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HERZ SPACES AND POINTWISE SUMMABILITY OF FOURIER SERIES

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Abstract: A general summability method, the so-called θ-summability is considered for multi-dimensional Fourier series. It is proved that if the kernel functions are uniformly bounded in a Herz space then the restricted maximal operator of the θ-means of a distribution is of weak type $(1,1)$, provided that the supremum in the maximal operator is taken over a cone-like set. From this it follows that $\sigma_n^\theta f \to f$ a.e. for all $f \in L_1(\mathbb{T}^d)$. Moreover, $\sigma_n^\theta f(x)$ converges to $f(x)$ over a cone-like set at each Lebesgue point of $f \in L_1(\mathbb{T}^d)$ if and only if the kernel functions are uniformly bounded in a suitable Herz space. The Cesàro, Riesz and Weierstrass summations are investigated as special cases of the θ-summation.

1. Introduction

The well-known Lebesgue [8] theorem says that for every integrable function f the Fejér means $\sigma_n f(x) = \frac{1}{n}\sum_{k=0}^{n-1} s_k f(x)$ converge to $f(x)$ as $n \to \infty$ at each Lebesgue point of f, where $s_k f$ denotes the kth partial sum of the Fourier series of f. Almost every point is a Lebesgue point

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of f. Later Alexits [1] generalized this result and gave a sufficient and necessary condition such that the singular integrals converge at every Lebesgue point.

For multi-dimensional trigonometric-Fourier series Marcinkievicz and Zygmund [9, 17] proved that the Fejér means $\sigma_n f$ of a function $f \in L_1(\mathbb{T}^d)$ converge a.e. to f as $n \to \infty$ provided that n is in a cone, i.e., $\tau^{-1} \leq n_k/n_j \leq \tau$ for every $k, j = 1, \ldots, d$ and for some $\tau \geq 1$ ($n = (n_1, \ldots, n_d) \in \mathbb{N}^d$). We have extended this result to the θ-summation in [14]. The so called θ-summation is a general method of summation and it is intensively studied in the literature (see e.g. Butzer and Nessel [3], Trigub and Belinsky [13] and Weisz [14, 4, 5] and the references therein). Similar results for so-called cone-like sets can be found in Gát [6] and Weisz [15, 16].

In this paper we extend the results concerning the Lebesgue points to cone-like sets defined by a function γ. We introduce a new version of the Hardy–Littlewood maximal function depending on γ and show that if the kernel functions of the θ-summation are uniformly bounded in a modified Herz space, then the maximal function $\sigma_\gamma^\theta f$ can be estimated by the Hardy–Littlewood maximal function $M_p^\gamma f$, provided that the supremum in the maximal operator is taken over a cone-like set. Since M_p^γ is of weak type (p, p) we obtain $\sigma_n^\theta f \to f$ a.e. over a cone-like set for all $f \in L_p(\mathbb{T}^d)$. The set of convergence is also characterized, the convergence holds at every p-Lebesgue point of f. The converse holds also, more exactly, $\sigma_n^\theta f(x) \to f(x)$ over a cone-like set at each p-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if and only if the kernel functions are uniformly bounded in the Herz space. As special cases five examples of the θ-summation are considered, amongst others the Cesàro, Riesz and Weierstrass summations. Similar results for Fourier transforms can be found in Feichtinger and Weisz [5, 15].

2. Wiener algebra

Let us fix $d \geq 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set $u \cdot x := \sum_{k=1}^d u_k x_k$.

We briefly write L_p or $L_p(\mathbb{T}^d)$ instead of $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p := (\int_{\mathbb{T}^d} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leq \infty)$, where $\mathbb{T} = [-\pi, \pi]$ is the torus and λ is the Lebesgue measure.

The *weak* L_p space, $L_{p,\infty}(\mathbb{T}^d)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\rho>0} \rho\lambda(|f| > \rho)^{1/p} < \infty,$$

while we set $L_{\infty,\infty}(\mathbb{T}^d) = L_\infty(\mathbb{T}^d)$. Note that $L_{p,\infty}(\mathbb{T}^d)$ is a quasi-normed space (see Bergh and Löfström [2]). It is easy to see that for each $0 < < p \leq \infty$,

$$L_p(\mathbb{T}^d) \subset L_{p,\infty}(\mathbb{T}^d) \quad \text{and} \quad \|\cdot\|_{p,\infty} \leq \|\cdot\|_p.$$

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{T}^d)$.

A measurable function f belongs to the *Wiener amalgam space* $W(L_\infty, \ell_1)(\mathbb{R}^d)$ if

$$\|f\|_{W(L_\infty,\ell_1)} := \sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,1)^d} |f(x+k)| < \infty.$$

It is easy to see that $W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$. The closed subspace of $W(L_\infty, \ell_1)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, \ell_1)(\mathbb{R}^d)$ and is called *Wiener algebra*. It is used quite often in Gabor analysis, because it provides a convenient and general class of windows (see e.g. Gröchenig [7]). It turned out in Feichtinger and Weisz [4, 5] that it can be well applied in summability theory, too.

3. θ-summability of Fourier series

We will consider the *θ-summation* defined by a multi-parameter sequence. Let

$$\theta = \big(\theta(k,n), k \in \mathbb{Z}^d, n \in \mathbb{N}^d\big) \tag{1}$$

be a $2d$-parameter sequence of real numbers satisfying

$$\theta(0,\dots 0, n) = 1, \quad \lim_{n\to\infty} \theta(k,n) = 1 \quad (\theta(k,n))_{k\in\mathbb{Z}^d} \in \ell_1 \tag{2}$$

for each $n \in \mathbb{N}^d$. Recall that for a distribution $f \in \mathcal{S}'(\mathbb{T}^d)$ the nth *Fourier coefficient* is defined by $\widehat{f}(n) := f(e^{-\imath n\cdot x})$ $(n \in \mathbb{Z}^d, \imath = \sqrt{-1})$. In special case, if $f \in L_1(\mathbb{T}^d)$ then

$$\widehat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(t) e^{-\imath n\cdot t}\, dt \quad (n \in \mathbb{Z}^d).$$

The θ-*means* of a distribution $f \in \mathcal{S}'(\mathbb{T}^d)$ are defined by

(3)
$$\sigma_n^\theta f(x) := \sum_{k_1=-\infty}^{\infty} \dots \sum_{k_d=-\infty}^{\infty} \theta(-k,n)\widehat{f}(k)e^{\imath k\cdot x} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}} f(x-t)K_n^\theta(t)\,dt$$

($x \in \mathbb{T}^d, n \in \mathbb{N}^d$), where K_n^θ denotes the θ-*kernel*

$$K_n^\theta(t) := \sum_{k_1=-\infty}^{\infty} \dots \sum_{k_d=-\infty}^{\infty} \theta(-k,n)e^{\imath k\cdot t} \qquad (t \in \mathbb{T}^d).$$

Observe that (2) ensures that $K_n^\theta \in L_1(\mathbb{T})$.

