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ON ANTIMATROIDS OF INFINITE CHARACTER

Hua **Mao**

Department of Mathematics, Hebei University, Baoding 071002, China

Sanyang **Liu**

Department of Mathematics, Xidian University, Xi'an 710071, China

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Abstract: This article provides a new construction of infinite antimatroids–antimatroid of infinite character. Comparing the new construction with a famous construction of infinite antimatroids–antimatroids of finite character, it obtains that there is no one-to-one corresponding between antimatroids of infinite character and antimatroids of finite character. It also deals with some properties for antimatroids of infinite character. At the final part, with the assistance of poset theory, it introduces a one-element extension of antimatroids of infinite character and discover methods to build up antimatroids of infinite character.

1. Introduction and preliminaries

Many researchers are interested in infinite antimatroids though there is no unique construction that is called an infinite antimatroid. Generally, people refer to a finite antimatroid-like model defined on in-

E-mail address: yushengmao@263.net

finite set as an infinite antimatroid. Therefore, there are many infinite antimatroid structures which are provided by people from different motivations by generalizing the structure of finite antimatroids, and meanwhile, many results relative to infinite antimatroids are provided (cf. [1, 6, 8, 9]).

In this paper, we present a new construction of infinite antimatroid–antimatroid of infinite character and discuss some relations among finite antimatroids, antimatroids of finite character and antimatroids of infinite character. Afterwards, we research some properties on antimatroids of infinite character with the assistance of poset theory. At the final part, we generalize the definitions of lifting and reduction from finite antimatroids to antimatroids of infinite character. Using these generalizations, we provide some methods to construct antimatroids of infinite character.

We begin this article by giving some basic definitions. Let E denote a non-empty–perhaps infinite–set; 2^E be the powerset of E and $\mathcal{L} \subseteq 2^E$; $\mathcal{L} - p = \{Y \setminus p : Y \in \mathcal{L}\}$ if $p \in E$; $\mathcal{L} + q = \{Y \cup q : Y \in \mathcal{L}\}$ if $q \notin E$; $X \subset\subset Y$ if X is a finite subset of Y.

Definition 1. (1) [1 & 2, 3] For a poset $\mathbf{P} = (S, \leq)$, a *filter* of $\mathbf{P}$ is a subset K of S such that if $x \in K$ and $x \leq y$ for $y \in S$, then $y \in K$. For distinct elements $x, y \in S$ with $x \leq y$, if $x \leq z \leq y$ necessarily implies $x = z$ or $z = y$, then x is *covered* by y, in notation, $x \prec y$.

(2) [5, 8.2.6 & 4] Let E be a finite set and $\mathcal{F} \subseteq 2^E$ be a non-empty family. A pair $(E, \mathcal{F})$ is an *antimatroid* if it satisfies the following properties.

(G1) For every non-empty $X \in \mathcal{F}$, there is an $x \in X$ such that $X \setminus x \in \mathcal{F}$.

(G2) For $X, Y \in \mathcal{F}$ such that $|X| > |Y|$, there is an $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{F}$.

(G3) If $A \subseteq B, A, B \in \mathcal{F}, x \in E \setminus B$, and $A \cup x \in \mathcal{F}$, then $B \cup x \in \mathcal{F}$.

The sets in $\mathcal{F}$ are called *feasible*. A *basis* of a subset $A \subseteq E$ is a maximal feasible subset of A.

(3) [4, p. 20 & 6] A *closure system* on E is a pair $(E, \mathcal{N})$ where $\mathcal{N}$ is a collection of possibly infinite subsets of E such that (I.1) and (I.2) as follows:

(I.1) $\emptyset, E \in \mathcal{N}$;

(I.2) $X, Y \in \mathcal{N}$ implies $X \cap Y \in \mathcal{N}$.

The elements of $\mathcal{N}$ are called *closed sets*.

