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A PROPERTY OF OPPOSITELY SIMILAR TRIANGLES

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Abstract: We prove the following theorem: if ABC and UVW are two oppositely similar triangles with homologous vertices A and U, B and V, C and W, with no parallel sides and if we denote $AB \cap VW = D$, $AB \cap WU = E$, $BC \cap WU = F$, $BC \cap UV = G$, $CA \cap UV = H$, $CA \cap VW = I$, then the three pairwise radical axis of the circumscribed circles of the inscribed quadrilaterals $AHUE$, $BGVD$, $CWIF$ coincide. Conversely, let us suppose that the circles $(\mathcal{C}_a)$, $(\mathcal{C}_b)$ and $(\mathcal{C}_c)$ have the common radical axis. Let us choose the points $A \in (\mathcal{C}_a)$, $B \in (\mathcal{C}_b)$ and $C \in (\mathcal{C}_c)$ so that the points A, B, C are not collinear and do not coincide with one of the point of intersections of the circles where appropriate. We consider the following intersection points: $AB \cap (\mathcal{C}_b) = D$, $AB \cap (\mathcal{C}_a) = E$, $BC \cap (\mathcal{C}_c) = F$, $BC \cap (\mathcal{C}_b) = G$, $CA \cap (\mathcal{C}_a) = H$, $CA \cap (\mathcal{C}_c) = I$ and $FE \cap GH = U$, $GH \cap DI = V$, $DI \cap FE = W$. It follows that: (a) $U \in (\mathcal{C}_a)$, $V \in (\mathcal{C}_b)$ and $W \in (\mathcal{C}_c)$. (b) The triangles ABC and UVW are oppositely similar, with homologous vertices A and U, B and V, C and W. Some applications are presented.

1. Introduction

Two similar triangles are oppositely similar if, when we read the vertices of the first triangle in the clockwise sense, then we read the homologous vertices of the second triangle in the counter-clockwise sense.

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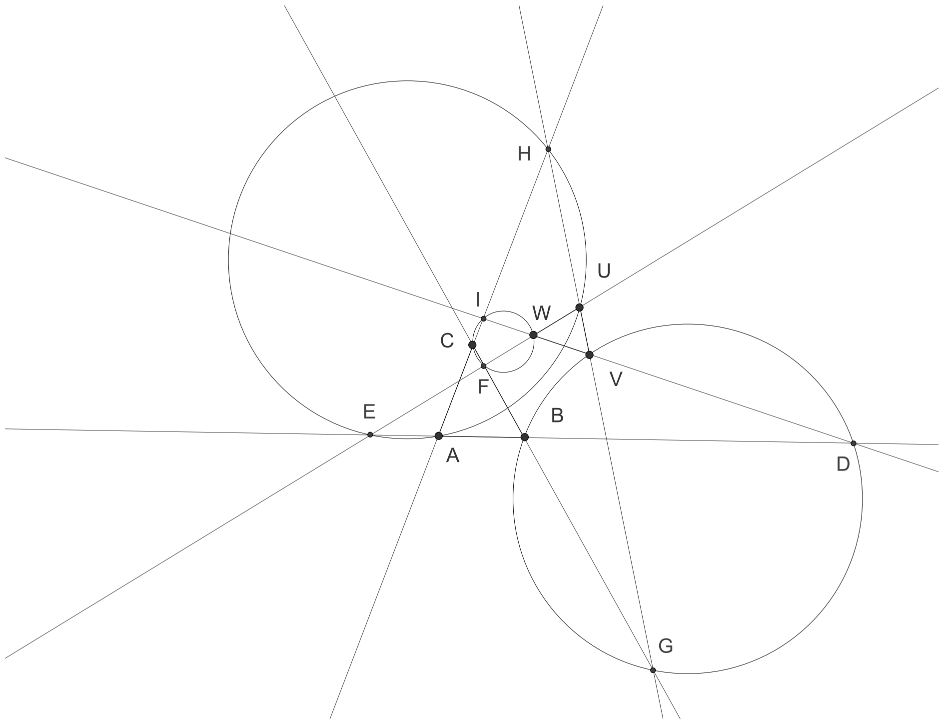

Figure 1

In plane transformation terms, two triangles are oppositely similar if one of them is the image of the other one through the product between a reflection on a straight line (the axis) and a homothety with the center on the straight line. This product is called opposite similarity. When we draw two oppositely similar triangles and consider the intersection points of the lines of the sides which are not homologous, we easily notice the existence of three inscribed quadrilaterals (see Fig. 1). The goal of the paper is to prove the following result concerning these quadrilaterals, which we have not seen stated.

Theorem 1. *We consider the oppositely similar triangles ABC and UVW, with homologous vertices A and U, B and V, C and W and with no parallel sides. Let us denote the following intersection points: $AB \cap VW = D$, $AB \cap WU = E$, $BC \cap WU = F$, $BC \cap UV = G$, $CA \cap UV = H$, $CA \cap WV = I$. It follows that:*

(a) *The quadrilaterals $AHUE$, $BGVD$, and $CWIF$ are inscribed quadrilaterals.*

(b) *The three pairwise radical axis of their circumscribed circles coincide.*

Conversely, let us suppose that the circles $\left(\mathcal{C}_a\right), \left(\mathcal{C}_b\right)$ *and* $\left(\mathcal{C}_c\right)$ *have the property that the three pairwise radical axis coincide. Let us choose the points* $A \in \left(\mathcal{C}_a\right)$, $B \in \left(\mathcal{C}_b\right)$ *and* $C \in \left(\mathcal{C}_c\right)$ *so that the points* A, B, C *are not collinear and do not coincide with one intersection point of the circles in case the circles intersect themselves. We consider the following intersection points:* $AB \cap \left(\mathcal{C}_b\right) = D$, $AB \cap \left(\mathcal{C}_a\right) = E$, $BC \cap \left(\mathcal{C}_c\right) = F$, $BC \cap \left(\mathcal{C}_b\right) = G$, $CA \cap \left(\mathcal{C}_a\right) = H$, $CA \cap \left(\mathcal{C}_c\right) = I$ *and* $FE \cap GH = U$, $GH \cap DI = V$, $DI \cap FE = W$. *It follows that:*

(a) $U \in \left(\mathcal{C}_a\right)$, $V \in \left(\mathcal{C}_b\right)$ *and* $W \in \left(\mathcal{C}_c\right)$.