We can also define a θ-summation by one single function θ defined on $\mathbb{R}^d$. In this case we define the sequence in (1) by

$$\theta(k,n) := \theta\Big(\frac{k_1}{n_1}, \dots, \frac{k_d}{n_d}\Big) \qquad (k \in \mathbb{Z}^d, n \in \mathbb{N}^d).$$

If $\theta(0) = 1$ and $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ then (2) is satisfied, because

$$\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_d=-\infty}^{\infty} \Big|\theta\Big(\frac{k_1}{n_1}, \dots, \frac{k_d}{n_d}\Big)\Big| \le \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_d=-\infty}^{\infty} \Big(\prod_{j=1}^d n_j\Big) \sup_{x\in[0,1)} |\theta(x+l)| =$$
$$= \Big(\prod_{j=1}^d n_j\Big)\|\theta\|_{W(C,\ell_1)} < \infty.$$

The *Fourier transform* of $f \in L_1(\mathbb{R}^d)$ is given by

$$\widehat{f}(x) := \int_{\mathbb{R}^d} f(t)e^{-2\pi\imath x\cdot t}\,dt \qquad (x \in \mathbb{R}^d).$$

If θ is a function and $\widehat{\theta} \in L_1(\mathbb{R}^d)$ then

(4)
$$\sigma_n^\theta f(x) = \Big(\prod_{j=1}^d n_j\Big)\int_{\mathbb{R}^d} f(x-t)\widehat{\theta}(n_1t_1, \dots, n_dt_d)\,dt$$

for all $x \in \mathbb{T}^d$, $n \in \mathbb{N}^d$ and $f \in L_1(\mathbb{T}^d)$, where f is extended periodically to $\mathbb{R}^d$ (see Feichtinger and Weisz [4]).

4. Hardy–Littlewood inequality and cone-like sets

Suppose that for all $j = 2, \dots, d$, $\gamma_j : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing and continuous functions such that $\gamma_j(1) = 1$, $\lim_\infty \gamma_j = \infty$ and $\lim_{+0} \gamma_j = 0$. Moreover, suppose that there exist $c_{j,1}, c_{j,2}, \xi > 1$ such that

(5)
$$c_{j,1}\gamma_j(x) \le \gamma_j(\xi x) \le c_{j,2}\gamma_j(x) \qquad (x > 0).$$

Note that this is satisfied if γ_j is a power function. For convenience we extend the notations for $j = 1$ by $\gamma_1 := \mathcal{I}$ and $c_{1,1} = c_{1,2} = \xi$. Here $\mathcal{I}$ denotes the identity function $\mathcal{I}(x) = x$. Let $\gamma = (\gamma_1, \dots, \gamma_d)$ and $\tau = (\tau_1, \dots, \tau_d)$ with $\tau_1 = 1$ and fixed $\tau_j \geq 1$ $(j = 2, \dots, d)$. We will investigate the Hardy–Littlewood maximal operator and later the maximal operator of the θ-summation over a *cone-like set* (with respect to the first dimension)

$$(6) \qquad \mathbb{R}^d_{\tau,\gamma} := \left\{ x \in \mathbb{R}^d_+ : \tau_j^{-1}\gamma_j(n_1) \leq n_j \leq \tau_j \gamma_j(n_1), j = 2, \dots, d \right\}.$$

If each γ_j is the identity, $j = 2, \dots, d$, then we get the cone defined by τ. The condition on γ_j seems to be natural, because Gát [6] proved in the two-dimensional case that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if (5) holds.

$L_p^{loc}(\mathbb{T}^d)$ $(1 \leq p \leq \infty)$ denotes the space of measurable functions f for which $|f|^p$ is locally integrable, resp. f is locally bounded if $p = \infty$. In [15] we have introduced the *Hardy–Littlewood maximal function* on a cone-like set by

$$M_p^{\tau,\gamma} f(x) := \sup_{x \in I, (|I_1|, \dots, |I_d|) \in \mathbb{R}^d_{\tau,\gamma}} \left(\frac{1}{|I|} \int_I |f|^p \, d\lambda \right)^{1/p} \qquad (x \in \mathbb{T}^d)$$

with the usual modification for $p = \infty$, where $f \in L_1^{loc}(\mathbb{T}^d)$ and the supremum is taken over all rectangles $I := I_1 \times \cdots \times I_d \subset \mathbb{T}^d$ with sides parallel to the axes. Taking the supremum over rectangles with $|I_j| = \gamma_j(|I_1|), j = 2, \dots, d$, (i.e. $\tau_j = 1$, $j = 1, \dots, d$), we obtain the maximal operator M_p^γ. The inequality

$$M_p^\gamma f \leq M_p^{\tau,\gamma} f \leq C M_p^\gamma f$$

was shown in Weisz [15]. In case $p = 1$ we write simply $M^{\tau,\gamma}$ and M^γ. If each γ_j is the identity function then we get back the classical Hardy–Littlewood maximal function defined on a cone. The following theorem was proved in [15].

Theorem 1. *The maximal operator $M_p^{\tau,\gamma}$ $(1 \leq p \leq \infty)$ is of weak type (p, p), i.e.*

$$\|M_p^{\tau,\gamma} f\|_{p,\infty} = \sup_{\rho > 0} \rho \lambda (M_p^{\tau,\gamma} f > \rho)^{1/p} \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, if $1 \leq p < r \leq \infty$ then

$$\|M_p^{\tau,\gamma} f\|_r \leq C_r \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Since the set of continuous functions are dense in $L_1(\mathbb{T}^d)$, the usual density argument due to Marcinkiewicz and Zygmund [9] implies

Corollary 1. *If $f \in L_1(\mathbb{T}^d)$ then*

$$\lim_{\substack{x\in I, (|I_1|,\ldots,|I_d|)\in \mathbb{R}^d_{\tau,\gamma} \\ |I_j|\to 0, j=1,\ldots,d}} \frac{1}{|I|}\int_I f\,d\lambda = f(x) \qquad \textit{for a.e. } x\in\mathbb{T}^d.$$

5. Herz spaces

The $E_q(\mathbb{R}^d)$ $(1 \le q \le \infty)$ spaces were used recently by Feichtinger and Weisz [5] in the summability theory of Fourier transforms. A function belongs to the *(homogeneous) Herz space* $E_q(\mathbb{R}^d)$ $(1 \le q \le \infty)$ if

$$\|f\|_{E_q} := \sum_{k=-\infty}^{\infty} 2^{kd(1-1/q)} \|f \mathbf{1}_{\{x\in\mathbb{R}^d : 2^{k-1}\pi \le \|x\|_\infty < 2^k\pi\}}\|_q < \infty.$$

Here we introduce a generalization of the $E_q(\mathbb{R}^d)$ spaces depending on the function γ (see [15]). A function $f \in L_q^{loc}(\mathbb{R}^d)$ is in the space $E_q^\gamma(\mathbb{R}^d)$ $(1 \le q \le \infty)$ if

$$\|f\|_{E_q^\gamma} := \sum_{k=-\infty}^{\infty} \Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)^{1-1/q} \|f \mathbf{1}_{P_k}\|_q < \infty, \tag{7}$$

where ξ and γ_j are defined in (5) and

$$P_k := \prod_{j=1}^d \Big(-\gamma_j(\xi^k)\pi, \gamma_j(\xi^k)\pi\Big) \setminus \prod_{j=1}^d \Big(-\gamma_j(\xi^{k-1})\pi, \gamma_j(\xi^{k-1})\pi\Big) \quad (k\in\mathbb{Z}).$$

