HERZ SPACES AND POINTWISE SUMMABILITY OF FOURIER SERIES

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Abstract: A general summability method, the so-called \( \theta \)-summability is considered for multi-dimensional Fourier series. It is proved that if the kernel functions are uniformly bounded in a Herz space then the restricted maximal operator of the \( \theta \)-means of a distribution is of weak type \((1, 1)\), provided that the supremum in the maximal operator is taken over a cone-like set. From this it follows that \( \sigma_n^\theta f \to f \) a.e. for all \( f \in L_1(\mathbb{T}^d) \). Moreover, \( \sigma_n^\theta f(x) \) converges to \( f(x) \) over a cone-like set at each Lebesgue point of \( f \in L_1(\mathbb{T}^d) \) if and only if the kernel functions are uniformly bounded in a suitable Herz space. The Cesàro, Riesz and Weierstrass summations are investigated as special cases of the \( \theta \)-summation.

1. Introduction

The well-known Lebesgue [8] theorem says that for every integrable function \( f \) the Fejér means \( \sigma_n f(x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x) \) converge to \( f(x) \) as \( n \to \infty \) at each Lebesgue point of \( f \), where \( s_k f \) denotes the \( k \)th partial sum of the Fourier series of \( f \). Almost every point is a Lebesgue point.

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of $f$. Later Alexits [1] generalized this result and gave a sufficient and necessary condition such that the singular integrals converge at every Lebesgue point.

For multi-dimensional trigonometric-Fourier series Marcinkiewicz and Zygmund [9, 17] proved that the Fejér means $\sigma_n f$ of a function $f \in L^1(\mathbb{T}^d)$ converge a.e. to $f$ as $n \to \infty$ provided that $n$ is in a cone, i.e., $\frac{1}{\tau} \leq \frac{n_k}{n_j} \leq \tau$ for every $k, j = 1, \ldots, d$ and for some $\tau \geq 1$ $(n = (n_1, \ldots, n_d) \in \mathbb{N}^d)$. We have extended this result to the $\theta$-summation in [14]. The so called $\theta$-summation is a general method of summation and it is intensively studied in the literature (see e.g. Butzer and Nessel [3], Trigub and Belinsky [13] and Weisz [14, 4, 5] and the references therein). Similar results for so-called cone-like sets can be found in Gát [6] and Weisz [15, 16].

In this paper we extend the results concerning the Lebesgue points to cone-like sets defined by a function $\gamma$. We introduce a new version of the Hardy–Littlewood maximal function depending on $\gamma$ and show that if the kernel functions of the $\theta$-summation are uniformly bounded in a modified Herz space, then the maximal function $\sigma_n^\theta f$ can be estimated by the Hardy–Littlewood maximal function $M_p^\gamma f$, provided that the supremum in the maximal operator is taken over a cone-like set. Since $M_p^\gamma$ is of weak type $(p, p)$ we obtain $\sigma_n^\theta f \to f$ a.e. over a cone-like set for all $f \in L_p(\mathbb{T}^d)$. The set of convergence is also characterized, the convergence holds at every $p$-Lebesgue point of $f$. The converse holds also, more exactly, $\sigma_n^\theta f(x) \to f(x)$ over a cone-like set at each $p$-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if and only if the kernel functions are uniformly bounded in the Herz space. As special cases five examples of the $\theta$-summation are considered, amongst others the Cesàro, Riesz and Weierstrass summations. Similar results for Fourier transforms can be found in Feichtinger and Weisz [5, 15].

2. Wiener algebra

Let us fix $d \geq 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d-times. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set $u \cdot x := \sum_{k=1}^d u_k x_k$.

We briefly write $L_p$ or $L_p(\mathbb{T}^d)$ instead of $L_p(\mathbb{T}^d, \lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p := (\int_{\mathbb{T}^d} |f|^p d\lambda)^{1/p}$ $(0 < p \leq \infty)$, where $\mathbb{T} = [-\pi, \pi]$ is the torus and $\lambda$ is the Lebesgue measure.
The weak $L_p$ space, $L_{p,\infty} (\mathbb{T}^d)$ ($0 < p < \infty$) consists of all measurable functions $f$ for which
\[ \|f\|_{p,\infty} := \sup_{\rho>0} \rho \lambda(\{|f| > \rho\})^{1/p} < \infty, \]
while we set $L_{\infty,\infty} (\mathbb{T}^d) = L_\infty (\mathbb{T}^d)$. Note that $L_{p,\infty} (\mathbb{T}^d)$ is a quasi-normed space (see Bergh and Lofström [2]). It is easy to see that for each $0 < p \leq \infty$,
\[ L_p (\mathbb{T}^d) \subset L_{p,\infty} (\mathbb{T}^d) \quad \text{and} \quad \|\cdot\|_{p,\infty} \leq \|\cdot\|_p. \]
The space of continuous functions with the supremum norm is denoted by $C(\mathbb{T}^d)$.

A measurable function $f$ belongs to the Wiener amalgam space $W(L_\infty, \ell_1)(\mathbb{R}^d)$ if
\[ \|f\|_{W(L_\infty, \ell_1)} := \sum_{k \in \mathbb{Z}^d} \sup_{x \in [0,1)^d} |f(x + k)| < \infty. \]
It is easy to see that $W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$ for all $1 \leq p \leq \infty$. The closed subspace of $W(L_\infty, \ell_1)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, \ell_1)(\mathbb{R}^d)$ and is called Wiener algebra. It is used quite often in Gabor analysis, because it provides a convenient and general class of windows (see e.g. Gröchenig [7]). It turned out in Feichtinger and Weisz [4, 5] that it can be well applied in summability theory, too.

3. $\theta$-summability of Fourier series

We will consider the $\theta$-summation defined by a multi-parameter sequence. Let
\[ \theta = (\theta(k,n), k \in \mathbb{Z}^d, n \in \mathbb{N}^d) \]
be a $2d$-parameter sequence of real numbers satisfying
\[ \theta(0, \ldots, 0, n) = 1, \quad \lim_{n \to \infty} \theta(k,n) = 1 \quad (\theta(k,n))_{k \in \mathbb{Z}^d} \in \ell_1 \]
for each $n \in \mathbb{N}^d$. Recall that for a distribution $f \in \mathcal{S}'(\mathbb{T}^d)$ the $n$th Fourier coefficient is defined by $\hat{f}(n) := f(e^{-inx})$ ($n \in \mathbb{Z}^d, i = \sqrt{-1}$). In special case, if $f \in L_1(\mathbb{T}^d)$ then
\[ \hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(t)e^{-int} dt \quad (n \in \mathbb{Z}^d). \]
The \( \theta \)-means of a distribution \( f \in \mathcal{S}'(\mathbb{T}^d) \) are defined by
\[
\sigma^\theta_n f(x) := \sum_{k_1 = -\infty}^{\infty} \ldots \sum_{k_d = -\infty}^{\infty} \theta(-k, n) \hat{f}(k) e^{i k \cdot x} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}} f(x - t) K^\theta_n(t) \, dt
\]
\((x \in \mathbb{T}^d, n \in \mathbb{N}^d)\), where \( K^\theta_n \) denotes the \( \theta \)-kernel
\[
K^\theta_n(t) := \sum_{k_1 = -\infty}^{\infty} \ldots \sum_{k_d = -\infty}^{\infty} \theta(-k, n) e^{i k \cdot t} \quad (t \in \mathbb{T}^d).
\]
Observe that (2) ensures that \( K^\theta_n \in L_1(\mathbb{T}) \).