(b) *The triangles* ABC *and* UVW *are oppositely similar, with homologous vertices* A *and* U, B *and* V, C *and* W.

We shall use in proofs the complex number formalism. Next, we denote with the corresponding small letter the affix of a point which is denoted with a capital letter. Let us remind some basic facts (see [1]).

(1) If N is the intersection point of the lines AB and CD, then its affix is

$$n = \frac{(c\bar{d} - \bar{c}d)(a - b) - (a\bar{b} - \bar{a}b)(c - d)}{(\bar{a} - \bar{b})(c - d) - (a - b)(\bar{c} - \bar{d})}. \tag{1.1}$$

(2) The complex equation of the circumscribed circle of the triangle ABC is

$$z\bar{z} + \beta z + \bar{\beta}\bar{z} + \varepsilon = 0,$$

where

$$\alpha = \frac{a - b}{c - b},\ \beta = \frac{\bar{\alpha}\bar{c} - \alpha\bar{a}}{\alpha - \bar{\alpha}},\ \varepsilon = \frac{\alpha\bar{a}c - \bar{\alpha}a\bar{c}}{\alpha - \bar{\alpha}}. \tag{1.2}$$

(3) The radical axis of the circles $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$, described by the equations $z\bar{z} + \beta_{1,2}z + \bar{\beta}_{1,2}\bar{z} + \varepsilon_{1,2} = 0$ is

$$\left(\beta_1 - \beta_2\right)z + \left(\bar{\beta}_1 - \bar{\beta}_2\right)\bar{z} + \varepsilon_1 - \varepsilon_2 = 0. \tag{1.3}$$

2. Lemmas

In this paragraph we prove some identities between the affixes of the homologous vertices of two oppositely similar triangles which we intensively utilize in the next paragraph, where we prove the main result. Let us consider the oppositely similar triangles ABC and UVW, with homologous vertices A and U, B and V, C and W. The relation between the affixes of the homologous vertices can be stated as:

$$\frac{u-v}{\bar{a}-\bar{b}} = \frac{v-w}{\bar{b}-\bar{c}} = \frac{w-u}{\bar{c}-\bar{a}} = \zeta,$$

therefore

$$u-v=\zeta(\bar{a}-\bar{b});\ v-w=\zeta(\bar{b}-\bar{c});\ w-u=\zeta(\bar{c}-\bar{a}). \tag{2.1}$$

We also denote

$$\eta=(a-b)(b-c)(c-a).$$

Lemma 1. *The following relations are true:*

(2.2)
$$\begin{aligned}(b-c)(\bar{c}-\bar{a})\bar{u}-(\bar{b}-\bar{c})(c-a)\bar{v} &= (a-b)(\bar{b}-\bar{c})\bar{w}-(\bar{a}-\bar{b})(b-c)\bar{u}= \\ &=(c-a)(\bar{a}-\bar{b})\bar{v}-(\bar{c}-\bar{a})(a-b)\bar{w}.\end{aligned}$$

$$\begin{aligned}&(c-a)(\bar{b}-\bar{c})b\bar{v}-(\bar{c}-\bar{a})(b-c)a\bar{u}+ \\ &\quad +(a-b)(\bar{b}-\bar{c})c\bar{w}-(\bar{a}-\bar{b})(b-c)a\bar{u}+\eta(\bar{b}-\bar{c})\bar{\zeta}=0.\end{aligned} \tag{2.3}$$

Proof. As $w=\frac{\bar{c}-\bar{b}}{\bar{a}-\bar{b}}\cdot u-\frac{\bar{c}-\bar{a}}{\bar{a}-\bar{b}}\cdot v$, it follows that:

$$\begin{aligned}&(a-b)(\bar{b}-\bar{c})\bar{w}-(\bar{a}-\bar{b})(b-c)\bar{u}= \\ &=(a-b)(\bar{b}-\bar{c})\left(\frac{c-b}{a-b}\cdot\bar{u}-\frac{c-a}{a-b}\cdot\bar{v}\right)-(\bar{a}-\bar{b})(b-c)\bar{u}= \\ &=(c-b)(\bar{b}-\bar{c}+\bar{a}-\bar{b})\bar{u}-(\bar{b}-\bar{c})(c-a)\bar{v}= \\ &=(b-c)(\bar{c}-\bar{a})\bar{u}-(\bar{b}-\bar{c})(c-a)\bar{v},\end{aligned}$$

therefore (2.2) is true. Next, we get:

$$\begin{aligned}&(c-a)(\bar{b}-\bar{c})b\bar{v}-(\bar{c}-\bar{a})(b-c)a\bar{u}+ \\ &+(a-b)(\bar{b}-\bar{c})c\bar{w}-(\bar{a}-\bar{b})(b-c)a\bar{u}+\eta(\bar{b}-\bar{c})\bar{\zeta}= \\ &=(\bar{b}-\bar{c})\big[(c-a)b\bar{v}+(a-b)c\bar{w}\big]-(b-c)(\bar{c}-\bar{a}+\bar{a}-\bar{b})a\bar{u}+\eta(\bar{b}-\bar{c})\bar{\zeta}= \\ &=(\bar{b}-\bar{c})\big[(c-a)b\bar{v}+(a-b)c\bar{w}+(b-c)a\bar{u}+\eta\bar{\zeta}\big].\end{aligned}$$