If $\gamma_j = \mathcal{I}$ for all $j = 1,\ldots,d$ and $\xi = 2$ then we get back the original spaces $E_q(\mathbb{R}^d)$. However, it is easy to see that the spaces are equivalent for all $\xi > 1$, whenever each γ_j is the identity function. If we modify the definition of P_k,

$$P_k' = \prod_{j=1}^d \Big(-\gamma_j(\xi^k)\pi, \gamma_j(\xi^k)\pi\Big) \setminus \prod_{j=1}^d \Big(-\gamma_j(\xi^{k-1})\pi, \gamma_j(\xi^{k-1})\pi\Big) \bigcap \mathbb{T}^d \quad (k\in\mathbb{Z}),$$

then we get the definition of the space $E_q^\gamma(\mathbb{T}^d)$. This means that we have to take the sum in (7) for $k \le 0$, only, because $\gamma_j(1) = 1$ for all $j = 1,\ldots,d$. Observe that

$$|P_k| \sim \prod_{j=1}^d \gamma_j(\xi^k) \qquad (k\in\mathbb{Z}).$$

Indeed,

$$|P_k| = (2\pi)^d \Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)\Big(1 - \prod_{j=1}^d \frac{\gamma_j(\xi^{k-1})}{\gamma_j(\xi^k)}\Big)$$

and

$$\frac{1}{c_{j,2}}\gamma_j(\xi^k) \le \gamma_j(\xi^{k-1}) \le \frac{1}{c_{j,1}}\gamma_j(\xi^k)$$

because of (5). Thus

$$(2\pi)^d\Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)\Big(1-\prod_{j=1}^d \frac{1}{c_{j,1}}\Big) \le |P_k| \le (2\pi)^d\Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)\Big(1-\prod_{j=1}^d \frac{1}{c_{j,2}}\Big).$$

This implies easily that

$$L_1(\mathbb{X}^d) = E_1^\gamma(\mathbb{X}^d) \hookleftarrow E_q^\gamma(\mathbb{X}^d) \hookleftarrow E_{q'}^\gamma(\mathbb{X}^d) \hookleftarrow E_\infty^\gamma(\mathbb{X}^d) \quad (1<q<q'<\infty),$$

where $\mathbb{X}$ denotes either $\mathbb{R}$ or $\mathbb{T}$. Moreover,

$$E_q^\gamma(\mathbb{T}^d) \hookleftarrow L_q(\mathbb{T}^d) \qquad (1 \le q \le \infty). \tag{8}$$

Indeed, we have

$$\gamma_j(\xi^k) \le \frac{1}{c_{j,1}}\gamma_j(\xi^{k+1}) \le \dots \le \frac{1}{c_{j,1}^{|k|}}$$

and

$$\begin{aligned}\|f\|_{E_q^\gamma(\mathbb{T}^d)} &\le \sum_{k=-\infty}^{0} \Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)^{1-1/q} \|f\mathbf{1}_{P_k}\|_q \le \\ &\le \sum_{k=-\infty}^{0} \Big(\prod_{j=1}^d \frac{1}{c_{j,1}}\Big)^{|k|(1-1/q)} \|f\mathbf{1}_{P_k}\|_q \le C_q\|f\|_q.\end{aligned}$$

6. Convergence of the θ-means of Fourier transforms

For a given τ, γ satisfying the above conditions the *restricted maximal θ-operator* are defined by

$$\sigma_\gamma^\theta f := \sup_{n\in\mathbb{R}^d_{\tau,\gamma}} |\sigma_n^\theta f|.$$

If $\gamma_j = \mathcal{I}$ for all $j = 2, \dots, d$ then we get a cone. This case was considered in Marcinkiewicz and Zygmund [9, 17] and more recently by the author [14]. Obviously, $\sigma_n^\theta f \to f$ in L_1- or C-norm if and only if the numbers

$\|K_n^\theta\|_1$ are uniformly bounded ($n \in \mathbb{R}^d_{\tau,\gamma}$). In [4, 15] we have proved if θ is a function then this condition is equivalent to $\widehat{\theta} \in L_1(\mathbb{R}^d)$.

Here we consider the pointwise convergence of the θ-means. In the one-dimensional case Alexits [1] and Torchinsky [12] proved that if there exists an even function η such that η is non-increasing on $\mathbb{R}_+$, $|\widehat{\theta}| \le \eta$, $\eta \in L_1(\mathbb{R})$ then the maximal operator of the θ-means is of weak type $(1,1)$. This condition is equivalent to $\widehat{\theta} \in E_\infty(\mathbb{R})$ (see [5]). Now we generalize this theorem as follows.

Theorem 2. *Let θ satisfy (2), $1 \le p \le \infty$ and $1/p + 1/q = 1$. If*

$$\sup_{n \in \mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C, \tag{9}$$

then

$$\sigma_\gamma^\theta f \le C\Big(\sup_{n \in \mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)}\Big) M_p^{\tau,\gamma} f \qquad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$.

Proof. By (3),

$$|\sigma_n^\theta f(x)| = \frac{1}{(2\pi)^d}\left|\int_{\mathbb{T}^d} f(x-t) K_n^\theta(t)\,dt\right| \le \frac{1}{(2\pi)^d} \sum_{k=-\infty}^{0} \int_{P_k} |f(x-t)||K_n^\theta(t)|\,dt.$$

Then

$$|\sigma_n^\theta f(x)| \le \frac{1}{(2\pi)^d} \sum_{k=-\infty}^{0} \left(\int_{P_k} |K_n^\theta(t)|^q\,dt\right)^{1/q} \left(\int_{P_k} |f(x-t)|^p\,dt\right)^{1/p}.$$

It is easy to see that if

$$G(u) := \left(\int_{|t_j|<u_j, j=1,\ldots,d} |f(x-t)|^p\,dt\right)^{1/p} \qquad (u \in \mathbb{R}^d_+)$$

then

$$\frac{G^p(u)}{\prod_{j=1}^d u_j} \le C (M_p^{\tau,\gamma} f)^p(x) \qquad (u \in \mathbb{R}^d_{\tau,\gamma}).$$

Therefore

$$\begin{aligned}|\sigma_n^\theta f(x)| &\le C \sum_{k=-\infty}^{0} \Big(\int_{P_k} |K_n^\theta(t)|^q\,dt\Big)^{1/q} G(\gamma_1(\xi^k)\pi, \ldots, \gamma_d(\xi^k)\pi) \le \\ &\le C \sum_{k=-\infty}^{0} \Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)^{1/p} \Big(\int_{P_k} |K_n^\theta(t)|^q\,dt\Big)^{1/q} M_p^{\tau,\gamma} f(x) = \\ &= C \|K_n^\theta\|_{E_q(\mathbb{T}^d)} M_p^{\tau,\gamma} f(x),\end{aligned}$$

which shows the theorem. ◊

Note that (2) implies $K_n^\theta \in L_\infty(\mathbb{T}^d) \subset L_q(\mathbb{T}^d) \subset E_q^\gamma(\mathbb{T}^d)$ for all $n \in \mathbb{N}^d$. Th. 1 implies immediately

Theorem 3. *Let θ satisfy* (2), $1 \le p \le \infty$ *and* $1/p + 1/q = 1$. *If*

$$\sup_{n \in \mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C,$$

then

$$\|\sigma_\gamma^\theta f\|_{p,\infty} \le C_p \Big(\sup_{n \in \mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \Big) \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, for every $p < r \le \infty$

$$\|\sigma_\gamma^\theta f\|_r \le C \Big(\sup_{n \in \mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \Big) \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

These inequalities and the usual density theorem due to Marcinkiewicz–Zygmund [9] imply

Corollary 2. *Let θ satisfy (2),* $1 \le p \le \infty$ *and* $1/p + 1/q = 1$. *If*

$$\sup_{n \in \mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C,$$

then

$$\lim_{n \to \infty, n \in \mathbb{R}^d_{\tau,\gamma}} \sigma_n^\theta f = f \qquad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$ whenever $1 \le p < \infty$ and for all $f \in C(\mathbb{T}^d)$ whenever $p = \infty$.

