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PROOFS OF ELEMENTARY THEOREMS USING MODELS OF HYPERBOLIC GEOMETRY

J. **Horváth**

University of Pécs, Faculty of Sciences, Institute of Mathematics and Informatics, Pécs, Ifjúság útja 6, H–7624 Hungary

Á. H. **Temesvári**

University of Pécs, Faculty of Sciences, Institute of Mathematics and Informatics, Pécs, Ifjúság útja 6, H–7624 Hungary

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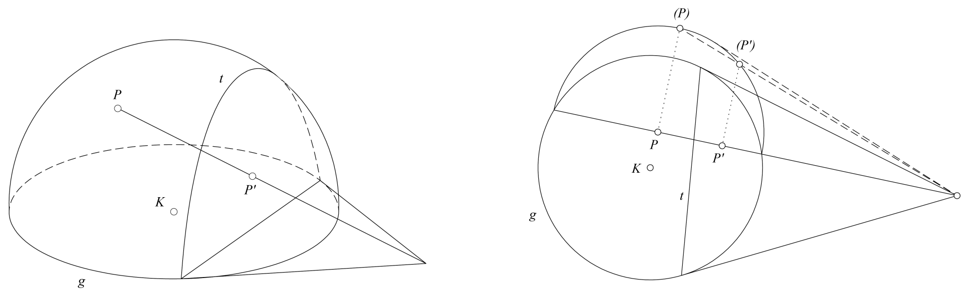
Abstract: The aim of our consideration is to give elementary and simple proofs for several theorems of the hyperbolic triangle geometry using models of hyperbolic geometry. We transform a point of a figure into a special point of the model.

1. Introduction

The hyperbolic plane or Bolyai–Lobachevsky plane (B-L plane) has different models. We use the well-known Poincaré hemisphere model (PH model) and the Cayley–Klein disc model (CK model). Models are used for the illustration of the hyperbolic geometry, for the proof of relative consistency of the axioms and for proofs of theorems. The proofs are complicated in general. They leave out of consideration that the above

E-mail address: temesvari@gamma.ttk.pte.hu



(a) (b)

Figure 1

models are in the euclidean plane or space and have special hyperbolic points (h-points) and hyperbolic lines (h-lines). For example the center of the disc in case of the CK model or the pole of the hemisphere of the PH model. We denote by K these special points and by g the base circles of the models.

The models of Poincaré can be considered as elementary. The orthogonal projection of the Poincaré's hemisphere model in the plane of the base circle g is the Cayley–Klein disc model. In this sense is this model also elementary.

The subject of our consideration is to give elementary and simple proofs for several theorems of the hyperbolic triangle geometry.

We extend the points of the hyperbolic plane with the ideal and ultra-ideal points. We use the Cayley–Klein disc model, the ideal point are the point of g and the ultra-ideal points lie outside the disc. In case of the real points (interior of the disc) we take in consideration that the Cayley–Klein disc model is the orthogonal projection of the Poincaré's hemisphere model on the plane of g. The constructions show this.

The reflections of the CK disc model are central axial collineations whose axes are real. The base circle g of the CK model is a fix circle of the collineation (reflection) (Fig. 1). Using reflections we can transform a point of a figure into a special point of the model.

2. Midpoints of line segments

The real (ordinary) points A and B determine two segments (Fig. 2). One of them is the real segment AB.

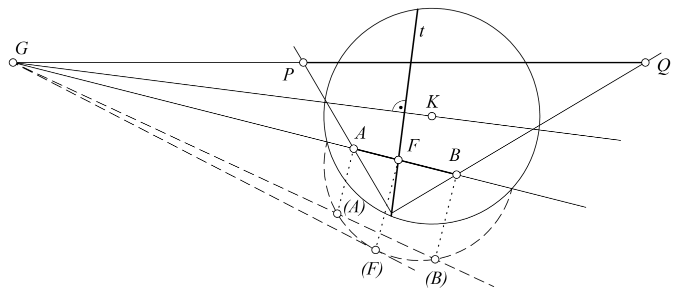


Figure 2

The segment AB has two midpoints.

We consider the perpendicular bisector t of the real segment AB. The line t is the symmetry axis of AB. The reflection in t is a central axial collineation in the model.