As $\eta\bar{\zeta} = (a-b)(c-a)(\bar{v}-\bar{w})$, one obtains

$$\begin{aligned}&(c-a)(\bar{b}-\bar{c})b\bar{v} - (\bar{c}-\bar{a})(b-c)a\bar{u}+\\ &\quad + (a-b)(\bar{b}-\bar{c})a\bar{u} - (\bar{a}-\bar{b})(b-c)a\bar{u} + \eta(\bar{b}-\bar{c})\bar{\zeta} =\\ &= (\bar{b}-\bar{c})\big[(c-a)b\bar{v} + (a-b)c\bar{w}+\\ &\quad + (b-c)a\bar{u} + (a-b)(c-a)\bar{v} - (a-b)(c-a)\bar{w}\big] =\\ &= a(\bar{b}-\bar{c})\big[(b-c)\bar{u} + (c-a)\bar{v} + (a-b)\bar{w}\big].\end{aligned}$$

But

$$\begin{aligned}&(b-c)\bar{u} + (c-a)\bar{v} + (a-b)\bar{w} =\\ &= (b-c)\bar{u} + (c-a)\bar{v} + (a-b)\left(\frac{c-b}{a-b}\cdot\bar{u} - \frac{c-a}{a-b}\cdot\bar{v}\right) = 0,\end{aligned}$$

therefore (2.3) is true. $\Diamond$

Lemma 2. *If*

$$\begin{aligned}\Xi_{ab} = &\big[(\bar{c}-\bar{a})(u-v)\bar{u}-(c-a)(\bar{u}-\bar{v})\bar{a}\big]\big[(a-b)(\bar{v}-\bar{w})-(\bar{a}-\bar{b})(v-w)\big]-\\ &- \big[(\bar{a}-\bar{b})(v-w)\bar{v}-(a-b)(\bar{v}-\bar{w})\bar{b}\big]\big[(c-a)(\bar{u}-\bar{v})-(\bar{c}-\bar{a})(u-v)\big]\end{aligned}$$

and

$$\begin{aligned}\Xi_{ac} = &\big[(\bar{c}-\bar{a})(u-v)\bar{u}-(c-a)(\bar{u}-\bar{v})\bar{a}\big]\big[(b-c)(\bar{w}-\bar{u})-(\bar{b}-\bar{c})(w-u)\big]-\\ &- \big[(\bar{b}-\bar{c})(w-u)\bar{w}-(b-c)(\bar{w}-\bar{u})\bar{c}\big]\big[(c-a)(\bar{u}-\bar{v})-(\bar{c}-\bar{a})(u-v)\big],\end{aligned}$$

then

$$|c-a|^2\Xi_{ab} + |a-b|^2\Xi_{ac} = 0. \tag{2.4}$$

Proof. Replacing in Ξ_{ab} with (2.1), we get

$$\begin{aligned}\Xi_{ab} &= \big[(\bar{c}-\bar{a})(\bar{a}-\bar{b})\zeta\bar{u}-(c-a)(a-b)\bar{\zeta}\bar{a}\big]\big[(a-b)(b-c)\bar{\zeta}-(\bar{a}-\bar{b})(\bar{b}-\bar{c})\zeta\big]-\\ &\quad- \big[(\bar{a}-\bar{b})(\bar{b}-\bar{c})\zeta\bar{v}-(a-b)(b-c)\bar{\zeta}\bar{b}\big]\big[(c-a)(a-b)\bar{\zeta}-(\bar{c}-\bar{a})(\bar{a}-\bar{b})\zeta\big] =\\ &= |a-b|^2|\zeta|^2\big[(\bar{c}-\bar{a})(b-c)\bar{u}-(c-a)(\bar{b}-\bar{c})\bar{v}\big]+\\ &\qquad + |a-b|^2|\zeta|^2\big[(c-a)(\bar{b}-\bar{c})\bar{a}-(\bar{c}-\bar{a})(b-c)\bar{b}\big]+\\ &\qquad + \zeta^2(\bar{a}-\bar{b})(\bar{v}-\bar{u})\bar{\eta}+\bar{\zeta}^2(a-b)(\bar{b}-\bar{a})\eta.\end{aligned}$$

As $\bar{v}-\bar{u} = (b-a)\bar{\zeta}$, it follows that

$$\begin{aligned}\Xi_{ab} = |a-b|^2\Big\{&|\zeta|^2\big[(\bar{c}-\bar{a})(b-c)\bar{u} - (c-a)(\bar{b}-\bar{c})\bar{v}\big]+\\ &+ |\zeta|^2\big[(c-a)(\bar{b}-\bar{c})\bar{a} - (\bar{c}-\bar{a})(b-c)\bar{b}\big] - \zeta^2\bar{\zeta}\bar{\eta} - \bar{\zeta}^2\eta\Big\}.\end{aligned}$$

Analogously, we obtain

$$\Xi_{ac} = |c-a|^2\Big\{|\zeta|^2\big[(\bar a-\bar b)(b-c)\bar u-(a-b)(\bar b-\bar c)\bar w\big]+ \\ +|\zeta|^2\big[(a-b)(\bar b-\bar c)\bar a-(\bar a-\bar b)(b-c)\bar c\big]+\zeta^2\bar\zeta\bar\eta+\bar\zeta^2\eta\Big\}.$$