We can also define a \( \theta \)-summation by one single function \( \theta \) defined on \( \mathbb{R}^d \). In this case we define the sequence in (1) by
\[
\theta(k, n) := \theta\left(\frac{k_1}{n_1}, \ldots, \frac{k_d}{n_d}\right) \quad (k \in \mathbb{Z}^d, n \in \mathbb{N}^d).
\]
If \( \theta(0) = 1 \) and \( \theta \in W(C, \ell_1)(\mathbb{R}^d) \) then (2) is satisfied, because
\[
\sum_{k_1 = -\infty}^{\infty} \ldots \sum_{k_d = -\infty}^{\infty} \left| \theta\left(\frac{k_1}{n_1}, \ldots, \frac{k_d}{n_d}\right) \right| \leq \sum_{l_1 = -\infty}^{\infty} \ldots \sum_{l_d = -\infty}^{\infty} \left( \prod_{j=1}^{d} n_j \right) \sup_{x \in [0,1)} |\theta(x+l)| = \left( \prod_{j=1}^{d} n_j \right) \|\theta\|_{W(C,\ell_1)} < \infty.
\]
The Fourier transform of \( f \in L_1(\mathbb{R}^d) \) is given by
\[
\hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi i x \cdot t} \, dt \quad (x \in \mathbb{R}^d).
\]
If \( \theta \) is a function and \( \hat{\theta} \in L_1(\mathbb{R}^d) \) then
\[
\sigma^\theta_n f(x) = \left( \prod_{j=1}^{d} n_j \right) \int_{\mathbb{R}^d} f(x - t) \hat{\theta}(n_1 t_1, \ldots, n_d t_d) \, dt
\]
for all \( x \in \mathbb{T}^d, n \in \mathbb{N}^d \) and \( f \in L_1(\mathbb{T}^d) \), where \( f \) is extended periodically to \( \mathbb{R}^d \) (see Feichtinger and Weisz [4]).

4. Hardy–Littlewood inequality and cone-like sets

Suppose that for all \( j = 2, \ldots, d \), \( \gamma_j : \mathbb{R}_+ \to \mathbb{R}_+ \) are strictly increasing and continuous functions such that \( \gamma_j(1) = 1 \), \( \lim_{\infty} \gamma_j = \infty \) and \( \lim_{0^+} \gamma_j = 0 \). Moreover, suppose that there exist \( c_{j,1}, c_{j,2}, \xi > 1 \) such that
\[
c_{j,1} \gamma_j(x) \leq \gamma_j(\xi x) \leq c_{j,2} \gamma_j(x) \quad (x > 0).
\]
Note that this is satisfied if $\gamma_j$ is a power function. For convenience we extend the notations for $j = 1$ by $\gamma_1 := \mathcal{I}$ and $c_{1,2} = \xi$. Here $\mathcal{I}$ denotes the identity function $\mathcal{I}(x) = x$. Let $\gamma = (\gamma_1, \ldots, \gamma_d)$ and $\tau = (\tau_1, \ldots, \tau_d)$ with $\tau_1 = 1$ and fixed $\tau_j \geq 1$ $(j = 2, \ldots, d)$. We will investigate the Hardy–Littlewood maximal operator and later the maximal operator of the $\theta$-summation over a cone-like set (with respect to the first dimension)

$$\mathbb{R}_+^d := \{x \in \mathbb{R}_+^d : \tau_1^{-1}\gamma_1(n_1) \leq n_j \leq \tau_j \gamma_j(n_1), j = 2, \ldots, d\}.$$ 

If each $\gamma_j$ is the identity, $j = 2, \ldots, d$, then we get the cone defined by $\tau$. The condition on $\gamma_j$ seems to be natural, because Gát [6] proved in the two-dimensional case that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if (5) holds.

$L^p_{\text{loc}}(\mathbb{T}^d)$ $(1 \leq p \leq \infty)$ denotes the space of measurable functions $f$ for which $|f|^p$ is locally integrable, resp. $f$ is locally bounded if $p = \infty$. In [15] we have introduced the Hardy–Littlewood maximal function on a cone-like set by

$$M_p^{\gamma} f(x) := \sup_{x \in I, (|I_1|, \ldots, |I_d|) \in \mathbb{R}_+^d} \left(\frac{1}{|I|} \int_I |f|^p d\lambda\right)^{1/p} \quad (x \in \mathbb{T}^d)$$

with the usual modification for $p = \infty$, where $f \in L^p_{\text{loc}}(\mathbb{T}^d)$ and the supremum is taken over all rectangles $I := I_1 \times \cdots \times I_d \subset \mathbb{T}^d$ with sides parallel to the axes. Taking the supremum over rectangles with $|I_j| = \gamma_j(|I_1|)$, $j = 2, \ldots, d$, (i.e. $\tau_j = 1$, $j = 1, \ldots, d$), we obtain the maximal operator $M_{\gamma}^{\tau}$. The inequality

$$M^p \leq M_p^{\gamma} f \leq CM_p f$$

was shown in Weisz [15]. In case $p = 1$ we write simply $M^{\tau, \gamma}$ and $M^{\gamma}$. If each $\gamma_j$ is the identity function then we get back the classical Hardy–Littlewood maximal function defined on a cone. The following theorem was proved in [15].

**Theorem 1.** The maximal operator $M_p^{\tau, \gamma} (1 \leq p \leq \infty)$ is of weak type $(p, p)$, i.e.

$$\|M_p^{\tau, \gamma} f\|_{p, \infty} = \sup_{\rho > 0} \rho \lambda(M_p^{\tau, \gamma} f > \rho)^{1/p} \leq C_p \|f\|_p \quad (f \in L_p(\mathbb{T}^d)).$$

Moreover, if $1 \leq p < r \leq \infty$ then

$$\|M_p^{\tau, \gamma} f\|_r \leq C_r \|f\|_r \quad (f \in L_r(\mathbb{T}^d)).$$
Since the set of continuous functions are dense in \( L_1(\mathbb{T}^d) \), the usual density argument due to Marcinkiewicz and Zygmund [9] implies

**Corollary 1.** If \( f \in L_1(\mathbb{T}^d) \) then
\[
\lim_{x \in I, (|I_1|,\ldots,|I_d|) \in \mathbb{R}^d, \gamma} \frac{1}{|I|} \int_I f \, d\lambda = f(x) \quad \text{for a.e. } x \in \mathbb{T}^d.
\]

5. Herz spaces

The \( E_q(\mathbb{R}^d) \) \((1 \leq q \leq \infty)\) spaces were used recently by Feichtinger and Weisz [5] in the summability theory of Fourier transforms. A function belongs to the (homogeneous) Herz space \( E_q(\mathbb{R}^d) \) \((1 \leq q \leq \infty)\) if
\[
\|f\|_{E_q} := \sum_{k=-\infty}^{\infty} 2^{kd(1-1/q)} \|f1_{\{x \in \mathbb{R}^d, 2^k \leq \|x\|_\infty < 2^{k+1}\}}\|_q < \infty.
\]

Here we introduce a generalization of the \( E_q(\mathbb{R}^d) \) spaces depending on the function \( \gamma \) (see [15]). A function \( f \in L_{q, \text{loc}}(\mathbb{R}^d) \) is in the space \( E_{\gamma_q}(\mathbb{R}^d) \) \((1 \leq q \leq \infty)\) if
\[
\langle 7 \rangle \quad \|f\|_{E_{\gamma_q}} := \sum_{k=-\infty}^{\infty} \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right)^{1-1/q}\|f1_{P_k}\|_q < \infty,
\]
where \( \xi \) and \( \gamma_j \) are defined in (5) and
\[
P_k := \prod_{j=1}^{d} \left( -\gamma_j(\xi^k)\pi, \gamma_j(\xi^k)\pi \right) \cap \prod_{j=1}^{d} \left( -\gamma_j(\xi^{k-1})\pi, \gamma_j(\xi^{k-1})\pi \right) \quad (k \in \mathbb{Z}).
\]