A *closure operator* τ on E is a map from E to E such that the following (I.3)–(I.6):

(I.3) $A \subseteq \tau(A)$;

(I.4) $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$;

(I.5) $\tau(\tau(A)) = \tau(A)$;

(I.6) $\tau(\emptyset) = \emptyset$.

A closure system $(E, \mathcal{N})$ gives rise to the following closure operator:

$$\tau(A) = \cap\{X : A \subseteq X \quad \text{and} \quad X \in \mathcal{N}\}.$$

A closure system is called an *antimatroid* if the following anti-exchange property is satisfied: for all $A \in \mathcal{N}$, for all distinct $x, y \in E \setminus A$,

$$y \in \tau(A \cup x) \Rightarrow x \notin \tau(A \cup y).$$

Antimatroids are also known as *convex geometries*.

A closure system is said to be of *finite character* if and only if for any $A \subseteq E$,

$$\tau(A) = \cup\{\tau(B) : B \quad \text{is a finite subset of} \quad A\}.$$

[4, 6] show that the notions of closure system and closure operator are in fact equivalent.

For clear, we call an antimatroid defined on a finite set E, i.e. the definition in Def. 1 (2), a *finite antimatroid*.

We refer the reader to [2, 3] for further details on poset theory and lattice theory; [4, 5] for finite antimatroid theory; [4–7] for convex geometry.

From some results in [4, 5] relative to finite antimatroids, there are the following views:

(1.1) $(E, \mathcal{F})$ is a finite antimatroid if and only if

$(E, \mathcal{N})$ is a convex geometry, where $\mathcal{N} = \{E \setminus X : X \in \mathcal{F}\}$.

(1.2) $(E, \mathcal{F})$ is a finite antimatroid if and only if $\mathcal{F}$ satisfies (G1) and is closed under union.

(1.3) $(E, \mathcal{F})$ is a finite antimatroid if and only if $\emptyset \in \mathcal{F}$ and $\mathcal{F}$ satisfies that for $X, Y \in \mathcal{F}$ such that $X \not\subseteq Y$, there is an $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{F}$.

(1.4) $(E, \mathcal{F})$ is a finite antimatroid if and only if it has a unique basis and satisfies (G1), (G2) and

(G4) if $A \subseteq B \subseteq C, A, B, C \in \mathcal{F}, x \in E \setminus C, A \cup x \in \mathcal{F}$, and $C \cup x \in \mathcal{F}$, then $B \cup x \in \mathcal{F}$.

(1.5) Because of (1.1), some researchers also call a convex geometry $(E, \mathcal{N})$ an antimatroid sometimes.

We generalize the definition of finite antimatroid in [4, 5] to infinite case as follows.

Definition 2. $\mathcal{L}$ is an *antimatroid of infinite character* on E (sometimes, it is denoted by $(E, \mathcal{L})$) if it satisfies the following:

(L1) $\emptyset \in \mathcal{L}$;

(L2) if $X \neq \emptyset$ and $X \in \mathcal{L}$, then $X \setminus x \in \mathcal{L}$ for some $x \in X$;

(L3) if $X, Y \in \mathcal{L}$ and $X \nsubseteq Y$, then $Y \cup x \in \mathcal{L}$ for some $x \in X \setminus Y$.

Any element of $\mathcal{L}$ is called a *feasible set* and a maximal feasible set is called *basis*.

For infinite antimatroids, we need to point out the following results.

(1.6) By (1.1)–(1.3), Def. 1 and Def. 2, we may state that any finite antimatroid is an antimatroid of finite character, and also an antimatroid of infinite character.

2. Correlation

From now on, we will deal with some relations among antimatroids of infinite character, antimatroids of finite character and finite antimatroids. The aim is to illustrate the internal properties of antimatroids of infinite character.

We give two examples of antimatroids of infinite character in order to show some properties of antimatroids of infinite character clearly.

Example 1. Let $E = \{1, 2, \ldots\}$ and $\mathcal{L} = \{X : X \subset\subset E\}$. Then we easily receive some properties of $(E, \mathcal{L})$ as follows.