It is easy to check that, if a, b, c are complex number, then
$\big[(c-a)(\bar b-\bar c)\bar a-(\bar c-\bar a)(b-c)\bar b\big]+\big[(a-b)(\bar b-\bar c)\bar a-(\bar a-\bar b)(b-c)\bar c\big]=0.$
Taking into account this identity and (2.2), it follows that the relation (2.4) is true. $\Diamond$

Lemma 3. *If*

$$\Sigma_{ab}=\big[(c-a)(\bar u-\bar v)\bar a u-(\bar c-\bar a)(u-v)a\bar u\big]\big[(a-b)(\bar v-\bar w)-(\bar a-\bar b)(v-w)\big]- \\ -\big[(a-b)(\bar v-\bar w)\bar b v-(\bar a-\bar b)(v-w)b\bar v\big]\big[(c-a)(\bar u-\bar v)-(\bar c-\bar a)(u-v)\big]$$

and

$$\Sigma_{ac}=\big[(c-a)(\bar u-\bar v)\bar a u-(\bar c-\bar a)(u-v)a\bar u\big]\big[(b-c)(\bar w-\bar u)-(\bar b-\bar c)(w-u)\big]- \\ -\big[(b-c)(\bar w-\bar u)\bar c w-(\bar b-\bar c)(w-u)c\bar w\big]\big[(c-a)(\bar u-\bar v)-(\bar c-\bar a)(u-v)\big],$$

then

$$|c-a|^2\Sigma_{ab}+|a-b|^2\Sigma_{ac}=0. \tag{2.5}$$

Proof. Taking into account (2.1), it follows that

$$\Sigma_{ab}= \\ =\big[(c-a)(a-b)\bar\zeta\bar a u-(\bar c-\bar a)(\bar a-\bar b)\zeta a\bar u\big]\big[(a-b)(b-c)\bar\zeta-(\bar a-\bar b)(\bar b-\bar c)\zeta\big]- \\ -\big[(a-b)(b-c)\bar\zeta\bar b v-(\bar a-\bar b)(\bar b-\bar c)\zeta b\bar v\big]\big[(c-a)(a-b)\bar\zeta-(\bar c-\bar a)(\bar a-\bar b)\zeta\big]= \\ =|\zeta|^2|a-b|^2\big[(c-a)(\bar b-\bar c)b\bar v-(\bar c-\bar a)(b-c)a\bar u+ \\ +(\bar c-\bar a)(b-c)\bar b v-(c-a)(\bar b-\bar c)\bar a u\big]+ \\ +\zeta^2(\bar a-\bar b)(a\bar u-b\bar v)\bar\eta+\bar\zeta^2(a-b)(\bar a u-\bar b v)\eta.$$

But $a\bar u-b\bar v=a\bar u-b\bar u+b\bar u-b\bar v=(a-b)\bar u+b(\bar u-\bar v)=(a-b)(\bar u+b\bar\zeta)$.
Let us denote $\Omega_{ab}=(c-a)(\bar b-\bar c)b\bar v-(\bar c-\bar a)(b-c)a\bar u$. Then we can rewrite
$\Sigma_{ab}=|a-b|^2|\zeta|^2\big[\Omega_{ab}+\bar\Omega_{ab}\big]+|a-b|^2\zeta^2\bar\eta(\bar u+b\bar\zeta)+|a-b|^2\bar\zeta^2\eta(u+\bar b\zeta),$
therefore

$$\Sigma_{ab}=|a-b|^2|\zeta|^2\big[\Omega_{ab}+\bar\Omega_{ab}+\eta\bar b\bar\zeta+\bar\eta b\zeta\big]+|a-b|^2(\zeta^2\bar\eta\bar u+\bar\zeta^2\eta u).$$

Analogously, taking into account (2.1) and $a\bar u-c\bar w=(a-c)(\bar u+c\bar\zeta)$, we get

$$\Sigma_{ac}=|c-a|^2|\zeta|^2\big[-(a-b)(\bar b-\bar c)\bar a u-(\bar a-\bar b)(b-c)a\bar u+ \\ +(b-c)(\bar a-\bar b)\bar c w+(\bar b-\bar c)(a-b)c\bar w\big]- \\ -|c-a|^2\zeta^2\bar\eta(\bar u+c\bar\zeta)-|c-a|^2\bar\zeta^2\eta(u+\bar c\zeta).$$

Let us denote $\Omega_{ac} = -(b-c)(\bar{a}-\bar{b})a\bar{u} + (\bar{b}-\bar{c})(a-b)c\bar{w}$. Then

$$\Sigma_{ac} = |c-a|^2|\zeta|^2\left[\Omega_{ac} + \bar{\Omega}_{ac} - \eta\bar{c}\bar{\zeta} - \bar{\eta}c\zeta\right] - |c-a|^2(\zeta^2\bar{\eta}\bar{u} + \bar{\zeta}^2\eta u).$$

We obtain

$$|c-a|^2\Sigma_{ab} + |a-b|^2\Sigma_{ac} =$$
$$= |a-b|^2|c-a|^2|\zeta|^2\left[\Omega_{ab} + \bar{\Omega}_{ab} + \Omega_{ac} + \bar{\Omega}_{ac} + \eta\bar{b}\bar{\zeta} + \bar{\eta}b\zeta - \eta\bar{c}\bar{\zeta} - \bar{\eta}c\zeta\right] =$$
$$= |a-b|^2|c-a|^2|\zeta|^2\left[\Omega_{ab} + \Omega_{ac} + \eta\bar{\zeta}(\bar{b}-\bar{c}) + \overline{\left(\Omega_{ab} + \Omega_{ac} + \eta\bar{\zeta}(\bar{b}-\bar{c})\right)}\,\right].$$