If \( \gamma_j = I \) for all \( j = 1, \ldots, d \) and \( \xi = 2 \) then we get back the original spaces \( E_q(\mathbb{R}^d) \). However, it is easy to see that the spaces are equivalent for all \( \xi > 1 \), whenever each \( \gamma_j \) is the identity function. If we modify the definition of \( P_k \),
\[
P_k' := \prod_{j=1}^{d} \left( -\gamma_j(\xi^k)\pi, \gamma_j(\xi^k)\pi \right) \cap \prod_{j=1}^{d} \left( -\gamma_j(\xi^{k-1})\pi, \gamma_j(\xi^{k-1})\pi \right) \cap \mathbb{T}^d \quad (k \in \mathbb{Z}),
\]
then we get the definition of the space \( E_{\gamma_q}(\mathbb{T}^d) \). This means that we have to take the sum in (7) for \( k \leq 0 \), only, because \( \gamma_j(1) = 1 \) for all \( j = 1, \ldots, d \). Observe that
\[
|P_k| \sim \prod_{j=1}^{d} \gamma_j(\xi^k) \quad (k \in \mathbb{Z}).
\]
Indeed,

$$|P_k| = (2\pi)^d \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right) \left( 1 - \prod_{j=1}^{d} \frac{\gamma_j(\xi^{k-1})}{\gamma_j(\xi^k)} \right)$$

and

$$\frac{1}{c_{j,2}} \gamma_j(\xi^k) \leq \gamma_j(\xi^{k-1}) \leq \frac{1}{c_{j,1}} \gamma_j(\xi^k)$$

because of (5). Thus

$$(2\pi)^d \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right) \left( 1 - \prod_{j=1}^{d} \frac{1}{c_{j,2}} \right) \leq |P_k| \leq (2\pi)^d \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right) \left( 1 - \prod_{j=1}^{d} \frac{1}{c_{j,1}} \right).$$

This implies easily that

$$E_q^\gamma(X^d) \leftrightarrow E_q^\gamma(X^d) \leftrightarrow E_q^\gamma(X^d) \leftrightarrow E_q^\gamma(X^d) \leftrightarrow E_q^\gamma(X^d) \quad (1 < q < q' < \infty),$$

where $X$ denotes either $\mathbb{R}$ or $\mathbb{T}$. Moreover,

$$(8) \quad E_q^\gamma(T^d) \leftrightarrow L_q(T^d) \quad (1 \leq q \leq \infty).$$

Indeed, we have

$$\gamma_j(\xi^k) \leq \frac{1}{c_{j,1}} \gamma_j(\xi^{k+1}) \leq \ldots \leq \frac{1}{c_{|k|,1}}$$

and

$$\|f\|_{E_q^\gamma(T^d)} \leq \sum_{k=-\infty}^{0} \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right)^{1-1/q} \|f1_{P_k}\|_q \leq$$

$$\leq \sum_{k=-\infty}^{0} \left( \prod_{j=1}^{d} \frac{1}{c_{j,1}} \right)^{|k|(1-1/q)} \|f1_{P_k}\|_q \leq C_q\|f\|_q.$$

### 6. Convergence of the $\theta$-means of Fourier transforms

For a given $\tau, \gamma$ satisfying the above conditions the restricted maximal $\theta$-operator are defined by

$$\sigma^n_\gamma f := \sup_{n \in \mathbb{R}^d, \gamma} |\sigma^n_\gamma f|.$$

If $\gamma_j = I$ for all $j = 2, \ldots, d$ then we get a cone. This case was considered in Marcinkiewicz and Zygmund [9, 17] and more recently by the author [14]. Obviously, $\sigma^n_\gamma f \rightarrow f$ in $L_1$- or $C$-norm if and only if the numbers
∥K_n^θ∥_1 are uniformly bounded (n ∈ R^d_τ,γ). In [4, 15] we have proved if θ is a function then this condition is equivalent to \( \hat{\theta} \in L_1(\mathbb{R}^d) \).

Here we consider the pointwise convergence of the θ-means. In the one-dimensional case Alexits [1] and Torchinsky [12] proved that if there exists an even function η such that η is non-increasing on \( \mathbb{R}_+ \), \( |\hat{\theta}| ≤ \eta \), \( \eta \in L_1(\mathbb{R}) \) then the maximal operator of the θ-means is of weak type (1, 1). This condition is equivalent to \( \hat{\theta} \in E_\infty(\mathbb{R}) \) (see [5]). Now we generalize this theorem as follows.

**Theorem 2.** Let θ satisfy (2), \( 1 \leq p \leq \infty \) and \( 1/p + 1/q = 1 \). If

\[
\sup_{n \in \mathbb{R}^d_τ,γ} ||K_n^θ||_{E_q(T^d)} \leq C, \tag{9}
\]

then

\[
\sigma_γ f ≤ C \left( \sup_{n \in \mathbb{R}^d_τ,γ} ||K_n^θ||_{E_q(T^d)} \right) M_τ^γ f \quad \text{a.e.}
\]

for all \( f \in L_p(T^d) \).

**Proof.** By (3),

\[
|\sigma_n^θ f(x)| = \frac{1}{(2\pi)^d} \left| \int_{T^d} f(x-t)K_n^θ(t) \, dt \right| ≤ \frac{1}{(2\pi)^d} \sum_{k=\infty}^0 \int_{P_k} |f(x-t)||K_n^θ(t)| \, dt.
\]

Then

\[
|\sigma_n^θ f(x)| ≤ \frac{1}{(2\pi)^d} \sum_{k=\infty}^0 \left( \int_{P_k} |K_n^θ(t)|^q \, dt \right)^{1/q} \left( \int_{P_k} |f(x-t)|^p \, dt \right)^{1/p}.
\]

It is easy to see that if

\[
G(u) := \left( \int_{|t_j| < u_j, j=1,\ldots,d} |f(x-t)|^p \, dt \right)^{1/p} \quad (u \in \mathbb{R}^d_+)
\]

then

\[
\frac{G^\gamma(u)}{\prod_{j=1}^d u_j} ≤ C(M_p^γ f)^\gamma(x) \quad (u \in \mathbb{R}^d_τ,γ).
\]

Therefore

\[
|\sigma_n^θ f(x)| ≤ C \sum_{k=\infty}^0 \left( \int_{P_k} |K_n^θ(t)|^q \, dt \right)^{1/q} G(\gamma_1(\xi_k)π, \ldots, \gamma_d(\xi_k)π) ≤
\]

\[
≤ C \sum_{k=\infty}^0 \left( \prod_{j=1}^d \gamma_j(\xi_k) \right)^{1/p} \left( \int_{P_k} |K_n^θ(t)|^q \, dt \right)^{1/q} M_p^γ f(x) =
\]

\[
= C ||K_n^θ||_{E_q(T^d)} M_p^γ f(x),
\]
Herz spaces and pointwise summability of Fourier series

243

which shows the theorem. ♦

Note that (2) implies $K_n^\theta \in L_\infty(\mathbb{T}^d) \subset L_q(\mathbb{T}^d) \subset E_\gamma^q(\mathbb{T}^d)$ for all $n \in \mathbb{N}^d$. Th. 1 implies immediately