(i) $(E, \mathcal{L})$ is an antimatroid of infinite character.

(ii) for any $X, Y \in \mathcal{L}$, it is obvious $X \cup Y \in \mathcal{L}$.

(iii) $(E, \mathcal{L})$ has no basis.

(iv) $(E, \mathcal{L})$ is not an antimatroid of finite character because if $X = E \setminus \{1\}$, then for any $Y \subset\subset X$, there is $Y \in \mathcal{L}$, but $X \notin \mathcal{L}$. However, $X = \cup_{Y \subset\subset X} Y$ is true.

(v) Let $\mathcal{N} = \{X : E \setminus X \in \mathcal{L}\}$. Then it satisfies $\emptyset \notin \mathcal{N}$ in virtue of $E \notin \mathcal{L}$. According to [6], $(E, \mathcal{N})$ will not be a convex geometry, and also not an antimatroid of finite character.

The (iv) and (v) in Ex. 1 illustrate that antimatroid of infinite character in infinite antimatroid constructions is new.

The (v) in Ex. 1 implies that for antimaroids of infinite character, (1.1) is not true.

On matroids, some authors (see e.g. [18] and [19]) talk about "merely finitary" matroids, by which they mean a closure system on an infinite set that is finitary: if p is in the closure of a set S, then p is in the closure of a finite subset of S. Combining with the results in [6] and Ex. 1, we arrive at the conclusion that $(E, \mathcal{L})$ in Ex. 1 owns a "matroid merely finitary character", but not an antimatroid of finite character. This analysis and comparison describe that some kind of infinite matroids are different from the new class of antimatroids–antimatroid of infinite character.

These descriptions state that it has importance and value to study on the new construction–antimatroid of infinite character.

Example 2. Let $E = \{1, 2, \ldots\}$ and $\mathcal{L} = 2^E$. Then it is easy to see that $(E, \mathcal{L})$ is an antimatorid of infinite character and E is the only basis.

From Ex. 1 and Ex. 2, we may explore the following property.

Property 1. *If an antimatroid of infinite character* $(E, \mathcal{L})$ *has bases, then the cardinality of the set of bases is* 1*.*

Proof. Otherwise, let B_1, B_2 be two bases of $(E, \mathcal{L})$. By the maximality of bases, it brings about $B_1 \nsubseteq B_2$ and $B_2 \nsubseteq B_1$. In light of (L3), it infers to $B_2 \cup b_1, B_1 \cup b_2 \in \mathcal{L}$ for some $b_1 \in B_1 \setminus B_2$ and $b_2 \in B_2 \setminus B_1$. These illustrate $B_1 \subset B_1 \cup b_2$ and $B_2 \subset B_2 \cup b_1$, a contradiction. $\Diamond$

There are many results relative to finite antimatroids (cf. [1, 4, 5, 10] and (1.1)–(1.5)). Considered them with antimatroids of finite character (see [6] for the details of antimatroids of finite character) and antimatroids of infinite character, we sum up the following results.

(2.1) [5, Prop. 8.2.8 & 4] points out that a finite antimatroid has a unique basis. Ex. 1 indicates that an antimatroid of infinite character perhaps has no base.

(2.2) The definition of a finite antimatroid and Def. 2 state that a finite antimatroid is an antimatroid of infinite character. (1.4) and (iii) in Ex. 1 show that the converse will not be true.

(2.3) (1.1) expresses that every finite antimatroid can be defined by a convex geometry. Any antimatroid of finite character is defined by convex geometry (cf. [6]). The (v) in Ex. 1 indicates that not every antimatroid of infinite character can be defined by convex geometry.

From the discussion above, we may infer to the following consequences.

(2.4) Def. 2 is an extension of a finite antimatroid.

(2.5) Antimatroids of infinite character and antimatroids of finite character are different generalizations of finite antimatroids under infinite cases. They are infinite models of finite antimatroids.