Let us notice that, following our previous notations, the relation (2.3) can be rewritten as:

$$\Omega_{ab} + \Omega_{ac} + \eta\bar{\zeta}(\bar{b}-\bar{c}) = 0,$$

therefore (2.5) is true. $\Diamond$

3. The Proof of Theorem 1

3.1. The Proof of Direct Theorem

The proof of point (a) is obvious. We remind that a small letter represents the affix of the point which is denoted with the corresponding capital letter. Let us denote $(\mathcal{C}_a), (\mathcal{C}_b)$ and $(\mathcal{C}_c)$ the circumscribed circles of the inscribed quadrilaterals $AHUE$, $BGVD$, and $CWIF$ respectively. If we denote their equations in the complex plane with

$$\left(\mathcal{C}_a\right):\ z\bar{z} + \beta_a z + \bar{\beta}_a\bar{z} + \varepsilon_a = 0,$$
$$\left(\mathcal{C}_b\right):\ z\bar{z} + \beta_b z + \bar{\beta}_b\bar{z} + \varepsilon_b = 0,$$
$$\left(\mathcal{C}_c\right):\ z\bar{z} + \beta_c z + \bar{\beta}_c\bar{z} + \varepsilon_c = 0,$$

then we have to prove that the relation (see (1.3))

$$\frac{\beta_a - \beta_b}{\beta_a - \beta_c} = \frac{\bar{\beta}_a - \bar{\beta}_b}{\bar{\beta}_a - \bar{\beta}_c} = \frac{\varepsilon_a - \varepsilon_b}{\varepsilon_a - \varepsilon_c} \tag{3.1}$$

is true. According to (1.1), the affixes of the points of intersection are

$$d = \frac{(v\bar{w} - \bar{v}w)(a-b) - (a\bar{b} - \bar{a}b)(v-w)}{(\bar{a}-\bar{b})(v-w) - (a-b)(\bar{v}-\bar{w})},$$
$$e = \frac{(w\bar{u} - \bar{w}u)(a-b) - (a\bar{b} - \bar{a}b)(w-u)}{(\bar{a}-\bar{b})(w-u) - (a-b)(\bar{w}-\bar{u})},$$
$$f = \frac{(w\bar{u} - \bar{w}u)(b-c) - (b\bar{c} - \bar{b}c)(w-u)}{(\bar{b}-\bar{c})(w-u) - (b-c)(\bar{w}-\bar{u})},$$

$$g=\frac{(u\bar{v}-\bar{u}v)(b-c)-(b\bar{c}-\bar{b}c)(u-v)}{(\bar{b}-\bar{c})(u-v)-(b-c)(\bar{u}-\bar{v})},$$
$$h=\frac{(u\bar{v}-\bar{u}v)(c-a)-(c\bar{a}-\bar{c}a)(u-v)}{(\bar{c}-\bar{a})(u-v)-(c-a)(\bar{u}-\bar{v})},$$
$$i=\frac{(v\bar{w}-\bar{v}w)(c-a)-(c\bar{a}-\bar{c}a)(v-w)}{(\bar{c}-\bar{a})(v-w)-(c-a)(\bar{v}-\bar{w})}.$$

According to (1.2), the coefficients of the equations of the circumscribed circles are

$$\beta_a=\frac{\bar{\alpha}_a\bar{u}-\alpha_a\bar{a}}{\alpha_a-\bar{\alpha}_a},\ \varepsilon_a=\frac{\alpha_a\bar{a}u-\bar{\alpha}_a a\bar{u}}{\alpha_a-\bar{\alpha}_a},$$
$$\beta_b=\frac{\bar{\alpha}_b\bar{v}-\alpha_b\bar{b}}{\alpha_b-\bar{\alpha}_b},\ \ \varepsilon_b=\frac{\alpha_b\bar{b}v-\bar{\alpha}_b b\bar{v}}{\alpha_b-\bar{\alpha}_b},$$
$$\beta_c=\frac{\bar{\alpha}_c\bar{w}-\alpha_c\bar{c}}{\alpha_c-\bar{\alpha}_c},\ \varepsilon_c=\frac{\alpha_c\bar{c}w-\bar{\alpha}_c c\bar{w}}{\alpha_c-\bar{\alpha}_c},$$

where

$$\alpha_a=\frac{a-h}{u-h},\quad \alpha_b=\frac{b-d}{v-d},\quad \alpha_c=\frac{c-f}{w-f}.$$

Replacing with the affixes of the corresponding points, we obtain

$$\alpha_a=\frac{(c-a)\alpha'_a}{(u-v)\alpha''_a},\quad \alpha_b=\frac{(a-b)\alpha'_b}{(v-w)\alpha''_b},\quad \alpha_c=\frac{(b-c)\alpha'_c}{(w-u)\alpha''_c},$$

where

$$\alpha'_a=\bar{a}(u-v)-a(\bar{u}-\bar{v})-(u\bar{v}-\bar{u}v),\qquad \alpha''_a=u(\bar{c}-\bar{a})-\bar{u}(c-a)+(c\bar{a}-\bar{c}a),$$
$$\alpha'_b=\bar{b}(v-w)-b(\bar{v}-\bar{w})-(v\bar{w}-\bar{v}w),\quad \alpha''_b=v(\bar{a}-\bar{b})-\bar{v}(a-b)+(a\bar{b}-\bar{a}b),$$
$$\alpha'_c=\bar{c}(w-u)-c(\bar{w}-\bar{u})-(w\bar{u}-\bar{w}u),\quad \alpha''_c=w(\bar{b}-\bar{c})-\bar{w}(b-c)+(b\bar{c}-\bar{b}c).$$