We extend this central axial collineation with axis t and center G to the whole plane. This is a reflection in line in the plane extended with ideal and ultra-ideal elements. Two figures **A** and **B** are congruent if there are a finite number of reflections which carry **A** into **B**. The image of the ultra-ideal point P is the ultra-ideal point Q. The point $F = t \cap AB$ is one of the midpoint (interior midpoint). We consider the reflection in t and the central axial collineation corresponding to this reflection. The other midpoint (exterior) of AB is the centre G (ultra-ideal point) of this collineation. The image of the line segment AG is the line segment BG. We say that $AG = BG$. Fig. 2 shows the construction of the midpoints of AB. (We use that the orthogonal projection of the Poincaré's hemisphere model to the plane of g is the Cayley–Klein disc model.)

Let A and B be ultra-ideal points but AB a real line that is the line AB intersects the base circle of the CK model.

There are two midpoints also in this case, a real and an ultra-ideal midpoint.

Let a and b be the polars of A and B. We denote by t the real symmetry axis of the lines a and b. The point $F = t \cap AB$ is one of the midpoints. We consider the reflection in t and the central axial collineation corresponding to this reflection. The other midpoint of AB is the centre G (ultra-ideal point) of this collineation. Fig. 3 shows the construction of the midpoints of AB in this case.

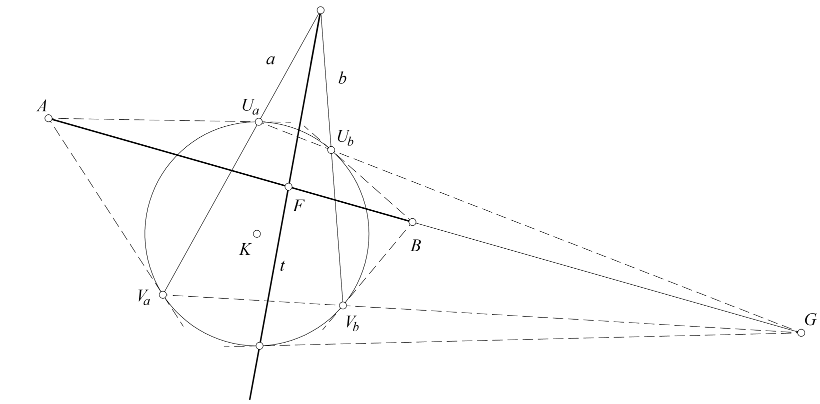


Figure 3

Let the line AB be ultra-ideal (Fig. 4). Let a and b be the polars of the points A and B. The intersection point of a and b is real. The lines a and b have two symmetry axis (angle bisectors). The reflections in these axes carry the point A into B and reversed.

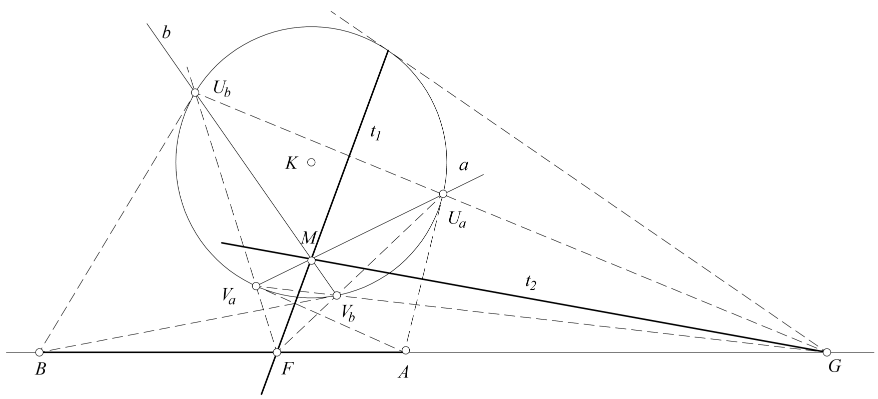


Figure 4

The line segment AB has two midpoints in this case, too.

The construction of the symmetry axis (angle bisectors) t_1 and t_2 is very simple: $F = U_aV_b \cap V_aU_b$, $G = U_aU_b \cap V_aV_b$, $M = a \cap b$, $t_1 = FM$, $t_2 = GM$. The midpoints of AB are F and G.

3. Angle bisectors

Let the triangle ABC be real. Using reflections we transform the center of the incircle of the triangle ABC into the center of the CK model

(Fig. 5). In this case the exterior and interior angle bisectors of ABC are angle bisectors also in euclidean sense.