(2.6) An antimatroid of infinite character can not be defined by a convex geometry.

An antimatroid of infinite character have not some characters that antimatroids of finite character own. That is, antimatroids of infinite character possess itself special features.

(2.7) It is necessary to study on antimatroids of infinite character.

3. Properties with posets

In this section, we will find out some properties of antimatroids of infinite character. All of them are true for finite antimatroids (for finite cases, see [1, 4, 5, 10]). All the discussion are assisted with the method of poset theory.

Theorem 1. *Let $(E,\mathcal{L})$ be an antimatroid of infinite character and $X, Y \in \mathcal{L}$ and $X \subset Y$. $X \prec Y$ in $(\mathcal{L}, \subseteq)$ if and only if $X \cup p \in \mathcal{L}$ and $X \cup p = Y$ for some $p \in Y \setminus X$.*

Proof. Evidently, $(\mathcal{L}, \subseteq)$ is a poset.

($\Rightarrow$) If $X \cup p \in \mathcal{L}$ for some $p \in Y \setminus X$, then $X \cup p = Y$ because X is covered by Y in $(\mathcal{L}, \subseteq)$. Thus, the following is to prove $X \cup p \in \mathcal{L}$ for some $p \in Y \setminus X$.

Otherwise, for any $t \in Y \setminus X$, it has $X \cup t \notin \mathcal{L}$.

According to $Y \in \mathcal{L}$ and (L2), there exists $y \in Y$ satisfying $Y \setminus y \in \mathcal{L}$. Obviously, $Y \setminus y$ is covered by Y. If $y \notin X$, then $X \subseteq Y \setminus y \prec Y$, a contradiction to $X \prec Y$. That is, $y \in X$ is true. This infers to $Y \setminus y \nsubseteq X$. So, $X \cup q \in \mathcal{L}$ holds for some $q \in (Y \setminus y) \setminus X$ because of (L3), a contradiction to the supposition.

($\Leftarrow$) Exercise. $\Diamond$

Though Ex. 1 describes that an antimatroid of infinite character perhaps does not have a basis, we still have the following property.

Property 2. *Let $(E,\mathcal{L})$ be an antimatroid of infinite character.*

(1) *If $(E,\mathcal{L})$ has a basis B, then the length $h(\mathcal{L}, \subseteq)$ of $(\mathcal{L}, \subseteq)$, i.e. $\max\{|l| : l$ is a chain in $(\mathcal{L}, \subseteq)\}$, is the cardinality $|B|$ of B.*

(2) *Assume that $(E,\mathcal{L})$ satisfies that for any $x \in E$, there is $A \in \mathcal{L}$ satisfying $x \in A$. Then*

$$h(\mathcal{L}, \subseteq) < \infty \text{ if and only if } |E| < \infty.$$

Proof. (1) It is straightforward from the definition of base in Def. 2, Prop. 1, Th. 1 and (L1).

(2) ($\Leftarrow$) Obviously.

($\Rightarrow$) Taken (L2), (L3) and $h(\mathcal{L}, \subseteq) < \infty$ together, it infers to the existence of bases in $(\mathcal{L}, \subseteq)$. Using Prop. 1, it follows that $(E, \mathcal{L})$ has a unique base B. The above (1) and $h(\mathcal{L}, \subseteq) < \infty$ will compel $|B| < \infty$. Moreover, $|2^B| < \infty$ is correct. Thus, $|\mathcal{L}| \leq |2^B| < \infty$ is correct. Applying (L3) with the given condition, it yields out $|B| = |E|$. Furthermore, $(E, \mathcal{L})$ is a finite antimatroid, and hence, $|E| < \infty$, the required results follow. $\Diamond$

Using the above results, we may state the following.