Taking into account

$\overline{\alpha'}_a=-\alpha'_a,\ \overline{\alpha''}_a=-\alpha''_a,\ \overline{\alpha'}_b=-\alpha'_b,\ \overline{\alpha''}_b=-\alpha''_b,\ \overline{\alpha'}_c=-\alpha'_c,\ \overline{\alpha''}_c=-\alpha''_c,$

we get the following form for the coefficients of the circumscribed circles:

$$\beta_a=\frac{(\bar{c}-\bar{a})(u-v)\bar{u}-(c-a)(\bar{u}-\bar{v})\bar{a}}{(c-a)(\bar{u}-\bar{v})-(\bar{c}-\bar{a})(u-v)},$$
$$\varepsilon_a=\frac{(c-a)(\bar{u}-\bar{v})\bar{a}u-(\bar{c}-\bar{a})(u-v)a\bar{u}}{(c-a)(\bar{u}-\bar{v})-(\bar{c}-\bar{a})(u-v)},$$
$$\beta_b=\frac{(\bar{a}-\bar{b})(v-w)\bar{v}-(a-b)(\bar{v}-\bar{w})\bar{b}}{(a-b)(\bar{v}-\bar{w})-(\bar{a}-\bar{b})(v-w)},$$
$$\varepsilon_b=\frac{(a-b)(\bar{v}-\bar{w})\bar{b}v-(\bar{a}-\bar{b})(v-w)b\bar{v}}{(a-b)(\bar{v}-\bar{w})-(\bar{a}-\bar{b})(v-w)},$$

$$\beta_c = \frac{(\bar{b}-\bar{c})(v-w)\bar{w}-(b-c)(\bar{w}-\bar{u})\bar{c}}{(b-c)(\bar{w}-\bar{u})-(\bar{b}-\bar{c})(w-u)},$$
$$\varepsilon_c = \frac{(b-c)(\bar{w}-\bar{v})\bar{c}w-(\bar{b}-\bar{c})(w-u)c\bar{w}}{(b-c)(\bar{w}-\bar{u})-(\bar{b}-\bar{c})(w-u)}.$$

We have

$$\beta_a-\beta_b = \frac{(\bar{c}-\bar{a})(u-v)\bar{u}-(c-a)(\bar{u}-\bar{v})\bar{a}}{(c-a)(\bar{u}-\bar{v})-(\bar{c}-\bar{a})(u-v)} - \frac{(\bar{a}-\bar{b})(v-w)\bar{v}-(a-b)(\bar{v}-\bar{w})\bar{b}}{(a-b)(\bar{v}-\bar{w})-(\bar{a}-\bar{b})(v-w)}.$$

Let us denote

$$\Upsilon_1 = (c-a)(\bar{u}-\bar{v})-(\bar{c}-\bar{a})(u-v),$$
$$\Upsilon_2 = (a-b)(\bar{v}-\bar{w})-(\bar{a}-\bar{b})(v-w),$$
$$\Upsilon_3 = (b-c)(\bar{w}-\bar{u})-(\bar{b}-\bar{c})(w-u).$$

Taking into account the notation in Lemma 2, we obtain

$$\beta_a-\beta_b = \frac{\Xi_{ab}}{\Upsilon_1\cdot\Upsilon_2}.$$

Analogously, we obtain

$$\beta_a-\beta_c = \frac{\Xi_{ac}}{\Upsilon_1\cdot\Upsilon_3}.$$

Taking into account (2.4), it follows that

$$\frac{\beta_a-\beta_b}{\beta_a-\beta_c} = -\frac{|a-b|^2\Upsilon_3}{|c-a|^2\Upsilon_2}. \tag{3.2}$$

We have

$$\varepsilon_a-\varepsilon_b = \frac{(c-a)(\bar{u}-\bar{v})\bar{a}u-(\bar{c}-\bar{a})(u-v)a\bar{u}}{(c-a)(\bar{u}-\bar{v})-(\bar{c}-\bar{a})(u-v)} - \frac{(a-b)(\bar{v}-\bar{w})\bar{b}v-(\bar{a}-\bar{b})(v-w)b\bar{v}}{(a-b)(\bar{v}-\bar{w})-(\bar{a}-\bar{b})(v-w)}.$$

Then, taking into account the notation in Lemma 3, we obtain

$$\varepsilon_a-\varepsilon_b = \frac{\Sigma_{ab}}{\Upsilon_1\cdot\Upsilon_2}.$$

Analogously, we obtain

$$\varepsilon_a-\varepsilon_c = \frac{\Sigma_{ac}}{\Upsilon_1\cdot\Upsilon_3}.$$

Taking into account (2.5), it follows that

$$\frac{\varepsilon_a - \varepsilon_b}{\varepsilon_a - \varepsilon_c} = -\frac{|a-b|^2 \Upsilon_3}{|c-a|^2 \Upsilon_2}. \tag{3.3}$$

(3.2) and (3.3) show that the relation (3.1) is true. This concludes the proof of Direct Theorem.

3.2. The Proof of Converse Theorem

First, let us prove the following technical lemmas.

