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CONTINUOUS ORDER-PRESERVING FUNCTIONS FOR NONTOTAL PREORDERS ON NORMAL SPACES

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Abstract: We discuss the continuous real representability of a not necessarily total preorder on a normal topological space in connection with a suitable continuity assumption, called *C-continuity* in this paper. We show that a topology τ on a set X is normal if and only if the topological preordered space $(X, \precsim, \tau)$ is normally preordered for every C-continuous preorder $\precsim$ on (X, τ). We also prove that a C-continuous preorder $\precsim$ on a normal topological space (X, τ) is representable by means of a continuous order-preserving function u if and only if $\precsim$ verifies a suitable separability condition à la Nachbin.

1. Introduction

There is an extensive literature concerning the continuous representability of nontotal preorders on a topological space. Such an argument is interesting in the applications of mathematics to economics and social sciences since the assumption according to which the individual preferences are expressed by means of total preorders seems not to be realistic.

The first impressive mathematical contribution in this field was presented by Peleg [11]. Many other authors were concerned with the existence of continuous representations of nontotal preorders (see e,g, the book of Bridges and Mehta [3] and the deep results presented by Mehta [9] and Herden [5]).

In this paper we are concerned with the representability of nontotal preorders on normal topological spaces. We improve a result appearing in Bosi and Isler [2] by using the classical *Nachbin's approach* to mathematical utility theory (see Nachbin [10]). We refer to a suitable continuity assumption, called *C-continuity* in this paper, that already appeared in the literature (see McCartan [8] and Künzi [7]). According to such a condition, the smallest decreasing (increasing) superset of any closed set is also closed. We show that a topology τ on a set X is normal if and only if the topological preordered space $(X, \precsim, \tau)$ is normally preordered for every C-continuous preorder $\precsim$ on (X, τ). We also prove that a C-continuous preorder $\precsim$ on a normal topological space (X, τ) is representable by means of a continuous order-preserving function u if and only if $\precsim$ verifies a suitable separability condition à la Nachbin, which is referred to as *weak separability*.

2. Notation and preliminaries

A *preorder* $\precsim$ on a nonempty set X is a *reflexive* and *transitive* binary relation on X. The *asymmetric part* $\prec$ and the *symmetric part* $\sim$ of a preorder $\precsim$ are defined as usual.

A preorder $\precsim$ on a set X is said to be *total* if for any two elements $x, y \in X$ either $x \precsim y$ or $y \precsim x$.

If $(X, \precsim)$ is a preordered set and τ is a topology on X, then the triplet $(X, \precsim, \tau)$ will be referred to as a *topological preordered space*.

A subset A of a set X endowed with a preorder $\precsim$ is said to be *decreasing* (*increasing*) if $x \in A$ and $y \precsim x$ imply $y \in A$ (respectively, $x \in A$ and $x \precsim y$ imply $y \in A$).

If A is any subset of a set X endowed with a preorder $\precsim$, then denote by $d(A)$ ($i(A)$) the intersection of all the decreasing (increasing) subsets of X containing A.

A preorder $\precsim$ on a topological space (X, τ) is said to be *C-continuous* if $d(A)$ and $i(A)$ are closed sets for every closed subset A of X.

The concept of a C-continuous preorder was first introduced by McCartan [8] and then studied by other authors (see e.g. Künzi [7] and Bosi and Isler [2]).

If $(X, \precsim)$ is a preordered set, then a function $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ is said to be

(i) *increasing* if, for every $x, y \in X$, $[x \precsim y \Rightarrow u(x) \leq u(y)]$,

(ii) *order-preserving* if it is increasing and, for every $x, y \in X$, $[x \prec y \Rightarrow u(x) < u(y)]$.

Denote by τ_{nat} the *natural topology* on the real line $\mathbb{R}$. Given a topological preordered space $(X, \precsim, \tau)$, an interesting problem consists in determining a continuous order-preserving function $u : (X, \precsim, \tau) \longrightarrow$ $\longrightarrow (\mathbb{R}, \leq, \tau_{nat})$.

A *preorder* $\precsim$ on a topological space (X, τ) is said to be *weakly continuous* if for every pair $(x, y) \in X$ such that $x \prec y$ there exists a continuous increasing function $u_{xy} : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq, \tau_{nat})$ such that $u_{xy}(x) < u_{xy}(y)$ (see e.g. Bosi [1]).

A topology τ on a set X is said to satisfy the *Weakly Continuous Representation Property* if every weakly continuous preorder $\precsim$ on (X, τ) admits a continuous order-preserving function $u : (X, \precsim, \tau) \longrightarrow (\mathbb{R}, \leq \tau_{nat})$.

We recall that the weakly continuous representation property generalizes the *Continuous representation property*, which requires the continuous representability of all continuous total preorders (see e.g. Campión, Candeal and Induráin [4]).

From Nachbin [10], a topological preordered space $(X, \precsim, \tau)$ is said to be *normally preordered* if, given a closed decreasing set F_0 and a closed increasing set F_1 with $F_0 \cap F_1 = \emptyset$, there exist an open decreasing set A_0 containing F_0, and an open increasing set A_1 containing F_1 such that $A_0 \cap A_1 = \emptyset$.