**Theorem 3.** Let $\theta$ satisfy (2), $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. If

$$\sup_{n \in \mathbb{N}^d} \|K_n^\theta\|_{E_\gamma^q(\mathbb{T}^d)} \leq C,$$

then

$$\|\sigma_n^\theta f\|_{p,\infty} \leq C_p \left( \sup_{n \in \mathbb{N}^d} \|K_n^\theta\|_{E_\gamma^q(\mathbb{T}^d)} \right) \|f\|_p \quad (f \in L_p(\mathbb{T}^d)).$$

Moreover, for every $p < r \leq \infty$

$$\|\sigma_n^\theta f\|_r \leq C \left( \sup_{n \in \mathbb{N}^d} \|K_n^\theta\|_{E_\gamma^q(\mathbb{T}^d)} \right) \|f\|_r \quad (f \in L_r(\mathbb{T}^d)).$$

These inequalities and the usual density theorem due to Marcinkiewicz-Zygmund [9] imply

**Corollary 2.** Let $\theta$ satisfy (2), $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. If

$$\sup_{n \in \mathbb{N}^d} \|K_n^\theta\|_{E_\gamma^q(\mathbb{T}^d)} \leq C,$$

then

$$\lim_{n \to \infty, n \in \mathbb{N}^d} \sigma_n^\theta f = f \quad \text{a.e.}$$

for all $f \in L_p(\mathbb{T}^d)$ whenever $1 \leq p < \infty$ and for all $f \in C(\mathbb{T}^d)$ whenever $p = \infty$.

In case the summability method is defined by a function $\theta$ and $\hat{\theta} \in E_\gamma^q(\mathbb{R}^d)$ then the preceding theorems hold.

**Theorem 4.** Suppose that $c_j = c_{j,1} = c_{j,2}$ for all $j = 1, \ldots, d$. Let $\theta \in W(C, \ell_1)(\mathbb{R}^d)$, $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. If $\hat{\theta} \in E_\gamma^q(\mathbb{R}^d)$ then

$$\sigma_n^\theta f \leq C \|\hat{\theta}\|_{E_\gamma^q(\mathbb{R}^d)} M_p^{\gamma} f \quad \text{a.e.}$$

for all $f \in L_p(\mathbb{T}^d)$.

**Proof.** Since by (4)

$$\sigma_n^\theta f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t)K_n^\theta(t) \, dt =$$

$$= \left( \prod_{j=1}^d n_j \right) \int_{\mathbb{R}^d} f(x-t)\hat{\theta}(n_1t_1, \ldots, n_dt_d) \, dt,$$

we can see that
where

\[
\text{Suppose that } \xi \text{ we can choose } r, s.
\]

We will prove that \( \hat{\theta} \in E^0_\gamma(\mathbb{R}^d) \) implies

\[
\|K_n^d\|_{E^0_\gamma(\mathbb{T}^d)} \leq C_q \|\hat{\theta}\|_{E^0_\gamma(\mathbb{R}^d)} \quad \text{for all } n \in \mathbb{R}^d.
\]

Since \( n \in \mathbb{R}^d \) we have \( \tau_j^{-1} \gamma_j(n_1) \leq n_j \leq \tau_j \gamma_j(n_1) \) for all \( j = 1, \ldots, d \).

For the term \( j = 0 \) of the norm we observe by (6) that

\[
\left\Vert \left( \prod_{j=1}^d n_j \right) \hat{\theta}(n_1 t_1, \ldots, n_d t_d) \right\Vert_{E^0_\gamma(\mathbb{T}^d)} = \sum_{k=-\infty}^0 \left( \prod_{j=1}^d \gamma_j(\xi^k) \right)^{1-1/q} \left( \prod_{j=1}^d n_j \right) \left( \int_{P_k} |\hat{\theta}(n_1 t_1, \ldots, n_d t_d)|^q \, dt \right)^{1/q} \leq C_q \sum_{k=-\infty}^0 \left( \prod_{j=1}^d \gamma_j(\xi^k) \right)^{1-1/q} \left( \prod_{j=1}^d \gamma_j(n_1) \right)^{1-1/q} \left( \int_{Q_k} |\hat{\theta}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q},
\]

where

\[
Q_k := \prod_{j=1}^d \left( -\tau_j \gamma_j(n_1) \gamma_j(\xi^k) \pi, \tau_j \gamma_j(n_1) \gamma_j(\xi^k) \pi \right) \setminus \prod_{j=1}^d \left( -\tau_j^{-1} \gamma_j(n_1) \gamma_j(\xi^{k-1}) \pi, \tau_j^{-1} \gamma_j(n_1) \gamma_j(\xi^{k-1}) \pi \right).
\]

Suppose that \( \xi^{l-1} \leq n_1 < \xi^l \) for some \( l \in \mathbb{N} \). Then by (5),

\[
c_j^{l-1} = \gamma_j(\xi^{l-1}) \leq \gamma_j(n_1) \leq \gamma_j(\xi^l) = c_j^l.
\]

We can choose \( r, s \in \mathbb{N} \) such that \( \tau_j/c_j^r \leq 1 \) and \( c_j^s/\tau_j \geq 1 \) for all \( j = 1, \ldots, d \). This and (5) imply that

\[
\tau_j \gamma_j(n_1) \gamma_j(\xi^k) \leq \tau_j \gamma_j(\xi^l) \gamma_j(\xi^k) = \tau_j c_j^l \gamma_j(\xi^k) = \frac{\tau_j}{c_j^r} \gamma_j(\xi^{k+l+r}) \leq \gamma_j(\xi^{k+l+r})
\]

and

\[
\frac{1}{\tau_j} \gamma_j(n_1) \gamma_j(\xi^{k-1}) \geq \frac{1}{\tau_j} \gamma_j(\xi^{l-1}) \gamma_j(\xi^{k-1}) = \frac{1}{\tau_j} c_j^{l-1} \gamma_j(\xi^{k-1}) = \frac{c_j^s}{\tau_j} \gamma_j(\xi^{k+l-s-2}) \geq \gamma_j(\xi^{k+l-s-2}).
\]