In case the summability method is defined by a function θ and $\widehat{\theta} \in E_q^\gamma(\mathbb{R}^d)$ then the preceding theorems hold.

Theorem 4. *Suppose that $c_j = c_{j,1} = c_{j,2}$ for all $j = 1, \ldots, d$. Let $\theta \in W(C, \ell_1)(\mathbb{R}^d)$, $1 \le p \le \infty$ and $1/p + 1/q = 1$. If $\widehat{\theta} \in E_q^\gamma(\mathbb{R}^d)$ then*

$$\sigma_\gamma^\theta f \le C \|\widehat{\theta}\|_{E_q^\gamma(\mathbb{R}^d)} M_p^{\tau,\gamma} f \qquad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$.

Proof. Since by (4)

$$\sigma_n^\theta f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^\theta(t)\, dt =$$

$$= \Big(\prod_{j=1}^d n_j \Big) \int_{\mathbb{R}^d} f(x-t) \widehat{\theta}(n_1 t_1, \ldots, n_d t_d)\, dt,$$

we can see that

$$K_n^\theta(t) = (2\pi)^d \Big(\prod_{j=1}^d n_j\Big) \sum_{j\in\mathbb{Z}^d} \widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi)).$$

We will prove that $\widehat{\theta} \in E_q^\gamma(\mathbb{R}^d)$ implies

$$\|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C_q \|\widehat{\theta}\|_{E_q^\gamma(\mathbb{R}^d)} \qquad \text{for all} \qquad n \in \mathbb{R}^d_{\tau,\gamma}. \tag{10}$$

Since $n \in \mathbb{R}^d_{\tau,\gamma}$ we have $\tau_j^{-1}\gamma_j(n_1) \le n_j \le \tau_j\gamma_j(n_1)$ for all $j = 1,\ldots,d$. For the term $j = 0$ of the norm we observe by (6) that

$$\Big\|\Big(\prod_{j=1}^d n_j\Big)\widehat{\theta}(n_1t_1,\ldots,n_dt_d)\Big\|_{E_q^\gamma(\mathbb{T}^d)} =$$

$$= \sum_{k=-\infty}^0 \Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)^{1-1/q}\Big(\prod_{j=1}^d n_j\Big)\Big(\int_{P_k} |\widehat{\theta}(n_1t_1,\ldots,n_dt_d)|^q\,dt\Big)^{1/q} \le$$

$$\le C_q \sum_{k=-\infty}^0 \Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)^{1-1/q}\Big(\prod_{j=1}^d \gamma_j(n_1)\Big)^{1-1/q}\Big(\int_{Q_k} |\widehat{\theta}(t_1,\ldots,t_d)|^q\,dt\Big)^{1/q},$$

where

$$Q_k := \prod_{j=1}^d \Big(-\tau_j\gamma_j(n_1)\gamma_j(\xi^k)\pi, \tau_j\gamma_j(n_1)\gamma_j(\xi^k)\pi\Big)\setminus$$
$$\setminus \prod_{j=1}^d \Big(-\tau_j^{-1}\gamma_j(n_1)\gamma_j(\xi^{k-1})\pi, \tau_j^{-1}\gamma_j(n_1)\gamma_j(\xi^{k-1})\pi\Big).$$

Suppose that $\xi^{l-1} \le n_1 < \xi^l$ for some $l \in \mathbb{N}$. Then by (5),

$$c_j^{l-1} = \gamma_j(\xi^{l-1}) \le \gamma_j(n_1) \le \gamma_j(\xi^l) = c_j^l.$$

We can choose $r, s \in \mathbb{N}$ such that $\tau_j/c_j^r \le 1$ and $c_j^s/\tau_j \ge 1$ for all $j = 1,\ldots,d$. This and (5) imply that

$$\tau_j\gamma_j(n_1)\gamma_j(\xi^k) \le \tau_j\gamma_j(\xi^l)\gamma_j(\xi^k) = \tau_j c_j^l\gamma_j(\xi^k) = \frac{\tau_j}{c_j^r}\gamma_j(\xi^{k+l+r}) \le \gamma_j(\xi^{k+l+r})$$

and

$$\frac{1}{\tau_j}\gamma_j(n_1)\gamma_j(\xi^{k-1}) \ge \frac{1}{\tau_j}\gamma_j(\xi^{l-1})\gamma_j(\xi^{k-1}) =$$
$$= \frac{1}{\tau_j}c_j^{l-1}\gamma_j(\xi^{k-1}) = \frac{c_j^s}{\tau_j}\gamma_j(\xi^{k+l-s-2}) \ge \gamma_j(\xi^{k+l-s-2}).$$

If

$$Q_{k,l} := \prod_{j=1}^{d}\big(-\gamma_j(\xi^{k+l+r})\pi, \gamma_j(\xi^{k+l+r})\pi\big) \setminus \prod_{j=1}^{d}\big(-\gamma_j(\xi^{k+l-s-2})\pi, \gamma_j(\xi^{k+l-s-2})\pi\big),$$

then

(11)

$$\begin{aligned}
&\Big\|\Big(\prod_{j=1}^{d} n_j\Big)\widehat{\theta}(n_1t_1,\ldots,n_dt_d)\Big\|_{E_q^\gamma(\mathbb{T}^d)} \leq \\
&\leq C_q \sum_{k=-\infty}^{0}\Big(\prod_{j=1}^{d}\gamma_j(\xi^k)\Big)^{1-1/q}\Big(\prod_{j=1}^{d}\gamma_j(\xi^l)\Big)^{1-1/q}\Big(\int_{Q_{k,l}}|\widehat{\theta}(t_1,\ldots,t_d)|^q\,dt\Big)^{1/q} \leq \\
&\leq C_q \sum_{k=-\infty}^{0}\Big(\prod_{j=1}^{d} c_j^{s+1}\Big)^{1-1/q}\Big(\prod_{j=1}^{d}\gamma_j(\xi^{k+l-s-1})\Big)^{1-1/q}\times \\
&\qquad \times\Big(\sum_{i=k+l-s-1}^{k+l+r}\int_{P_i}|\widehat{\theta}(t_1,\ldots,t_d)|^q\,dt\Big)^{1/q} \leq \\
&\leq C_q \sum_{k=-\infty}^{0}\sum_{i=k+l-s-1}^{k+l+r}\Big(\prod_{j=1}^{d}\gamma_j(\xi^i)\Big)^{1-1/q}\Big(\int_{P_i}|\widehat{\theta}(t_1,\ldots,t_d)|^q\,dt\Big)^{1/q} \leq \\
&\leq C_q \sum_{i=-\infty}^{l+r}\Big(\prod_{j=1}^{d}\gamma_j(\xi^i)\Big)^{1-1/q}\Big(\int_{P_i}|\widehat{\theta}(t_1,\ldots,t_d)|^q\,dt\Big)^{1/q} \leq \\
&\leq C_q\|\widehat{\theta}\|_{E_q^\gamma(\mathbb{R}^d)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\Big\|\Big(\prod_{j=1}^{d} n_j\Big)\sum_{j\in\mathbb{Z}^d, j\neq 0}\widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi))\Big\|_{E_q^\gamma(\mathbb{T}^d)} = \\
&= \sum_{k=-\infty}^{0}\Big(\prod_{j=1}^{d}\gamma_j(\xi^k)\Big)^{1-1/q}\Big(\prod_{j=1}^{d} n_j\Big)\times \\
&\qquad \times\Big(\int_{P_k}\Big|\sum_{j\in\mathbb{Z}^d, j\neq 0}\widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi))\Big|^q\,dt\Big)^{1/q} \leq \\
&\leq \sum_{k=-\infty}^{0}\Big(\prod_{j=1}^{d} c_j\Big)^{k(1-1/q)}\Big(\prod_{j=1}^{d} n_j\Big)\times
\end{aligned}$$