From Mehta [9, Th. 1], a topological preordered space $(X, \precsim, \tau)$ is *Nachbin separable* if there exists a countable family $\{A_n, B_n\}_{n \in \mathbf{N}}$ of pairs of closed disjoint subsets of X such that A_n is decreasing, B_n is increasing, and $\{(x,y) \in X \times X : x \prec y\} \subset \bigcup_{n \in \mathbf{N}^+} A_n \times B_n$.

We say that a preorder $\precsim$ on a topological space (X, τ) is *weakly Nachbin separable* if there exists a countable family $\{(A_n, B_n)\}_{n \in \mathbf{N}^+}$ of pairs of closed subsets of X satisfying the following conditions:

(i) $d(A_n) \cap i(B_n) = \emptyset$ for all $n \in \mathbf{N}^+$;

(ii) $\{(x,y) \in X \times X : x \prec y\} \subset \bigcup_{n \in \mathbf{N}^+} A_n \times B_n$.

It is immediate to check that a preorder is Nachbin separable whenever it is C-continuous and weakly Nachbin separable.

3. Existence of continuous order-preserving functions

We present a characterization of a normal topological space based on the requirement that every C-continuous preorder induces a normally preordered topological structure.

Theorem 3.1. *Let τ be a topology on a set X. Then the following conditions are equivalent:*

(i) *(X, τ) is a normal topological space;*

(ii) *$(X, \precsim, \tau)$ is a normally preordered topological space for every C-continuous preorder $\precsim$ on (X, τ).*

Proof. (ii) $\Rightarrow$ (i). Since the identity relation "=" on (X, τ) is a C-continuous preorder it follows that $(X, =, \tau) = (X, \tau)$ is a normal topological space.

(i) $\Rightarrow$ (ii). Let $D \subset X$ and $I \subset X$ be arbitrarily chosen disjoint closed and decreasing respectively, increasing subsets of X. Since (X, τ) is a normal space there exists an open subset O of X such that the inclusions $D \subset O \subset \overline{O} \subset X \setminus I$ hold. Now define two subsets U and V of X as follows:

$$V = X \setminus d(\overline{O}), \;\; U = X \setminus i(\overline{V}).$$

We have that $d(\overline{O}) \subset X \setminus I$ or equivalently $I \subset V$ since $X \setminus I$ is a decreasing set as a consequence of the fact that I is increasing. Further, C-continuity of $\precsim$ implies that $d(\overline{O})$ is closed. The definition of V implies with help of the fact that $d(\overline{O})$ is closed and decreasing that V is an open and increasing subset of X. Furthermore, it follows from the inclusions $V = X \setminus d(\overline{O}) \subset X \setminus \overline{O} \subset X \setminus O \subset X \setminus D$ that $\overline{V} \subset X \setminus O \subset X \setminus D$. Since $X \setminus D$ is increasing, we have that $i(\overline{V}) \subset X \setminus D$ or equivalently $D \subset U$. C-continuity of $\precsim$ implies that $i(\overline{V})$ is closed, so that U is an open and decreasing subset of X. The inclusion $V \subset i(\overline{V})$ implies, moreover, that $U = X \setminus i(\overline{V}) \subset X \setminus V$. Therefore the identity $U \cap V = \emptyset$ finishes the proof of the theorem. $\Diamond$

The next corollary, which generalizes Th. 3.1 in Bosi and Isler [2], furnishes a characterization of the existence of a continuous order-preserving function for a C-continuous preorder on a normal topological space.

The immediate proof is omitted since it is an immediate consequence of *Nachbin separation theorem* (see Nachbin [10]), which generalizes to topological preordered spaces the well known *Urysohn Lemma* in normal topological spaces. It is easy to show that if there exists a continuous order-preserving function u on a topological preordered space $(X, \precsim, \tau)$, then $(X, \precsim, \tau)$ is weakly Nachbin separable. On the other hand, Nachbin separation theorem guarantees the existence of an order-preserving function u on a normally preordered topological space $(X, \precsim, \tau)$ provided that $\precsim$ is Nachbin separable.

Corollary 3.2. *Let (X, τ) be a normal topological space and let $\precsim$ be a C-continuous preorder on (X, τ). Then in order that there exists a continuous order-preserving u on $(X, \precsim, \tau)$ it is necessary and sufficient that $\precsim$ is weakly Nachbin separable.*

Finally, we observe that Cor. 3.2 can be used in order to present a sufficient condition for the existence of a continuous order-preserving function for a C-continuous preorder on a completely regular T_1 topological space (X, τ) in case that τ satisfies the weakly continuous representation property (this is the case, for example, when the product topology $\tau \times \tau$ on the Cartesian product $X \times X$ is *hereditarily Lindelöf*). Indeed, in this case we have that (X, τ) is a normal topological space (see Bosi [1, Prop. 3.8]).

Corollary 3.3. *Let (X, τ) be a completely regular T_1 topological space and assume that τ satisfies the weakly continuous representation property. Then every weakly Nachbin separable C-continuous preorder $\precsim$ on (X, τ) is representable by means of a continuous order-preserving function u on $(X, \precsim, \tau)$.*

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