If
\[
Q_{k,l} := \prod_{j=1}^{d} (-\gamma_j (\xi^{k-l+r})_\pi, \gamma_j (\xi^{k-l+r})_\pi) \setminus \prod_{j=1}^{d} (-\gamma_j (\xi^{k-l-s-2})_\pi, \gamma_j (\xi^{k-l-s-2})_\pi),
\]
then
\[
\left\| \left( \prod_{j=1}^{d} n_j \right) \widehat{\theta(n_1 t_1, \ldots, n_d t_d)} \right\|_{E_q^v(\mathbb{T}^d)} \leq 
\]
\[
\leq C_q \sum_{k=-\infty}^{0} \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right)^{1-1/q} \left( \prod_{j=1}^{d} \gamma_j(\xi^l) \right)^{1-1/q} \left( \int_{Q_{k,l}} |\widehat{\theta}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} \leq 
\]
\[
\leq C_q \sum_{k=-\infty}^{0} \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right)^{1-1/q} \left( \prod_{j=1}^{d} \gamma_j(\xi^{k-l-s-1}) \right)^{1-1/q} \times 
\]
\[
\times \left( \sum_{i=k+l-s-1}^{k+l+r} \int_{P_i} |\widehat{\theta}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} \leq 
\]
\[
\leq C_q \sum_{k=-\infty}^{0} \sum_{i=k+l-s-1}^{k+l+r} \left( \prod_{j=1}^{d} \gamma_j(\xi^i) \right)^{1-1/q} \left( \int_{P_i} |\widehat{\theta}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} \leq 
\]
\[
\leq C_q \sum_{i=-\infty}^{l+r} \left( \prod_{j=1}^{d} \gamma_j(\xi^i) \right)^{1-1/q} \left( \int_{P_i} |\widehat{\theta}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} \leq 
\]
\[
\leq C_q \left\| \widehat{\theta} \right\|_{E_q^v(\mathbb{R}^d)}. 
\]
Moreover,
\[
\left\| \left( \prod_{j=1}^{d} n_j \right) \sum_{j \in \mathbb{Z}^d, j \neq 0} \widehat{\theta(n_1 (t_1 + 2j_1 \pi), \ldots, n_d (t_d + 2j_d \pi))} \right\|_{E_q^v(\mathbb{T}^d)} = 
\]
\[
= \sum_{k=-\infty}^{0} \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right)^{1-1/q} \left( \prod_{j=1}^{d} n_j \right) \times 
\]
\[
\times \left( \int_{P_k} \left| \sum_{j \in \mathbb{Z}^d, j \neq 0} \widehat{\theta(n_1 (t_1 + 2j_1 \pi), \ldots, n_d (t_d + 2j_d \pi))} \right|^q \, dt \right)^{1/q} \leq 
\]
\[
\leq \sum_{k=-\infty}^{0} \left( \prod_{j=1}^{d} c_j \right)^{k(1-1/q)} \left( \prod_{j=1}^{d} n_j \right) \times 
\]
\begin{align*}
&\times \left( \int_{\mathbb{T}^d} \left| \sum_{j \in \mathbb{Z}^d, j \neq 0} \hat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right|^q dt \right)^{1/q} \\
&\leq C_q \left( \prod_{j=1}^d n_j \left( \int_{\mathbb{T}^d} \left| \sum_{j \in \mathbb{Z}^d, j \neq 0} \hat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right|^q dt \right)^{1/q} \right).
\end{align*}

Let
\[ R_i := \{ j \in \mathbb{Z}^d : j \neq 0, n_1(\mathbb{T} + 2j_1\pi) \times \ldots \times n_d(\mathbb{T} + 2j_d\pi) \cap P_i \neq 0 \}. \]
Since
\[ |n_j(t_j + 2j_\pi)| \geq \frac{1}{\tau_j} \gamma_j(n_1) \pi \geq \frac{1}{\tau_j} \gamma_j(\xi^{l-1}) \pi = \frac{1}{\tau_j} \epsilon_j^{l-1} \pi = \frac{1}{\tau_j} \gamma_j(\xi^{l-s-1}) \geq \gamma_j(\xi^{l-s-1}), \]
we conclude
\begin{align*}
&\left\| \left( \prod_{j=1}^d n_j \right) \sum_{j \in \mathbb{Z}^d, j \neq 0} \hat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right\|_{E_q^q} \leq \\
&\leq C_q \left( \prod_{j=1}^d n_j \left( \int_{\mathbb{T}^d} \left| \sum_{i=(l-s)\cup 0, j \in R_i} \sum_{j \in \mathbb{Z}^d, j \neq 0} \hat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right|^q dt \right)^{1/q} \right) \\
&\leq C_q \sum_{i=(l-s)\cup 0} \left( \prod_{j=1}^d n_j \left( \int_{\mathbb{T}^d} \left| \sum_{j \in \mathbb{Z}^d, j \neq 0} \hat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right|^q dt \right)^{1/q} \right).
\end{align*}
Since $R_i$ has at most $C \prod_{j=1}^d \frac{\gamma_j(\xi^i)}{n_j}$ members, we get that
\begin{align*}
\left( \prod_{j=1}^d n_j \right) \sum_{j \in \mathbb{Z}^d, j \neq 0} \hat{\theta}(n_1(t_1 + 2j_1\pi), \ldots, n_d(t_d + 2j_d\pi)) \right\|_{E_q^q} \leq \\
&\leq C_q \sum_{i=(l-s)\cup 0} \left( \prod_{j=1}^d n_j \left( \sum_{j \in R_i} \left( \prod_{m=1}^d \frac{\gamma_m(\xi^i)}{n_m} \right)^{-q-1} \times \right. \right.
\end{align*}
\[ \times \int_{n_1(T+2i\pi) \times \ldots \times n_d(T+2i\pi)} |\hat{\theta}(t_1, \ldots, t_d)|^q dt \] \[ \leq C_q \sum_{i=(l-s)/\nu_0}^{d} \left( \prod_{j=1}^{d} \gamma_j(\xi_i) \right)^{1-1/q} \left( \int_{P_i} |\hat{\theta}(t_1, \ldots, t_d)|^q dt \right)^{1/q} \leq C_q \|\hat{\theta}\|_{E^\gamma_q(\mathbb{R}^d)}, \]

which proves (10). The theorem follows from Th. 2. ♦

**Theorem 5.** Let \( \theta \in W(C, \ell_1)(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \) and \( 1/p + 1/q = 1 \). If \( \hat{\theta} \in E^\gamma_q(\mathbb{R}^d) \), then

\[ \|\sigma^\theta_n f\|_{P, \infty} \leq C_p \|\hat{\theta}\|_{E^\gamma_q(\mathbb{R}^d)} \|f\|_{P} \quad (f \in L_p(\mathbb{T}^d)). \]

Moreover, for every \( p < r \leq \infty \)

\[ \|\sigma^\theta_n f\|_r \leq C \|\hat{\theta}\|_{E^\gamma_q(\mathbb{R}^d)} \|f\|_r \quad (f \in L_r(\mathbb{T}^d)). \]

**Corollary 3.** Let \( \theta \in W(C, \ell_1)(\mathbb{R}^d) \), \( \theta(0) = 1 \), \( 1 \leq p \leq \infty \) and \( 1/p + 1/q = 1 \). If \( \hat{\theta} \in E^\gamma_q(\mathbb{R}^d) \), then

\[ \lim_{n \to \infty} \sigma^\theta_n f = f \quad \text{a.e.} \]

for all \( f \in L_p(\mathbb{T}^d) \) whenever \( 1 \leq p < \infty \) and for all \( f \in C(\mathbb{T}^d) \) whenever \( p = \infty \).

If \( f \in L_p(\mathbb{T}^d) \) \( (1 \leq p \leq 2) \) implies the a.e. convergence of Cor. 2, then \( \sigma^\theta_n f \) is bounded from \( L_p(\mathbb{T}^d) \) to \( L_p,\infty(\mathbb{T}^d) \), as in Th. 3 (see Stein [10]). The partial converse of Th. 2 is given in the next result. More exactly, if \( \sigma^\theta_n f \) can be estimated pointwise by \( M^\tau,\gamma f \), then (9) holds.