$$\times \Big(\int_{\mathbb{T}^d} \Big| \sum_{j\in\mathbb{Z}^d, j\neq 0} \widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi)) \Big|^q \, dt \Big)^{1/q} \leq$$

$$\leq C_q \Big(\prod_{j=1}^d n_j \Big) \Big(\int_{\mathbb{T}^d} \Big| \sum_{j\in\mathbb{Z}^d, j\neq 0} \widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi)) \Big|^q dt \Big)^{1/q}.$$

Let

$$R_i := \{ j \in \mathbb{Z}^d : j \neq 0, n_1(\mathbb{T}+2j_1\pi) \times \ldots \times n_d(\mathbb{T}+2j_d\pi) \cap P_i \neq 0 \}.$$

Since

$$|n_j(t_j+2j_j\pi)| \geq \frac{1}{\tau_j}\gamma_j(n_1)\pi \geq \frac{1}{\tau_j}\gamma_j(\xi^{l-1})\pi = $$
$$= \frac{1}{\tau_j} c_j^{l-1}\pi = \frac{c_j^s}{\tau_j}\gamma_j(\xi^{l-s-1}) \geq \gamma_j(\xi^{l-s-1}),$$

we conclude

$$\Big\| \Big(\prod_{j=1}^d n_j \Big) \sum_{j\in\mathbb{Z}^d, j\neq 0} \widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi)) \Big\|_{E_q^\gamma(\mathbb{T}^d)} \leq$$

$$\leq C_q \Big(\prod_{j=1}^d n_j \Big) \Big(\int_{\mathbb{T}^d} \Big| \sum_{i=(l-s)\vee 0}^{\infty} \sum_{j\in R_i} \widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi)) \Big|^q dt \Big)^{1/q} \leq$$

$$\leq C_q \sum_{i=(l-s)\vee 0}^{\infty} \Big(\prod_{j=1}^d n_j \Big) \Big(\int_{\mathbb{T}^d} \Big| \sum_{j\in R_i} \widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi)) \Big|^q dt \Big)^{1/q}.$$

Since R_i has at most $C\prod_{j=1}^d \frac{\gamma_j(\xi^i)}{n_j}$ members, we get that

$$\Big\| \Big(\prod_{j=1}^d n_j \Big) \sum_{j\in\mathbb{Z}^d, j\neq 0} \widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi)) \Big\|_{E_q^\gamma(\mathbb{T}^d)} \leq \tag{12}$$

$$\leq C_q \sum_{i=(l-s)\vee 0}^{\infty} \Big(\prod_{j=1}^d n_j \Big) \Big(\sum_{j\in R_i} \Big(\prod_{m=1}^d \frac{\gamma_m(\xi^i)}{n_m} \Big)^{q-1} \times$$

$$\times \int_{\mathbb{T}^d} |\widehat{\theta}(n_1(t_1+2j_1\pi),\ldots,n_d(t_d+2j_d\pi))|^q \, dt \Big)^{1/q} \leq$$

$$\leq C_q \sum_{i=(l-s)\vee 0}^{\infty} \Big(\prod_{j=1}^d \gamma_j(\xi^i) \Big)^{1-1/q} \Big(\prod_{j=1}^d n_j \Big) \Big(\sum_{j\in R_i} \Big(\prod_{m=1}^d n_m \Big)^{-q} \times$$

$$\times \int_{n_1(\mathbb{T}+2j_1\pi)\times\ldots\times n_d(\mathbb{T}+2j_d\pi)} |\widehat{\theta}(t_1,\ldots,t_d)|^q \, dt\Big)^{1/q} \le$$

$$\le C_q \sum_{i=(l-s)\vee 0}^{\infty} \Big(\prod_{j=1}^{d} \gamma_j(\xi^i)\Big)^{1-1/q} \Big(\int_{P_i} |\widehat{\theta}(t_1,\ldots,t_d)|^q \, dt\Big)^{1/q} \le$$

$$\le C_q \|\widehat{\theta}\|_{E_q^\gamma(\mathbb{R}^d)},$$

which proves (10). The theorem follows from Th. 2. $\Diamond$

Theorem 5. *Let $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $1 \le p \le \infty$ and $1/p + 1/q = 1$. If $\widehat{\theta} \in E_q^\gamma(\mathbb{R}^d)$, then*

$$\|\sigma_\gamma^\theta f\|_{p,\infty} \le C_p \|\widehat{\theta}\|_{E_q^\gamma(\mathbb{R}^d)} \|f\|_p \qquad (f \in L_p(\mathbb{T}^d)).$$

Moreover, for every $p < r \le \infty$

$$\|\sigma_\gamma^\theta f\|_r \le C \|\widehat{\theta}\|_{E_q^\gamma(\mathbb{R}^d)} \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

Corollary 3. *Let $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$, $1 \le p \le \infty$ and $1/p + 1/q = 1$. If $\widehat{\theta} \in E_q^\gamma(\mathbb{R}^d)$, then*

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \sigma_n^\theta f = f \qquad a.e.$$

for all $f \in L_p(\mathbb{T}^d)$ whenever $1 \le p < \infty$ and for all $f \in C(\mathbb{T}^d)$ whenever $p = \infty$.

If $f \in L_p(\mathbb{T}^d)$ $(1 \le p \le 2)$ implies the a.e. convergence of Cor. 2, then σ_γ^θ is bounded from $L_p(\mathbb{T}^d)$ to $L_{p,\infty}(\mathbb{T}^d)$, as in Th. 3 (see Stein [10]). The partial converse of Th. 2 is given in the next result. More exactly, if $\sigma_\gamma^\theta f$ can be estimated pointwise by $M_p^{\tau,\gamma} f$, then (9) holds.

