \end{cases}$$

and

$$B_i(x) = \begin{cases} \frac{1}{2}, & \text{for } i = 1, \\ \frac{3^{1/2}x^2}{2}, & \text{for } i = 3, \\ 0, & \text{for } i = 2, 4, \end{cases}$$

at any point $x \in M$. In our (M^4, g), (1.6) reduces with these associated scalars and 1-forms to the following equation:

$$S_{12} = ag_{12} + b[A_1B_2 + A_2B_1]. \tag{8.2}$$

It can be easily proved that the equation (8.2) is true.

We shall now show that the 1-forms are unit and orthogonal.

Here,

$$g^{ij}A_iA_j = 1, \;\; g^{ij}B_iB_j = 1, \;\; g^{ij}A_iB_j = 0.$$

So, the manifold under consideration is a $M(QE)_4$.

Example 2. We consider a Riemannian manifold $(\mathbb{R}^4, g)$ endowed with the metric g given by

$$ds^2 = g_{ij}dx^idx^j = (dx^1)^2+(x^1)^2(dx^2)^2+(x^1\sin x^2)^2(dx^3)^2+(dx^4)^2, \tag{8.3}$$

where $x^1 \neq 0$ and $0 < x^2 < \frac{\pi}{2}$. Then the non-vanishing components of the Christoffel symbols and the curvature tensor are

$$\Gamma^1_{22} = -x^1, \;\; \Gamma^1_{33} = -x^1(\sin x^2)^2, \;\; \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{x^1},$$

$$\Gamma^3_{23} = \cot x^2, \;\; \Gamma^2_{33} = -\sin x^2\cos x^2, \;\; R_{2332} = -(x^1\sin x^2)^2,$$

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

$$S_{22} = -1, \;\; S_{33} = -(\sin x^2)^2.$$

Also it can be easily found that the scalar curvature of the manifold is non-constant and is equal to $-\frac{2}{(x^1)^2} \neq 0$.

We take the associated scalars as follows:

$$a = -\frac{1}{(x^1)^2}, \;\; b = x^1x^2.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} x^1\sin x^2, & \text{for } i = 3, \\ 0, & \text{for } i = 1, 2, 4, \end{cases}$$

and

$$B_i(x) = \begin{cases} x^1, & \text{for } i = 2, \\ 0, & \text{for } i = 1, 3, 4, \end{cases}$$

at any point $x \in M$. In our (M^4, g), (1.6) reduces with these associated scalars and 1-forms to the following equations:

$$S_{22} = ag_{22} + b[A_2B_2 + A_2B_2], \tag{8.4}$$

$$S_{33} = ag_{33} + b[A_3B_3 + A_3B_3]. \tag{8.5}$$

It can be easily proved that the equations (8.4) and (8.5) are true.

We shall now show that the 1-forms are unit and orthogonal.

Here,

$$g^{ij}A_iA_j = 1, \ \ g^{ij}B_iB_j = 1, \ \ g^{ij}A_iB_j = 0.$$

So, the manifold under consideration is a $M(QE)_4$.

Acknowledgement. The authors wish to express their sincere thanks and gratitude to the referee for his valuable suggestions towards the improvement of the paper.

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