**Theorem 6.** Let \( \theta \) satisfy (2), \( 1 \leq p \leq \infty \) and \( 1/p + 1/q = 1 \). Suppose that

\[ \sigma^\theta_n f(x) \leq C M^\tau,\gamma f(x) \]

for all \( x \in \mathbb{T}^d \) and for all \( f \in L_p(\mathbb{T}^d) \). Then

\[ \sup_{n \in \mathbb{Z}^d} \|K^\theta_n\|_{E^\gamma(\mathbb{T}^d)} \leq C. \]

**Proof.** Let us define the space \( D^\gamma_q(\mathbb{T}^d) \) \( (1 \leq p \leq \infty) \) by the norm

\[ \|f\|_{D^\gamma_q(\mathbb{T}^d)} := \sup_{0 < r \leq 1} \left( \frac{1}{\prod_{j=1}^{d} \gamma_j(r)} \int_{\prod_{j=1}^{d} (-\gamma_j(r)\pi, \gamma_j(r)\pi)} |f|^p d\lambda \right)^{1/p}. \]
Observe that the norm

\[
\|f\|_* = \sup_{k \leq 0} \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right)^{-1/p} \|f 1_{P_k}\|_p
\]

is an equivalent norm on \( D^\gamma_p(\mathbb{T}^d) \). Indeed, choosing \( r = \xi^k (k \leq 0) \) we conclude \( \|f\|_* \leq C \|f\|_{D^\gamma_p} \). On the other hand, if \( \xi^{n-1} < r \leq \xi^n \) for some \( n \leq 0 \) then

\[
\frac{1}{\prod_{j=1}^{d} \gamma_j(r)} \int_{\prod_{j=1}^{d} (-\gamma_j(r) \pi, \gamma_j(r) \pi)} |f|^p d\lambda \\
\leq \left( \prod_{j=1}^{d} \gamma_j(\xi^{n-1})^{-1} \right) \int_{\prod_{j=1}^{d} (-\gamma_j(\xi^n) \pi, \gamma_j(\xi^n) \pi)} |f|^p d\lambda = \\
= \left( \prod_{j=1}^{d} \gamma_j(\xi^{n-1})^{-1} \right) \sum_{k=-\infty}^{n} \int_{P_k} |f|^p d\lambda \\
\leq \left( \prod_{j=1}^{d} \gamma_j(\xi^{n-1})^{-1} \right) \sum_{k=-\infty}^{n} \left( \prod_{j=1}^{d} \gamma_j(\xi^k) \right) \|f\|^p_*. 
\]

Note that

\[
\gamma_j(\xi^k) \leq \frac{1}{\xi^{j,1}} \gamma_j(\xi^{k+1}) \leq \ldots \leq \frac{1}{\xi^{n-k,1}} \gamma_j(\xi^n) \quad \text{and} \quad \gamma_j(\xi^{n-1}) \geq \frac{1}{\xi^{j,2}} \gamma_j(\xi^n).
\]

Hence

\[
\frac{1}{\prod_{j=1}^{d} \gamma_j(r)} \int_{\prod_{j=1}^{d} (-\gamma_j(r) \pi, \gamma_j(r) \pi)} |f|^p d\lambda \\
\leq \left( \prod_{j=1}^{d} \frac{1}{\xi^{j,2}} \right) \sum_{k=-\infty}^{n} \left( \prod_{j=1}^{d} \frac{1}{\xi^{j,1}} \right)^{n-k-1} \|f\|^p_* \leq \\
\leq C \|f\|^p_*,
\]

or, in other words \( \|f\|_{D^\gamma_p(\mathbb{T}^d)} \leq C \|f\|_* \). Choosing \( r = 1 \) we can see that \( D^\gamma_p(\mathbb{T}^d) \subset L_p(\mathbb{T}^d) \) and \( \|f\|_p \leq C \|f\|_{D^\gamma_p(\mathbb{T}^d)} \). Taking the supremums in (14) and (15) for all \( 0 < r < \infty \) and \( k \in \mathbb{Z} \) then we obtain the space \( D^\gamma_p(\mathbb{R}^d) \).

It is easy to see by (15) that

\[
\sup_{\|f\|_{D^\gamma_p(\mathbb{T}^d)} \leq 1} \left| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t) \, dt \right| = \|K_n^\theta\|_{E^\gamma_n(\mathbb{T}^d)}. 
\]

There exists a function \( f \in D^\gamma_p(\mathbb{T}^d) \) with \( \|f\|_{D^\gamma_p} \leq 1 \) such that
\[
\frac{\|K_n^\theta\|_{L_p^\gamma}(\mathbb{T}^d)}{2} \leq \left| \int_{\mathbb{T}^d} f(-t)K_n^\theta(t) \, dt \right|.
\]

Since \( f \in L_p(\mathbb{R}^d) \), by (13),
\[
|\sigma_n^\theta f(0)| = \left| \int_{\mathbb{T}^d} f(-t)K_n^\theta(t) \, dt \right| \leq CM^\gamma f(0) \quad (n \in \mathbb{R}^d),
\]
which implies
\[
\|K_n^\theta\|_{L_p^\gamma}(\mathbb{T}^d) \leq CM^\gamma f(0) \leq CM^\gamma f(0) \leq C\|f\|_{D_p^\gamma} \leq C.
\]
This proves the result. ♦

Note that the norm of \( D_p^\gamma(\mathbb{T}^d) \) in (14) is equivalent to
\[
\|f\| = \sup_{r \in (0, 1)^d, r \in \mathbb{R}^d} \left( \frac{1}{\prod_{j=1}^d r_j} \prod_{j=1}^d \left( \frac{M_{\gamma_j}}{r_j} \right)^p \right)^{1/p}.
\]

We will characterize the points of convergence. To this end we generalize the concept of Lebesgue points. By Cor. 1,
\[
\lim_{\tau \to 0} \left( \frac{1}{\prod_{j=1}^d r_j} \prod_{j=1}^d \left( \int_{I_j} f(x + u) - f(x) \, du \right)^p \right)^{1/p} = 0 \quad (1 \leq p < \infty)
\]
resp.
\[
\lim_{\tau \to 0} \sup_{u \in I} \left| f(x + u) - f(x) \right| = 0 \quad (p = \infty).
\]

One can see that this definition is equivalent to
\[
\lim_{r \to 0} \left( \frac{1}{\prod_{j=1}^d \gamma_j(r)} \prod_{j=1}^d \left( \int_{I_j} \left| f(x + u) - f(x) \right|^p \, du \right)^{1/p} \right) = 0 \quad (1 \leq p < \infty)
\]
resp. to
\[
\lim_{r \to 0} \sup_{u \in \prod_{j=1}^d (-\gamma_j(r)\pi, \gamma_j(r)\pi)} \left| f(x + u) - f(x) \right| = 0 \quad (p = \infty).
\]

Usually the 1-Lebesgue points are considered in the case if each \( \gamma_j \) is the identity function (cf. Stein and Weiss [11] or Butzer and Nessel [3]). One can show in the usual way that almost every point \( x \in \mathbb{T}^d \) is a \( p \)-Lebesgue point of \( f \in L_p(\mathbb{T}^d) \) if \( 1 \leq p < \infty \). \( x \in \mathbb{T}^d \) is an \( \infty \)-Lebesgue point of
\( f \in L^{\infty}_{\mathrm{loc}}(\mathbb{T}^d) \) if and only if \( f \) is continuous at \( x \). Moreover, all \( r \)-Lebesgue points are \( p \)-Lebesgue points, whenever \( p < r \).

The next theorem generalizes Lebesgue’s theorem.

**Theorem 7.** Let \( \theta \) satisfy (2), \( 1 \leq p \leq \infty \), \( 1/p + 1/q = 1 \) and
\[
\sup_{n \in \mathbb{R}^d} \| K_n^\theta \|_{L^q(\mathbb{T}^d)} \leq C.
\]
If for all \( \delta > 0 \)
\[
\lim_{n \to \infty, n \in \mathbb{R}_{\tau, \gamma}} \| K_n^\theta \|_{L^q(\mathbb{T}^d \setminus (-\delta, \delta)^d)} = 0,
\]
then
\[
\lim_{n \to \infty, n \in \mathbb{R}_{\tau, \gamma}} \sigma_n^\theta f(x) = f(x)
\]
for all \( p \)-Lebesgue points of \( f \in L_p(\mathbb{T}^d) \).