Theorem 6. *Let θ satisfy* (2), $1 \le p \le \infty$ *and* $1/p + 1/q = 1$. *Suppose that*

$$\sigma_\gamma^\theta f(x) \le C M_p^{\tau,\gamma} f(x) \tag{13}$$

for all $x \in \mathbb{T}^d$ and for all $f \in L_p(\mathbb{T}^d)$. Then

$$\sup_{n\in\mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C.$$

Proof. Let us define the space $D_p^\gamma(\mathbb{T}^d)$ $(1 \le p \le \infty)$ by the norm

$$\|f\|_{D_p^\gamma(\mathbb{T}^d)} := \sup_{0<r\le 1} \Big(\frac{1}{\prod_{j=1}^d \gamma_j(r)} \int_{\prod_{j=1}^d(-\gamma_j(r)\pi,\gamma_j(r)\pi)} |f|^p \, d\lambda\Big)^{1/p}. \tag{14}$$

Observe that the norm

$$\|f\|_* = \sup_{k\le 0}\Big(\prod_{j=1}^d \gamma_j(\xi^k)\Big)^{-1/p}\|f\mathbf{1}_{P_k}\|_p \tag{15}$$

is an equivalent norm on $D_p^\gamma(\mathbb{T}^d)$. Indeed, choosing $r=\xi^k$ $(k\le 0)$ we conclude $\|f\|_*\le C\|f\|_{D_p^\gamma}$. On the other hand, if $\xi^{n-1}<r\le\xi^n$ for some $n\le 0$ then

$$\begin{aligned}
&\frac{1}{\prod_{j=1}^d \gamma_j(r)}\int_{\prod_{j=1}^d(-\gamma_j(r)\pi,\gamma_j(r)\pi)}|f|^p\,d\lambda\le\\
&\le\Big(\prod_{j=1}^d\gamma_j(\xi^{n-1})\Big)^{-1}\int_{\prod_{j=1}^d(-\gamma_j(\xi^n)\pi,\gamma_j(\xi^n)\pi)}|f|^p\,d\lambda=\\
&=\Big(\prod_{j=1}^d\gamma_j(\xi^{n-1})\Big)^{-1}\sum_{k=-\infty}^{n}\int_{P_k}|f|^p\,d\lambda\le\\
&\le\Big(\prod_{j=1}^d\gamma_j(\xi^{n-1})\Big)^{-1}\sum_{k=-\infty}^{n}\Big(\prod_{j=1}^d\gamma_j(\xi^k)\Big)\|f\|_*^p.
\end{aligned}$$

Note that

$$\gamma_j(\xi^k)\le\frac{1}{c_{j,1}}\gamma_j(\xi^{k+1})\le\ldots\le\frac{1}{c_{j,1}^{n-k}}\gamma_j(\xi^n)\quad\text{and}\quad\gamma_j(\xi^{n-1})\ge\frac{1}{c_{j,2}}\gamma_j(\xi^n).$$

Hence

$$\frac{1}{\prod_{j=1}^d\gamma_j(r)}\int_{\prod_{j=1}^d(-\gamma_j(r)\pi,\gamma_j(r)\pi)}|f|^p\,d\lambda\le\Big(\prod_{j=1}^d c_{j,2}\Big)\sum_{k=-\infty}^{n}\Big(\prod_{j=1}^d\frac{1}{c_{j,1}}\Big)^{n-k}\|f\|_*^p\le$$
$$\le C\|f\|_*^p,$$

or, in other words $\|f\|_{D_p^\gamma(\mathbb{T}^d)}\le C\|f\|_*$. Choosing $r=1$ we can see that $D_p^\gamma(\mathbb{T}^d)\subset L_p(\mathbb{T}^d)$ and $\|f\|_p\le C\|f\|_{D_p^\gamma(\mathbb{T}^d)}$. Taking the supremums in (14) and (15) for all $0<r<\infty$ and $k\in\mathbb{Z}$ then we obtain the space $D_p^\gamma(\mathbb{R}^d)$.

It is easy to see by (15) that

$$\sup_{\|f\|_{D_p^\gamma(\mathbb{T}^d)}\le 1}\Big|\int_{\mathbb{T}^d}f(-t)K_n^\theta(t)\,dt\Big|=\|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)}. \tag{16}$$

There exists a function $f\in D_p^\gamma(\mathbb{T}^d)$ with $\|f\|_{D_p^\gamma}\le 1$ such that

$$\frac{\|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)}}{2} \le \Big| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t)\, dt \Big|.$$

Since $f \in L_p(\mathbb{R}^d)$, by (13),

$$|\sigma_n^\theta f(0)| = \Big| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t)\, dt \Big| \le C M_p^{\tau,\gamma} f(0) \qquad (n \in \mathbb{R}^d_{\tau,\gamma}),$$

which implies

$$\|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C M_p^{\tau,\gamma} f(0) \le C M_p^{\gamma} f(0) \le C\|f\|_{D_p^\gamma} \le C.$$

This proves the result. $\Diamond$

Note that the norm of $D_p^\gamma(\mathbb{T}^d)$ in (14) is equivalent to

$$\|f\| = \sup_{r \in (0,1]^d, r \in \mathbb{R}^d_{\tau,\gamma}} \left(\frac{1}{\prod_{j=1}^d r_j} \int_{\prod_{j=1}^d (-r_j\pi, r_j\pi)} |f|^p \, d\lambda \right)^{1/p}.$$

We will characterize the points of convergence. To this end we generalize the concept of Lebesgue points. By Cor. 1,

$$\lim_{\substack{0 \in I, (|I_1|,\ldots,|I_d|) \in \mathbb{R}^d_{\tau,\gamma} \\ |I_j| \to 0, j=1,\ldots,d}} \frac{1}{|I|} \int_I f(x+u)\, du = f(x) \qquad \text{for a.e. } x \in \mathbb{T}^d,$$

where $f \in L_1^{loc}(\mathbb{T}^d)$. A point $x \in \mathbb{T}^d$ is called a *p-Lebesgue point* (or a Lebesgue point of order p) of $f \in L_p^{loc}(\mathbb{T}^d)$ if

$$\lim_{\substack{0 \in I, (|I_1|,\ldots,|I_d|) \in \mathbb{R}^d_{\tau,\gamma} \\ |I_j| \to 0, j=1,\ldots,d}} \left(\frac{1}{|I|} \int_I |f(x+u) - f(x)|^p \, du \right)^{1/p} = 0 \qquad (1 \le p < \infty)$$

resp.

$$\lim_{\substack{0 \in I, (|I_1|,\ldots,|I_d|) \in \mathbb{R}^d_{\tau,\gamma} \\ |I_j| \to 0, j=1,\ldots,d}} \sup_{u \in I} |f(x+u) - f(x)| = 0 \qquad (p = \infty).$$

One can see that this definition is equivalent to

$$\lim_{r \to 0} \left(\frac{1}{\prod_{j=1}^d \gamma_j(r)} \int_{\prod_{j=1}^d (-\gamma_j(r)\pi, \gamma_j(r)\pi)} |f(x+u) - f(x)|^p \, du \right)^{1/p} = 0 \quad (1 \le p < \infty)$$

resp. to

$$\lim_{r \to 0} \sup_{u \in \prod_{j=1}^d (-\gamma_j(r)\pi, \gamma_j(r)\pi)} |f(x+u) - f(x)| = 0 \qquad (p = \infty).$$

Usually the 1-Lebesgue points are considered in the case if each γ_j is the identity function (cf. Stein and Weiss [11] or Butzer and Nessel [3]). One can show in the usual way that almost every point $x \in \mathbb{T}^d$ is a p-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if $1 \le p < \infty$. $x \in \mathbb{T}^d$ is an ∞-Lebesgue point of

$f \in L_\infty^{loc}(\mathbb{T}^d)$ if and only if f is continuous at x. Moreover, all r-Lebesgue points are p-Lebesgue points, whenever $p < r$.

The next theorem generalizes Lebesgue's theorem.