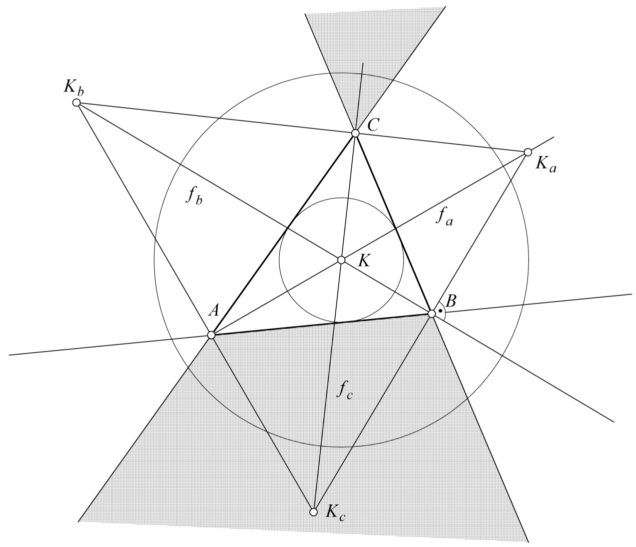


Figure 5

The interior angle bisectors have a common (real) point. Two exterior and the third interior angle bisectors are concurrent in the euclidean geometry.

Therefore two exterior and the third interior angle bisectors form a pencil in the hyperbolic geometry.

The vertex of the pencil can be real, ideal or ultra-ideal.

4. Perpendicular bisectors

Let the line segment AB be real. We consider the perpendicular bisectors of AB (Fig. 6). Let F be the interior and G the exterior midpoint of AB. Let C^* denote the pole of the line AB. The line FC^* is the interior perpendicular bisector of AB and GC^* is the exterior perpendicular bisector of AB. (It holds $\alpha = \beta$ because of the reflection.) It is clear that FC^* is the polar of the point G and GC^* is the polar of F.

We assume that the triangle ABC is real. Let F_c, F_a, F_b respectively G_c, G_a, G_b denote the interior respectively the exterior midpoints of the sides AB, BC and CA.

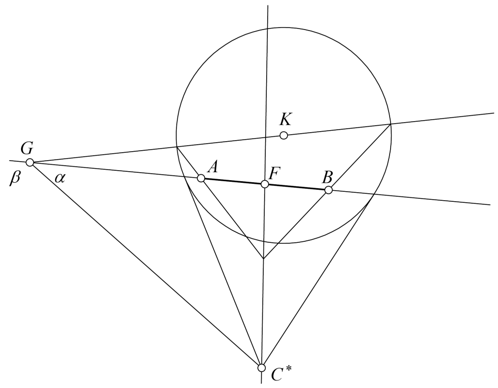


Figure 6

We prove that the interior perpendicular bisectors of the triangle ABC have a common point. Furthermore we show that two exterior perpendicular bisectors and the interior perpendicular bisector belonging to the third side of ABC are concurrent.

Using reflections we transform the midpoint F_c of the side AB into the center K of the CK model (Fig. 7). We proved in [1] that $F_aF_b \parallel AB$

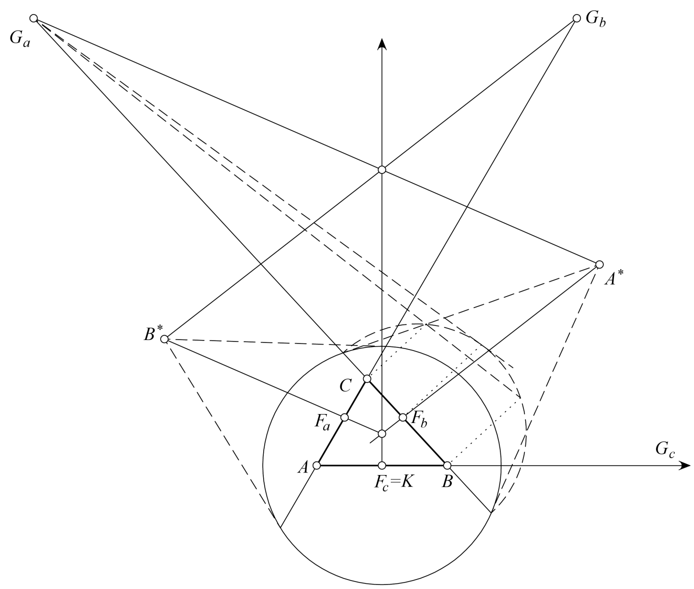


Figure 7

in euclidean sense.