Lemma 4. *Let us consider the circles $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ and the lines d_1 and d_2 which intersect the circles in the points $\{A, B\} = d_1 \cap (\mathcal{C}_1)$, $\{M, N\} = d_1 \cap (\mathcal{C}_2)$, $\{C, D\} = d_2 \cap (\mathcal{C}_1)$ and $\{P, Q\} = d_2 \cap (\mathcal{C}_2)$. If $BD \parallel MP$, then $AC \parallel QN$.*

Proof. We prefer to use complex numbers, although the synthetic proof is not difficult. The conditions from hypothesis can be written

$$\frac{c-b}{a-b} : \frac{c-d}{a-d} \in \mathbb{R};\ \frac{p-n}{m-n} : \frac{p-q}{m-q} \in \mathbb{R};\ \frac{a-b}{m-n} \in \mathbb{R};\ \frac{c-d}{p-q} \in \mathbb{R}.$$

We get $\frac{(c-b)(a-d)}{(p-n)(m-q)} \in \mathbb{R}$. As $\frac{m-p}{b-d} \in \mathbb{R}$, it follows that

$$(*) \quad \frac{\frac{(b-c)(a-d)}{b-d}}{\frac{(n-p)(m-q)}{m-p}} \in \mathbb{R}.$$

But the condition of concyclicity of the points A, B, C, D and M, N, P, Q can be rewritten

$$\frac{b-c}{a-c} : \frac{b-d}{a-d} \in \mathbb{R};\ \ \frac{n-p}{m-p} : \frac{m-q}{n-q} \in \mathbb{R}.$$

Taking into account $(*)$, it follows that $\frac{n-q}{a-c} \in \mathbb{R}$, which means $AC \parallel QN$. $\Diamond$

Lemma 5. *Let us consider the inscribed quadrilateral $ABCD$ and the points $M \in AB$ and $P \in CD$.*

(a) *Let us consider the points $N \in AC$ and $Q \in BD$, so that the triangles AMN and DPQ are similar, with homologous vertices A and D, M and P and N and Q. If the triangles AMN and DPQ are directly similar, then $CD \perp BD$.*

(b) *Let us consider the points $N \in AD$ and $Q \in BC$, so that he triangles AMN and CPQ are similar, with homologous vertices A and C, M and P and N and Q. If the triangles AMN and CPQ are directly similar, then $AB \perp AD$.*

Proof. (a) There exist the real numbers $\alpha, \beta, \gamma, \delta$ so that
$(*)$ $m = (1-\alpha)a+\alpha b,\ p = (1-\beta)d+\beta c,\ n = (1-\gamma)a+\gamma c,\ q = (1-\delta)d+\delta b.$
Let us suppose that the triangles AMN and DPQ are directly similar, with homologous vertices A and D, M and P and N and Q. Then, replacing in $\frac{m-a}{n-a} = \frac{p-d}{q-d}$ with $(*)$, we get $\frac{\alpha(b-a)}{\gamma(c-a)} = \frac{\beta(c-d)}{\delta(b-d)}$. As

$$\frac{b-a}{c-a} : \frac{b-d}{c-d} \in \mathbb{R},$$

it follows that

$$\left(\frac{c-d}{b-d}\right)^2 \in \mathbb{R}.$$

If $\frac{c-d}{b-d} \in \mathbb{R}$, it would follow that the points B, C and D are collinear, which is false. It follows that

$$\frac{c-d}{b-d} \in i\mathbb{R},$$

therefore $CD \perp BD$.

(b) The proof goes on the same line as the proof of the point (a), so we omit it. $\Diamond$

As $F \in (\mathcal{C}_c)$ and $E \in (\mathcal{C}_a)$, there exist the points $U_1 = FE \cap (\mathcal{C}_a)$ and $W_1 = FE \cap (\mathcal{C}_c)$. We claim that the lines HU_1 and IW_1 are not parallel. Indeed, argumentum ad absurdum, let us suppose that they are parallel. Then, applying Lemma 4 to the circles $(\mathcal{C}_a)$, $(\mathcal{C}_c)$ and the lines FE and HI, it follows that the lines AE and CF are parallel too, which is false, because $AE \cap CF = B$. We consider the point $V_1 = HU_1 \cap IW_1$. We get the points $H, A, U_1, E \in (\mathcal{C}_a)$ and $I, C, W_1, F \in$ $\in (\mathcal{C}_c)$. Then $\angle HU_1E \equiv \angle BAC$ and $\angle IW_1F \equiv \angle ACB$, so the triangles ABC and $U_1V_1W_1$ are similar, with the homologous vertices A and U_1, B and V_1, C and W_1. We prove that the triangles ABC and $U_1V_1W_1$ are oppositely similar. Indeed, argumentum ad absurdum, let us suppose that they are directly similar. Then, applying Lemma 5 to the inscribed quadrilaterals HAU_1E and ICW_1F, it would follow that both angles $\angle HU_1E$ and $\angle IW_1F$ are right angles, which implies that the angles A and C of the triangle ABC are right, which is false. Consequently, the triangles ABC and $U_1V_!W_1$ are oppositely similar, with the fore-cited homologous vertices. According to the Direct Theorem, if we consider the intersection points $AB \cap V_1W_1 = D_1$, $AB \cap W_1U_1 = E_1$, $BC \cap$ $\cap W_1U_1 = F_1$, $BC \cap U_1V_1 = G_1$, $CA \cap U_1V_1 = H_1$, $CA \cap W_1V_1 = I_1$, it follows that the quadrilaterals $AH_1U_1E_1$, $BG_1V_1D_1$ and $CW_1I_1F_1$ are inscribed and their circumscribed circles, denoted respectively $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$ have the radical axis in common. Let us notice that, according to hypothesis and with our construction, $H \in U_1V_1 \cap AC$, therefore $H \equiv H_1$. Analogously, one obtains $I_1 \equiv I$. The lines U_1W_1 and EF coincide, therefore $\{F_1\} = BC \cap W_1U_1 = BC \cap FE = \{F\}$ and $\{E_1\} = = AB \cap W_1U_1 = AB \cap FE = \{E\}$. It follows that the circles $(\mathcal{C}_1)$ and $(\mathcal{C}_a)$ coincide, because they are the circumscribed circle of the triangle AHE. Analogously, one obtains the circles $(\mathcal{C}_3)$ and $(\mathcal{C}_c)$ coincide. It follows that $U_1 \equiv U$ and $W_1 \equiv W$. The circles $(\mathcal{C}_2)$ and $(\mathcal{C}_b)$ have in common the point B and their common axis coincides with the common axis of the circles $(\mathcal{C}_1) \equiv (\mathcal{C}_a)$ and $(\mathcal{C}_3) \equiv (\mathcal{C}_c)$. If they did not coincide, then the point B would belong to their radical axis, therefore B would belong to the radical axis of the circles $(\mathcal{C}_a)$ and $(\mathcal{C}_c)$, which contradicts the hypothesis. It follows that $(\mathcal{C}_2) \equiv (\mathcal{C}_b)$, therefore $V \equiv V_1$ and this concludes the proof of the Converse Theorem.