Theorem 7. *Let θ satisfy (2), $1 \le p \le \infty$, $1/p + 1/q = 1$ and*

$$\sup_{n \in \mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C.$$

If for all $\delta > 0$

$$\lim_{n\to\infty, n \in \mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{L_q(\mathbb{T}^d \setminus (-\delta,\delta)^d)} = 0, \tag{17}$$

then

$$\lim_{n\to\infty, n \in \mathbb{R}^d_{\tau,\gamma}} \sigma_n^\theta f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{T}^d)$.

Proof. Now denote by

$$G(u) := \left(\int_{|t_j|<u_j, j=1,\dots,d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \qquad (u \in \mathbb{R}_+).$$

Since x is a p-Lebesgue point of f, for all $\epsilon > 0$ there exists $m \in \mathbb{Z}$, $m \le 0$ such that

$$\frac{G(\gamma_1(r)\pi, \dots, \gamma_d(r)\pi)}{\left(\prod_{j=1}^d \gamma_j(r) \right)^{1/p}} \le \epsilon \qquad \text{if} \qquad 0 < r \le \xi^m. \tag{18}$$

Note that

$$\sigma_n^\theta f(x) - f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (f(x-t) - f(x)) K_n^\theta(t) \, dt.$$

Thus

$$\begin{aligned} |\sigma_n^\theta f(x) - f(x)| &\le C \int_{\mathbb{T}^d} |f(x-t) - f(x)||K_n^\theta(t)| \, dt = \\ &= C \int_{\prod_{j=1}^d (-\gamma_j(\xi^m)\pi, \gamma_j(\xi^m)\pi)} |f(x-t) - f(x)||K_n^\theta(t)| \, dt + \\ &\quad + C \int_{\mathbb{T}^d \setminus \prod_{j=1}^d (-\gamma_j(\xi^m)\pi, \gamma_j(\xi^m)\pi)} |f(x-t) - f(x)||K_n^\theta(t)| \, dt =: \\ &=: A_0(x) + A_1(x). \end{aligned}$$

We estimate $A_0(x)$ by

$$A_0(x) = C \sum_{k=-\infty}^{m} \int_{P_k} |f(x-t) - f(x)||K_n^\theta(t)|\, dt \le$$
$$\le C \sum_{k=-\infty}^{m} \Big(\int_{P_k} |K_n^\theta(t)|^q\, dt\Big)^{1/q} \Big(\int_{P_k} |f(x-t) - f(x)|^p\, dt\Big)^{1/p} \le$$
$$\le C \sum_{k=-\infty}^{m} \Big(\int_{P_k} |K_n^\theta(t)|^q\, dt\Big)^{1/q} G(\gamma_1(\xi^k)\pi, \ldots, \gamma_d(\xi^k)\pi).$$

Then, by (18),

$$A_0(x) \le C_q \epsilon \sum_{k=-\infty}^{m} \Big(\prod_{j=1}^{d} \gamma_j(\xi^k)\Big)^{1/p} \Big(\int_{P_k} |K_n^\theta(t)|^q\, dt\Big)^{1/q} \le C_q \epsilon \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)}.$$

There exists $\delta > 0$ such that $(-\delta,\delta)^d \subset \prod_{j=1}^{d}(-\gamma_j(\xi^m)\pi, \gamma_j(\xi^m)\pi)$. Then

$$A_1(x) \le C \int_{\mathbb{T}^d \setminus (-\delta,\delta)^d} |f(x-t) - f(x)||K_n^\theta(t)|\, dt \le$$
$$\le C\Big(\int_{\mathbb{T}^d \setminus (-\delta,\delta)^d} |K_n^\theta(t)|^q\, dt\Big)^{1/q} (\|f\|_p + |f(x)|),$$

which tends to 0 as $n \to \infty, n \in \mathbb{R}^d_{\tau,\gamma}$. This completes the proof of the theorem. ◊

Observe that (8) and $(-\delta',\delta')^d \subset \prod_{j=1}^{d}(-\gamma_j(\xi^k)\pi, \gamma_j(\xi^k)\pi) \subset (-\delta,\delta)^d$ imply

$$(19) \quad \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d \setminus (-\delta,\delta)^d)} \le \|K_n^\theta\|_{L_q(\mathbb{T}^d \setminus \prod_{j=1}^{d}(-\gamma_j(\xi^k)\pi, \gamma_j(\xi^k)\pi))} \le$$
$$\le \Big(\sum_{l=k+1}^{0} \int_{P_l} |K_n^\theta(t)|^q\, dt\Big)^{1/q} \le$$
$$\le C_\delta \sum_{l=k+1}^{0} \Big(\prod_{j=1}^{d} \gamma_j(\xi^k)\Big)^{1-1/q} \Big(\int_{P_l} |K_n^\theta(t)|^q\, dt\Big)^{1/q} \le$$
$$\le C_\delta \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d \setminus \prod_{j=1}^{d}(-\gamma_j(\xi^k)\pi, \gamma_j(\xi^k)\pi))} \le$$
$$\le C_\delta \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d \setminus (-\delta',\delta')^d)}.$$

Condition (17) is trivially equivalent to

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{L_q(\mathbb{T}^d\setminus\prod_{j=1}^d(-\gamma_j(\xi^k)\pi,\gamma_j(\xi^k)\pi))} = 0$$

and hence to

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d\setminus(-\delta,\delta)^d)} = 0.$$

In case $\widehat{\theta} \in E_q^\gamma(\mathbb{R}^d)$ we can formulate a little bit simpler version of the preceding theorem.

Theorem 8. *Suppose that* $c_j = c_{j,1} = c_{j,2}$ *for all* $j = 1,\dots,d$. *Let* $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$, $\widehat{\theta} \in E_q^\gamma(\mathbb{R}^d)$, $1 \le p \le \infty$ *and* $1/p + 1/q = 1$. *Then*

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \sigma_n^\theta f(x) = f(x)$$

for all p*-Lebesgue points of* $f \in L_p(\mathbb{T}^d)$.

Proof. By (10) the first condition of Th. 7 is satisfied. On the other hand, let

$$\prod_{j=1}^d(-\gamma_j(\xi^{k_0})\pi, \gamma_j(\xi^{k_0})\pi) \subset (-\delta,\delta)^d, \qquad \tau_j/c_j^r \le 1, \qquad c_j^s/\tau_j \ge 1$$

and $\xi^{l-1} \le n_1 < \xi^l$ as in the proof of Th. 4. Obviously, if $n \to \infty, n \in \mathbb{R}^d_{\tau,\gamma}$ then $l \to \infty$. We get similarly to (11) and (12) that

$$\begin{aligned} &\|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d\setminus(-\delta,\delta)^d)} \le \\ &\le C_q \sum_{i=k_0+l-s-1}^\infty \Big(\prod_{j=1}^d \gamma_j(\xi^i)\Big)^{1-1/q} \Big(\int_{P_i} |\widehat{\theta}(t_1,\dots,t_d)|^q\,dt\Big)^{1/q} + \\ &\quad + C_q \sum_{i=(l-s)\vee 0}^\infty \Big(\prod_{j=1}^d \gamma_j(\xi^i)\Big)^{1-1/q} \Big(\int_{P_i} |\widehat{\theta}(t_1,\dots,t_d)|^q\,dt\Big)^{1/q}, \end{aligned}$$

which tends to 0 as $n \to \infty, n \in \mathbb{R}^d_{\tau,\gamma}$, since $\widehat{\theta} \in E_q^\gamma(\mathbb{R}^d)$. Then (17) follows by (19), which finishes the proof of our theorem. $\Diamond$

Since each point of continuity is a p-Lebesgue point, we have

Corollary 4. *If the conditions of Th. 7 or Th. 8 are satisfied and if* $f \in L_1(\mathbb{T}^d)$ *is continuous at a point* x, *then*

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \sigma_n^\theta f(x) = f(x).$$

The converse of Th. 7 holds also.