It holds $(BCF_aG_a) = (ACF_bG_b) = -1$ that means $G_aG_b \parallel AB$ in euclidean sense.

It is clear that G_c is a point at infinity in euclidean sense and the points G_a, G_b, G_c lie on the same line. The polars of the exterior midpoints are the interior perpendicular bisectors. Therefore the interior perpendicular bisectors form a pencil. The vertex of this pencil can be real, ideal or ultra-ideal.

The points F_a, F_b and G_c are collinear, too. The polars of these points are the exterior perpendicular bisectors belonging to G_a and G_b further the interior perpendicular bisector through F_c. Hence these three perpendicular bisectors form a pencil.

By similar arguments we can prove the theorem in the remainder cases.

5. Medians

We use the notation of Sec. 4 for the real triangle ABC (Fig. 8).

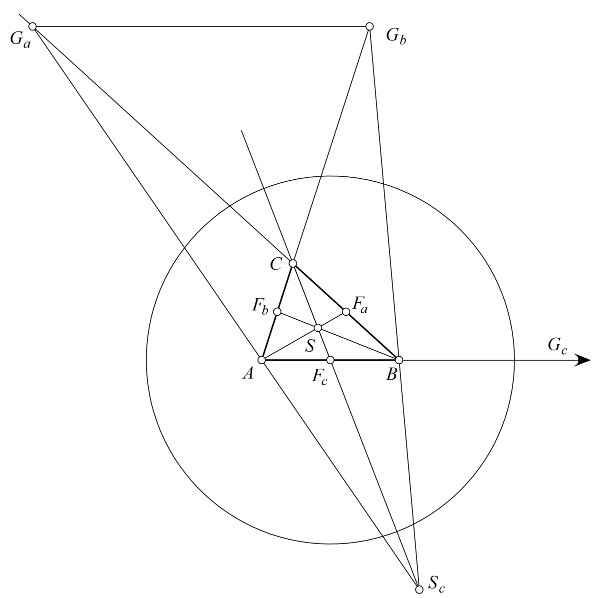


Figure 8

The interior medians AF_a, BF_b, CF_c of the real triangle ABC have a common point.

In order to prove this we transformed the point F_c in the center of the base circle using a reflection as in Sec. 4. We proved that $F_bF_a \parallel AB$

in euclidean sense. Using an elementary theorem for trapeziums it is easy to see that the interior medians are concurrent. (The straight line joining the point of intersection of the diagonals and the common point of the two non-parallel sides bisect the bases of the trapezium.)

In Sec. 4 we proved that the lines G_bG_a and AB are parallel to each other in euclidean sense. Therefore using the above theorem for trapezium the exterior medians BG_b, AG_a and the interior median CF_c have a common point. Hence *two exterior medians and the interior median belonging to the third vertex are concurrent.*

6. Medians, perpendicular bisectors

Let the triangle ABC be ultra-ideal. (The lines of the sides are ultra-ideal, too.) Analogous statements are true as in Sec. 4 and Sec. 5.

We consider the polar triangle of ABC and transform the center of the incircle of the polar triangle into the center K of the CK model using a reflection in line. Let A^*, B^*, C^* be the vertices of the polar triangle after the above reflections (Fig. 9). The angle bisectors of the polar triangle

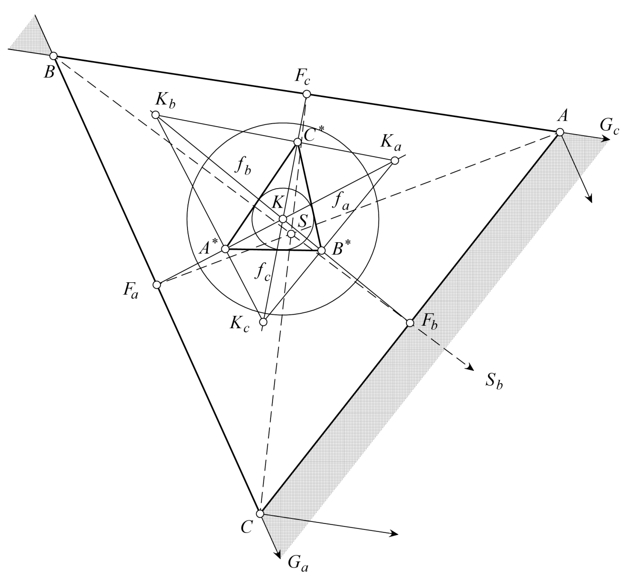


Figure 9

$A^*B^*C^*$ are bisectors in euclidean sense. The side of the triangle ABC is perpendicular to the corresponding interior angle bisector. Consequently the exterior angle bisectors are parallel to the corresponding sides of the triangle ABC that is one of the two midpoints is point at infinity in euclidean sense. It follows that the interior midpoint is midpoint in euclidean sense. The interior medians of the triangle ABC are medians in euclidean sense.