(3.1) The results in [4, 5] prescribe that finite antimatroids are "antipodal" to and different from finite matroids in many parts, and those properties make finite antimatroids be valuable to be studied. Prop. 2 (2) indicates some difference between an antimatroid of infinite character and a matroid of arbitrary cardinality, because from the results in [11], we know that for a matroid $(E, \mathcal{F})$ of arbitrary cardinality, $h(\mathcal{F}, \subseteq) < \infty$ holds no matter $|E| < \infty$ or $|E| \not< \infty$.

(3.2) More results of matroid of arbitrary cardinality can be seen [11–17]. We may search out more "antipodal" results to our antimatroids of infinite character. This is left rooms for the future. This also states the valuable of presenting and studying on antimatroids of infinite character.

4. Lifting and reduction

Similar to [1], we will define a one-element extension of antimatroids of infinite character and find out two methods to build up antimatroids of infinite character.

Definition 3. Let $\mathcal{L}_1, \mathcal{L}_2$ be the subfamilies of an antimatroid of infinite character $\mathcal{L}$. Suppose that they satisfy the following:

(E0) $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}$;

(E1) $\mathcal{L}_1$ is an antimatroid of infinite character;

(E2) $\mathcal{L}_2$ is a filter in $(\mathcal{L}, \subseteq)$;

(E3) $\mathcal{L}_2 = \{Y \in \mathcal{L} : X \subseteq Y \text{ for some } X \in \mathcal{L}_1 \cap \mathcal{L}_2\}$.

Let p be a new element not in E. Then

$$(\mathcal{L} \uparrow p)_{(\mathcal{L}_1, \mathcal{L}_2)} = \mathcal{L}_1 \cup (\mathcal{L}_2 + p) = \mathcal{L}_1 \cup \{Y \cup p : Y \in \mathcal{L}_2\},$$

which we call a *lifting* of $\mathcal{L}$ at $(\mathcal{L}_1, \mathcal{L}_2)$ by p. We write $\mathcal{L} \uparrow p$ to denote $(\mathcal{L} \uparrow p)_{(\mathcal{L}_1, \mathcal{L}_2)}$ when no confusion may occur.

Let $\mathcal{L}$ be an antimatroid of infinite character on E. In the poset $(\mathcal{L}, \subseteq)$, we easily receive the following (L4). Further, we will achieve with Th. 2.

(L4) Let $X, Y \in \mathcal{L}$. If $X \subseteq Y$ and $X \neq Y$, then there exists $Y_1 \in \mathcal{L}$ satisfying $X \subseteq Y_1 \prec Y$.

Theorem 2. *A lifting $(\mathcal{L} \uparrow p)_{(\mathcal{L}_1, \mathcal{L}_2)}$ is an antimatroid of infinite character on $E \cup p$.*

Proof. We will finish the proof step by step using Def. 2.

Step 1. (L1) is obviously true.

Step 2. It proves (L2) as follows. Let $Y \neq \emptyset$ and $Y \in \mathcal{L} \uparrow p$.

If $Y \in \mathcal{L}_1$, then (L2) is clear.

If $Y \in \mathcal{L}_2 + p$, i.e. $Y = Y' \cup p$ for some $Y' \in \mathcal{L}_2$. Then, there are the following two statuses to be dealt with.

Status 1. Suppose Y' is a minimal in $(\mathcal{L}_2, \subseteq)$.

It follows $Y' \in \mathcal{L}_1$ from (E3). Hence $Y \setminus p = Y' \in \mathcal{L}_1 \subseteq \mathcal{L} \uparrow p$.

Status 2. Suppose Y' is not a minimal in $(\mathcal{L}_2, \subseteq)$.

Then there is a minimal A in $(\mathcal{L}_2, \subseteq)$ such that $A \subset Y'$. By (L4), there is a chain $A \subseteq A_1 \subseteq \ldots \subseteq Y_1 \prec Y'$ between A and Y' in $(\mathcal{L}, \subseteq)$. (E2) will ask this chain to be true in $(\mathcal{L}_2, \subseteq)$. Moreover, by Th. 1, it implies that there is $y \in Y' \subset Y$ such that $Y' \setminus y = Y_1$. This means $Y_1 \in \mathcal{L}_2$ from (E2). Hence, we have $Y \setminus y = (Y' \setminus y) \cup p \in \mathcal{L} \uparrow p$. That is to say, (L2) is true.