**Proof.** Now denote by
\[
G(u) := \left( \int_{|t_j| < u, j = 1, \ldots, d} |f(x - t) - f(x)|^p \, dt \right)^{1/p} \quad (u \in \mathbb{R}_+).
\]
Since \( x \) is a \( p \)-Lebesgue point of \( f \), for all \( \epsilon > 0 \) there exists \( m \in \mathbb{Z} \), \( m \leq 0 \) such that
\[
\frac{G(\gamma_1(r)\pi, \ldots, \gamma_d(r)\pi)}{\left( \prod_{j=1}^d \gamma_j(r) \right)^{1/p}} \leq \epsilon \quad \text{if} \quad 0 < r \leq \xi^m.
\]
Note that
\[
\sigma_n^\theta f(x) - f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (f(x - t) - f(x)) K_n^\theta(t) \, dt.
\]
Thus
\[
|\sigma_n^\theta f(x) - f(x)| \leq C \int_{\mathbb{T}^d} |f(x - t) - f(x)||K_n^\theta(t)| \, dt =
\]
\[
= C \int_{\Pi_{j=1}^d (-\gamma_j(\xi^m)\pi, \gamma_j(\xi^m)\pi)} |f(x - t) - f(x)||K_n^\theta(t)| \, dt +
\]
\[
+ C \int_{\mathbb{T}^d \setminus \Pi_{j=1}^d (-\gamma_j(\xi^m)\pi, \gamma_j(\xi^m)\pi)} |f(x - t) - f(x)||K_n^\theta(t)| \, dt =:
\]
\[
= A_0(x) + A_1(x).
\]
We estimate $A_0(x)$ by

$$A_0(x) = C \sum_{k=-\infty}^{m} \int_{P_k} |f(x-t) - f(x)| |K_n^\theta(t)| \, dt \leq$$

$$\leq C \sum_{k=-\infty}^{m} \left( \int_{P_k} |K_n^\theta(t)|^q \, dt \right)^{1/q} \left( \int_{P_k} |f(x-t) - f(x)|^p \, dt \right)^{1/p} \leq$$

$$\leq C \sum_{k=-\infty}^{m} \left( \int_{P_k} |K_n^\theta(t)|^q \, dt \right)^{1/q} G(\gamma_1(\xi_k), \ldots, \gamma_d(\xi_k)).$$

Then, by (18),

$$A_0(x) \leq C q^\epsilon \sum_{k=-\infty}^{m} \left( \prod_{j=1}^{d} \gamma_j(\xi_k) \right)^{1/p} \left( \int_{P_k} |K_n^\theta(t)|^q \, dt \right)^{1/q} \leq C q^\epsilon \|K_n^\theta\|_{E_q^\gamma(T^d)}.$$

There exists $\delta > 0$ such that $(-\delta, \delta)^d \subset \prod_{j=1}^{d} (-\gamma_j(\xi_m) \pi, \gamma_j(\xi_m) \pi)$. Then

$$A_1(x) \leq C \int_{T^d \setminus (-\delta, \delta)^d} |f(x-t) - f(x)| |K_n^\theta(t)| \, dt \leq$$

$$\leq C \left( \int_{T^d \setminus (-\delta, \delta)^d} |K_n^\theta(t)|^q \, dt \right)^{1/q} (\|f\|_p + |f(x)|),$$

which tends to 0 as $n \to \infty$, $n \in \mathbb{R}^d_{\gamma, \gamma}$. This completes the proof of the theorem. ♦

Observe that (8) and $(-\delta', \delta')^d \subset \prod_{j=1}^{d} (-\gamma_j(\xi_k) \pi, \gamma_j(\xi_k) \pi) \subset (-\delta, \delta)^d$ imply

$$\|K_n^\theta\|_{E_q^\gamma(T^d \setminus (-\delta', \delta')^d)} \leq \|K_n^\theta\|_{L_q(T^d \setminus \prod_{j=1}^{d} (-\gamma_j(\xi_k) \pi, \gamma_j(\xi_k) \pi))} \leq$$

$$\leq \left( \sum_{l=k+1}^{0} \int_{P_l} |K_n^\theta(t)|^q \, dt \right)^{1/q} \leq$$

$$\leq C_\delta \left( \prod_{j=1}^{d} \gamma_j(\xi_k) \right)^{-1/q} \left( \int_{P_l} |K_n^\theta(t)|^q \, dt \right)^{1/q} \leq$$

$$\leq C_\delta \|K_n^\theta\|_{E_q^\gamma(T^d \setminus \prod_{j=1}^{d} (-\gamma_j(\xi_k) \pi, \gamma_j(\xi_k) \pi))} \leq$$

$$\leq C_\delta \|K_n^\theta\|_{E_q^\gamma(T^d \setminus (-\delta', \delta')^d)}.$$
Condition (17) is trivially equivalent to
\[ \lim_{n \to \infty, n \in \mathbb{R}^d_t} \| K_n^\theta \|_{L_q(\mathbb{T}^d \setminus \prod_{j=1}^d (-\gamma_j(\xi^k)\pi, \gamma_j(\xi^k)\pi))} = 0 \]
and hence to
\[ \lim_{n \to \infty, n \in \mathbb{R}^d_t} \| K_n^\theta \|_{E_q(\mathbb{T}^d \setminus (-\delta, \delta)^d)} = 0. \]

In case \( \hat{\theta} \in E_q^\gamma(\mathbb{R}^d) \) we can formulate a little bit simpler version of the preceding theorem.

**Theorem 8.** Suppose that \( c_j = c_{j,1} = c_{j,2} \) for all \( j = 1, \ldots, d \). Let \( \theta \in W(C, \ell_1)(\mathbb{R}^d), \theta(0) = 1, \hat{\theta} \in E_q^\gamma(\mathbb{R}^d), 1 \leq p \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Then
\[ \lim_{n \to \infty, n \in \mathbb{R}^d_{t, \gamma}} \sigma_n^\theta f(x) = f(x) \]
for all \( p \)-Lebesgue points of \( f \in L_p(\mathbb{T}^d) \).

**Proof.** By (10) the first condition of Th. 7 is satisfied. On the other hand, let
\[ \prod_{j=1}^d (-\gamma_j(\xi_{k_0})\pi, \gamma_j(\xi_{k_0})\pi) \subset (-\delta, \delta)^d, \quad \tau_j/c_j^\gamma \leq 1, \quad c_j^s/\tau_j \geq 1 \]
and \( \ell^{l-1} \leq n_j < \xi^l \) as in the proof of Th. 4. Obviously, if \( n \to \infty, n \in \mathbb{R}^d_{t, \gamma} \) then \( l \to \infty \). We get similarly to (11) and (12) that
\[ \| K_n^\theta \|_{E_q^\gamma(\mathbb{T}^d \setminus (-\delta, \delta)^d)} \leq \]
\[ \leq C_q \sum_{i=k_0+l-1}^{\infty} \left( \prod_{j=1}^d \gamma_j(\xi_{i_j}) \right)^{1-1/q} \left( \int_{P_i} |\hat{\theta}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q} + \]
\[ + C_q \sum_{i=(l-s) \vee 0}^{\infty} \left( \prod_{j=1}^d \gamma_j(\xi_{i_j}) \right)^{1-1/q} \left( \int_{P_i} |\hat{\theta}(t_1, \ldots, t_d)|^q \, dt \right)^{1/q}, \]
which tends to 0 as \( n \to \infty, n \in \mathbb{R}^d_{t, \gamma} \), since \( \hat{\theta} \in E_q^\gamma(\mathbb{R}^d) \). Then (17) follows by (19), which finishes the proof of our theorem. \( \diamond \)