Then *the interior medians of the triangle ABC are concurrent.*

It holds that $AG_a \parallel BC$, $CG_c \parallel AB$ in euclidean sense. We get a parallelogram where BF_b is the line of the diagonal.

Therefore *the two exterior medians CG_c and AG_a furthermore the interior median BF_b are concurrent.*

We remark that *the interior perpendicular bisectors respectively two exterior and the interior perpendicular bisectors of the triangle ABC which belongs to the third side of the triangle form a pencil.* The vertices of these pencils are the centers of the incircle and of the excircles of the triangle $A^*B^*C^*$. (One center is real the other three centers are not by all means real.)

7. Angle bisectors

Let a and b be two ultra-ideal straight lines (Fig. 10).

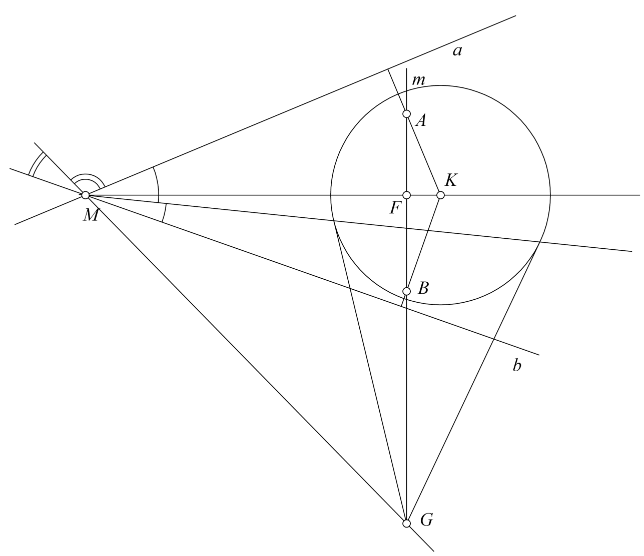


Figure 10

The intersection point of a and b is M.

There are two angle bisectors of a and b.

We consider the poles of a and b. The poles are real points. The polar of M is the line m where $m = AB$. The real line segment AB has two midpoints (F and G). The line FM is the perpendicular bisector of AB. Because of the pole-polar connection the reflection in the line MF carry the line a into b and reversed. The line MF is called one of the angle bisectors of a and b.

Using the above reflection the line GM is a fix line (the fix line of the collineation). Therefore the angle of GM and a is equal to the angle of GM and b. The line GM is the second angle bisector of a and b.

Let the triangle ABC be ultra-ideal. (The lines of the sides are also ultra-ideal.) We consider the angle bisectors. Analogous statements are true as in Sec. 3.