Step 3. We shall show (L3). Let $X, Y \in \mathcal{L} \uparrow p$ and $X \nsubseteq Y$.

If $X, Y \in \mathcal{L}_1$. Then (L3) is clear from (E1) and (L3).

If $X, Y \in \mathcal{L}_2 + p$, i.e. $X = X' \cup p, Y = Y' \cup p$, and $X', Y' \in \mathcal{L}_2$. Then $X' \nsubseteq Y'$ is true according to $X \nsubseteq Y$. Because of $\mathcal{L}_2 \subseteq \mathcal{L}$, we achieve $Y' \cup y \in \mathcal{L}$ for some $y \in X' \setminus Y'$ in view of (L3), and further, $Y' \cup y \in \mathcal{L}_2$ in view of (E2) and $Y' \subset Y' \cup y$. Thus $Y \cup y = (Y' \cup y) + p \in \mathcal{L} \uparrow p$ holds.

If $X \in \mathcal{L}_1$ and $Y = Y' + p \in \mathcal{L}_2 + p$. Then $X \nsubseteq Y$ brings about $X \nsubseteq Y'$. Considered with $X, Y' \in \mathcal{L}$ and (L3), it causes $Y' \cup x \in \mathcal{L}$ for some $x \in X \setminus Y$. Considering (E2) with $Y' \subseteq Y' \cup x \in \mathcal{L}$, we obtain $Y' \cup x \in \mathcal{L}_2$. Hence, $Y \cup x = (Y' \cup x) + p \in \mathcal{L} \uparrow p$ holds.

If $Y \in \mathcal{L}_1$ and $X = X' + p \in \mathcal{L}_2 + p$. Then, $X \nsubseteq Y$ and $p \notin E$ together follows $X' \nsubseteq Y$. Thus, $Y \cup x' \in \mathcal{L}$ holds for some $x' \in X' \setminus Y$ according to the correct of (L3) for $\mathcal{L}$. $p \notin E$ leads to $x' \in X \setminus Y$. Therefore, $Y \cup x' \in \mathcal{L} \subseteq \mathcal{L} \uparrow p$ is true for some $x' \in X \setminus Y$. $\Diamond$

Next we introduce the converse operation of lifting. Take an element $p \in E$. Then $\mathcal{L} \downarrow p = \mathcal{L} - p = \{X \setminus p : X \in \mathcal{L}\} = \{Y \subseteq E \setminus p :$ there exists $X \in \mathcal{L}$ satisfying $Y = X \setminus p\}$.

Lemma 1. *$\mathcal{L} \downarrow p$ is an antimatroid of infinite character on $E \setminus p$. We call it a reduction of $\mathcal{L}$ at p.*

Proof. Exercise. $\Diamond$

The following theorem expresses that the reduction and the lifting are the converse of each other. It is also a different method to build up an antimatroid of infinite character from that appeared in Th. 2.

Theorem 3. (1) *For any $p \in E$, we have $((\mathcal{L} \downarrow p) \uparrow p)_{(\mathcal{L}_1,\mathcal{L}_2)} = \mathcal{L}$, where $\mathcal{L}_1 = \{X : X \in \mathcal{L}, p \notin X\}$ and $\mathcal{L}_2 = \{X \setminus p : X \in \mathcal{L}, p \in X\}$.*

(2) *Let q be a new element not in E. Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy* (E0)–(E3). *Then $((\mathcal{L} \uparrow q)_{(\mathcal{L}_1,\mathcal{L}_2)}) \downarrow q = \mathcal{L}$.*

Proof. Using Lemma 1 and Def. 2, similarly to the proof in [1, Th. 2.2], the required results follow. $\Diamond$

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