Theorem 9. *Suppose that* $1 \le p \le \infty$, $1/p + 1/q = 1$ *and* (2) *and* (17) *hold. If*

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \sigma_n^\theta f(x) = f(x)$$

for all p*-Lebesgue points of* $f \in L_p(\mathbb{T}^d)$, *then*

$$\sup_{n\in\mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C.$$

Proof. The space $D_p^{\gamma,0}(\mathbb{T}^d)$ consists of all functions $f \in D_p^\gamma(\mathbb{T}^d)$ for which $f(0) = 0$ and 0 is a p-Lebesgue point of f, in other words

$$\lim_{r\to 0}\left(\frac{1}{\prod_{j=1}^d \gamma_j(r)} \int_{\prod_{j=1}^d(-\gamma_j(r)\pi, \gamma_j(r)\pi)} |f(u)|^p \, du\right)^{1/p} = 0$$

with the usual modification for $p = \infty$. We can easily show that $D_p^{\gamma,0}(\mathbb{T}^d)$ is a Banach space. We get from the conditions of the theorem that

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \sigma_n^\theta f(0) = 0 \quad \text{for all} \quad f \in D_p^{\gamma,0}(\mathbb{T}^d).$$

Thus the operators

$$U_n : D_p^{\gamma,0}(\mathbb{T}^d) \to \mathbb{R}, \qquad U_n f := \sigma_n^\theta f(0) \qquad (n \in \mathbb{R}^d_{\tau,\gamma})$$

are uniformly bounded by the Banach–Steinhaus theorem. Observe that in (16) we may suppose that f is 0 in a neighborhood of 0. Then

$$\begin{aligned} C \ge \|U_n\| &= \\ &= \sup_{\|f\|_{D_p^{\gamma,0}(\mathbb{T}^d)}\le 1} \left| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t)\, dt \right| = \\ &= \sup_{\|f\|_{D_p^{\gamma}(\mathbb{T}^d)}\le 1} \left| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t)\, dt \right| = \\ &= \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \end{aligned}$$

for all $n \in \mathbb{R}^d_{\tau,\gamma}$. $\diamondsuit$

Corollary 5. *Suppose that* $1 \le p \le \infty$, $1/p + 1/q = 1$ *and* (2) *and* (17) *holds. Then*

$$\lim_{n\to\infty, n\in\mathbb{R}^d_{\tau,\gamma}} \sigma_n^\theta f(x) = f(x)$$

for all p*-Lebesgue points of* $f \in L_p(\mathbb{T}^d)$ *if and only if*

$$\sup_{n\in\mathbb{R}^d_{\tau,\gamma}} \|K_n^\theta\|_{E_q^\gamma(\mathbb{T}^d)} \le C.$$

A one-dimensional version of this theorem can be found in the book of Alexits [1].

7. Some summability methods

In this section we consider some summability methods as special cases of the θ-summation. The details can be found in [15]. Note that $q = \infty$ is the most important case in the results of Sec. 6. Let $\gamma_j(\xi x) = \xi^{\omega_j}\gamma_j(x)$ $(x > 0)$ and $\omega_1 = 1$.

Example 1 ((C, α) or Cesàro summation). Let $d = 1$ and

$$\theta(k,n) = \begin{cases} \frac{A^{\alpha}_{n-1-|k|}}{A^{\alpha}_{n-1}} & \text{if } |k| \le n-1, \\ 0 & \text{if } |k| \ge n \end{cases}$$

for some $0 < \alpha < \infty$, where

$$A_k^{\alpha} := \binom{k+\alpha}{k} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+k)}{k!} = O(k^{\alpha}) \qquad (k \in \mathbb{N}).$$

The Cesàro means are given by

$$\sigma_n^{\theta} f(x) := \frac{1}{A^{\alpha}_{n-1}} \sum_{k=-n+1}^{n-1} A^{\alpha}_{n-1-|k|} \hat{f}(k) e^{\imath kx}.$$

In case $\alpha = 1$ we get the *Fejér means*, i.e.

$$\sigma_n^1 f(x) = \sum_{k=-n+1}^{n-1} \Big(1 - \frac{|k|}{n}\Big) \hat{f}(k) e^{\imath kx} = \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x).$$

It is known that the kernel functions satisfy

$$|K_n^{\theta}(u)| \le C \min(n, n^{-\alpha} u^{-\alpha-1}) \qquad (n \in \mathbb{N},\ u \ne 0)$$

(see Zygmund [17]). It is easy to see that (9) and (17) holds as well as all theorems of this paper.

Example 2 (Riesz summation). Let

$$\theta(x) := \begin{cases} (1-|x|^2)^{\alpha} & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1 \end{cases} \qquad (x \in \mathbb{R}^d).$$

Then

$$|\widehat{\theta}(x)| \le C|x|^{-d/2-\alpha-1/2} \qquad (x \ne 0).$$

If

$$\text{(20)} \qquad \frac{\sum_{j=1}^{d} \omega_j}{\omega_i} - \frac{d}{2} - \frac{1}{2} < \alpha < \infty \qquad \text{for all } i = 1, \dots, d,$$

then $\widehat{\theta} \in E_{\infty}^{\gamma}(\mathbb{R}^d)$. Here $|\cdot|$ denotes the Euclidean norm.

Note that for cones, i.e. $\omega_j = 1$, $j = 1, \dots, d$, we get the well known parameter $(d-1)/2$ on the left hand side of (20). In case $d = 2$ we obtain the condition $(1/\omega_2 - 1/2) \vee (\omega_2 - 1/2) < \alpha < \infty$.

Example 3 (Weierstrass summation). If $\theta(x) = e^{-2\pi|x|^2}$ $(x \in \mathbb{R}^d)$, then $\widehat{\theta}(x) = e^{-2\pi|x|^2}$ and $\widehat{\theta} \in E_\infty^\gamma(\mathbb{R}^d)$.

Example 4. If $\theta(x) = e^{-2\pi|x|}$ $(x \in \mathbb{R}^d)$ then $\widehat{\theta}(x) = c_d/(1+|x|^2)^{(d+1)/2}$. Suppose that $\omega_d \le \omega_j$ for all $j = 2, \dots, d$. If $\omega_d \le 1$ and $\sum_{j=1}^{d-1} \omega_j < d\omega_d$ or if $\omega_d > 1$ and $\sum_{j=2}^{d} \omega_j < d$ then $\widehat{\theta} \in E_\infty^\gamma(\mathbb{R}^d)$. If $d = 2$ then we obtain $1/2 < \omega_2 < 2$.

Example 5 (Picard and Bessel summation). In case

$$\theta(x) = 1/(1+|x|^2)^{(d+1)/2} \qquad (x \in \mathbb{R}^d)$$

we have $\widehat{\theta}(x) = c_d e^{-2\pi|x|}$ and $\widehat{\theta} \in E_\infty^\gamma(\mathbb{R}^d)$.

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