Since each point of continuity is a \( p \)-Lebesgue point, we have

**Corollary 4.** If the conditions of Th. 7 or Th. 8 are satisfied and if \( f \in L_1(\mathbb{T}^d) \) is continuous at a point \( x \), then
\[ \lim_{n \to \infty, n \in \mathbb{R}^d_{t, \gamma}} \sigma_n^\theta f(x) = f(x). \]

The converse of Th. 7 holds also.
Theorem 9. Suppose that $1 \leq p \leq \infty$, $1/p + 1/q = 1$ and (2) and (17) hold. If
\[ \lim_{n \to \infty, n \in \mathbb{R}^d_{\tau, \gamma}} \sigma_n^\theta f(x) = f(x) \]
for all $p$-Lebesgue points of $f \in L_p(\mathbb{T}^d)$, then
\[ \sup_{n \in \mathbb{R}^d_{\tau, \gamma}} \| K_n^\theta \|_{E_q(\mathbb{T}^d)} \leq C. \]

Proof. The space $D_p^\gamma(\mathbb{T}^d)$ consists of all functions $f \in D_p^\gamma(\mathbb{T}^d)$ for which $f(0) = 0$ and 0 is a $p$-Lebesgue point of $f$, in other words
\[ \lim_{r \to 0} \left( \frac{1}{\prod_{j=1}^d \gamma_j(r)} \int_{\prod_{j=1}^d (-\gamma_j(r) \pi, \gamma_j(r) \pi)} |f(u)|^p du \right)^{1/p} = 0 \]
with the usual modification for $p = \infty$. We can easily show that $D_p^\gamma(\mathbb{T}^d)$ is a Banach space. We get from the conditions of the theorem that
\[ \lim_{n \to \infty, n \in \mathbb{R}^d_{\tau, \gamma}} \sigma_n^\theta f(0) = 0 \quad \text{for all} \quad f \in D_p^\gamma(\mathbb{T}^d). \]
Thus the operators
\[ U_n : D_p^\gamma(\mathbb{T}^d) \to \mathbb{R}, \quad U_n f := \sigma_n^\theta f(0) \quad (n \in \mathbb{R}^d_{\tau, \gamma}) \]
are uniformly bounded by the Banach–Steinhaus theorem. Observe that in (16) we may suppose that $f$ is 0 in a neighborhood of 0. Then
\[ C \geq \| U_n \| = \]
\[ \sup_{\| f \|_{D_p^\gamma(\mathbb{T}^d)} \leq 1} \left| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t) dt \right| = \]
\[ \sup_{\| f \|_{D_p^\gamma(\mathbb{T}^d)} \leq 1} \left| \int_{\mathbb{T}^d} f(-t) K_n^\theta(t) dt \right| = \]
\[ \| K_n^\theta \|_{E_q(\mathbb{T}^d)} \]
for all $n \in \mathbb{R}^d_{\tau, \gamma}$.

Corollary 5. Suppose that $1 \leq p \leq \infty$, $1/p + 1/q = 1$ and (2) and (17) holds. Then
\[ \lim_{n \to \infty, n \in \mathbb{R}^d_{\tau, \gamma}} \sigma_n^\theta f(x) = f(x) \]
for all $p$-Lebesgue points of $f \in L_p(\mathbb{T}^d)$ if and only if
\[ \sup_{n \in \mathbb{R}^d_{\tau, \gamma}} \| K_n^\theta \|_{E_q(\mathbb{T}^d)} \leq C. \]

A one-dimensional version of this theorem can be found in the book of Alexits [1].
7. Some summability methods

In this section we consider some summability methods as special cases of the $\theta$-summation. The details can be found in [15]. Note that $q = \infty$ is the most important case in the results of Sec. 6. Let $\gamma_j(x) = \xi \omega_j \gamma_j(x)$ ($x > 0$) and $\omega_1 = 1$.

Example 1 (\textit{C}, $\alpha$ or Cesàro summation). Let $d = 1$ and

$$\theta(k, n) = \begin{cases} \frac{A^\alpha_{n-1-|k|}}{A^\alpha_{n-1}} & \text{if } |k| \leq n - 1, \\ 0 & \text{if } |k| \geq n \end{cases}$$

for some $0 < \alpha < \infty$, where

$$A^\alpha_k := \binom{k + \alpha}{k} = \frac{(\alpha + 1)(\alpha + 2) \ldots (\alpha + k)}{k!} = O(k^\alpha) \quad (k \in \mathbb{N}).$$

The Cesàro means are given by

$$\sigma^\theta_n f(x) := \frac{1}{A^\alpha_{n-1}} \sum_{k=-n+1}^{n-1} A^\alpha_{n-1-|k|} \hat{f}(k) e^{ikx}.$$

In case $\alpha = 1$ we get the Fejér means, i.e.

$$\sigma^1_n f(x) = \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) \hat{f}(k) e^{ikx} = \frac{1}{n} \sum_{k=0}^{n-1} s_k f(x).$$

It is known that the kernel functions satisfy

$$|K^\theta_n(u)| \leq C \min(n, n^{-\alpha} u^{-\alpha-1}) \quad (n \in \mathbb{N}, u \neq 0)$$

(see Zygmund [17]). It is easy to see that (9) and (17) holds as well as all theorems of this paper.

Example 2 (Riesz summation). Let

$$\theta(x) := \begin{cases} (1 - |x|^2)^\alpha & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1 \end{cases} \quad (x \in \mathbb{R}^d).$$

Then

$$|\hat{\theta}(x)| \leq C |x|^{-d/2-\alpha-1/2} \quad (x \neq 0).$$

If

$$\sum_{j=1}^{d} \frac{\omega_j}{\omega_i} - \frac{d}{2} - \frac{1}{2} < \alpha < \infty \quad \text{for all } i = 1, \ldots, d,$$

then $\hat{\theta} \in E^\gamma_\infty(\mathbb{R}^d)$. Here $|\cdot|$ denotes the Euclidean norm.
Note that for cones, i.e. $\omega_j = 1, j = 1, \ldots, d$, we get the well known parameter $(d-1)/2$ on the left hand side of (20). In case $d = 2$ we obtain the condition $\frac{1}{\omega_2 - 1/2} \lor (\omega_2 - 1/2) < \alpha < \infty$.

**Example 3 (Weierstrass summation).** If $\theta(x) = e^{-2\pi|x|^2}$ ($x \in \mathbb{R}^d$), then $\hat{\theta}(x) = e^{-2\pi|x|^2}$ and $\hat{\theta} \in E_\gamma^\alpha(\mathbb{R}^d)$.

**Example 4.** If $\theta(x) = e^{-2\pi|x|}$ ($x \in \mathbb{R}^d$) then $\hat{\theta}(x) = c_d/(1 + |x|^2)^{(d+1)/2}$. Suppose that $\omega_d \leq \omega_j$ for all $j = 2, \ldots, d$. If $\omega_d \leq 1$ and $\sum_{j=1}^{d-1} \omega_j < d\omega_d$ or if $\omega_d > 1$ and $\sum_{j=2}^{d} \omega_j < d$ then $\hat{\theta} \in E_\gamma^\alpha(\mathbb{R}^d)$. If $d = 2$ then we obtain $1/2 < \omega_2 < 2$.

**Example 5 (Picard and Bessel summation).** In case $\theta(x) = 1/(1 + |x|^2)^{(d+1)/2}$ ($x \in \mathbb{R}^d$) we have $\hat{\theta}(x) = c_de^{-2\pi|x|}$ and $\hat{\theta} \in E_\gamma^\alpha(\mathbb{R}^d)$.

**References**