Let A^*, B^*, C^* (Fig. 11) the real vertices of the polar triangle. The

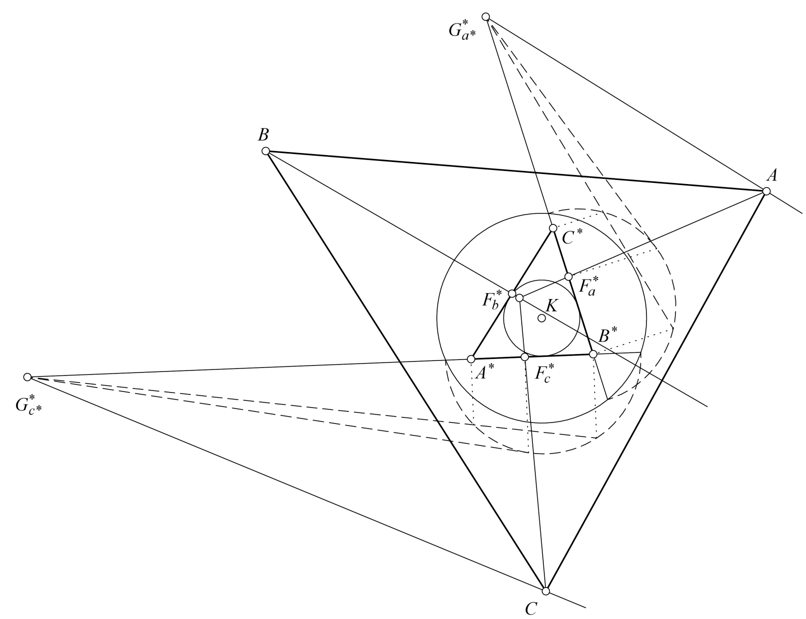


Figure 11

interior perpendicular bisectors of the triangle $A^*B^*C^*$ are the interior angle bisectors of ABC. From Sec. 4 it follows that they have a common point. Similarly the exterior perpendicular bisectors AG_{a^*} and CG_{c^*} are two exterior angle bisectors of ABC. The third interior perpendicular bisector BF_{b^*} is the third interior angle bisector of ABC. Hence the bisectors AG_{a^*}, CG_{c^*} and BF_{b^*} are concurrent.

8. Triangle with ultra-ideal vertices

You can have similar results for triangle with ultra-ideal vertices and real sides (Fig. 12).

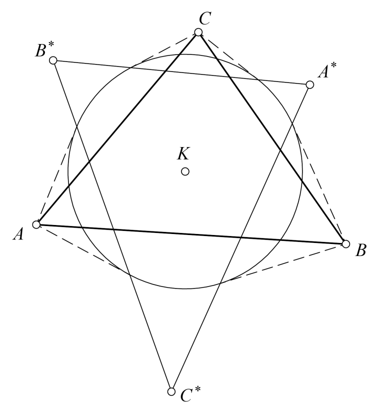


Figure 12

We can define the interior and exterior midpoints (Fig. 2), the interior and exterior perpendicular bisectors of a line segment (Fig. 3) in this case, too. *The interior perpendicular bisectors of a triangle, furthermore two exterior perpendicular bisectors and the interior perpendicular bisector belonging to the third side of the triangle form a pencil.* The proof is carried out analogously to the proof in Sec. 4 (Fig. 7). We transform the interior midpoint F_c of the side AB into the center K of the model (Fig. 13). We proved in [1] and [3] that $F_aF_b \parallel AB$ in euclidean sense. It follows that $G_aG_b \parallel AB$ in euclidean sense. By similar arguments we can continue the proof as in Sec. 4.

You can define the interior and the exterior medians of the triangle ABC. It is true that *the interior medians, and two exterior medians and the interior median belonging to the third vertex form a pencil* (Fig. 13). The proof is the same as in Sec. 5 (Fig. 8).

It can be defined the interior and exterior symmetry axes of the sides (angle bisectors). It holds that *the interior symmetry axes, and two exterior symmetry axes and the interior symmetry axis belonging to the third vertex form a pencil.* We consider namely the common perpendicular lines to the pairs of the sides in ABC. We get the triangle $A^*B^*C^*$ with ultra-ideal vertices and real sides (Fig. 12). The interior and exterior perpendicular bisectors of the triangle $A^*B^*C^*$ are the symmetry axes (angle bisectors) of ABC (Fig. 13).

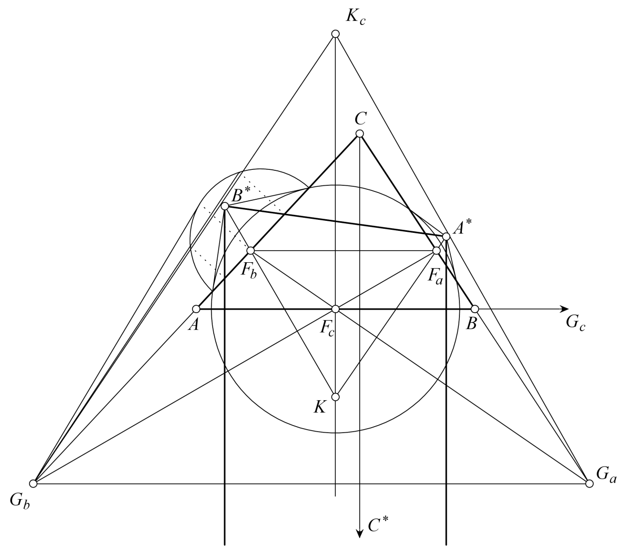


Figure 13

Remark. The referee of our paper pointed out the paper [6]. He noticed that [6] "contains similar results but different proofs". The author of [6] considers only triangles with real vertices. You can find his results for altitudes and midlines in [1], too.

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