3.3. A Remark

We have a better understanding of the Converse Theorem 1 if one considers three circles in general position. It is known that, if the centers of the circles $(\mathcal{C}_a)$, $(\mathcal{C}_b)$ and $(\mathcal{C}_c)$ are not collinear, then the three pairwise radical axis have a common intersection point, called the radical center. If the centers are collinear, then the three pairwise radical axis coincide or are parallel. Let us consider that the centers are not collinear. Let us choose the points $A \in (\mathcal{C}_a)$, $B \in (\mathcal{C}_b)$ and $C \in (\mathcal{C}_c)$ and let us consider the following intersection points: $AB \cap (\mathcal{C}_b) = D$, $AB \cap (\mathcal{C}_a) = E$, $BC \cap (\mathcal{C}_c) = F$, $BC \cap (\mathcal{C}_b) = G$, $CA \cap (\mathcal{C}_a) = H$ and $CA \cap (\mathcal{C}_c) = I$. Next, let us consider the intersection points $DI \cap (\mathcal{C}_b) = S$, $DI \cap (\mathcal{C}_c) = T$ and $GS \cap FT = R$. Obviously the triangles ABC and RST are oppositely similar, with homologous vertices A and R, B and S, C and T. According to Direct Theorem 1, if we consider the intersection points $M = AC \cap RS$ and $N = AB \cap TR$, then the quadrilateral $MARN$ is inscribed and its circumscribed circle, denoted $(\mathcal{C}_1)$ and the circles $(\mathcal{C}_b)$ and $(\mathcal{C}_c)$ have the property that the three pairwise radical axis coincide. If we repeat the construction with respect to the lines FE and GH, we finally obtain three directly similar triangles RST, UVW and XYZ, each of them oppositely similar with ABC (see Fig. 2). Let us notice that the circles

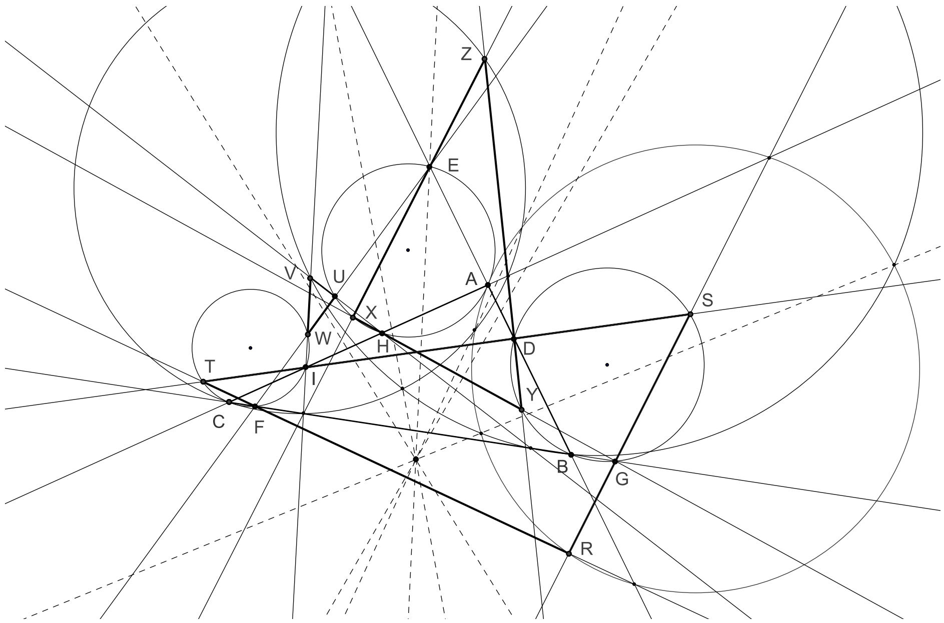

Figure 2

$(\mathcal{C}_a), (\mathcal{C}_b), (\mathcal{C}_c)$, together with the new circles $(\mathcal{C}_1), (\mathcal{C}_2)$ and $(\mathcal{C}_3)$ have the common radical center, which is the common center of the direct similarities which transform the triangles RST, UVW and XYZ one another. The Converse Theorem 1 states: if the initial circles have the common radical axis, then the triangles RST, UVW and ZYX coincide.

4. Applications

The case when the circles intersect themselves presents a particularly interesting in applications. First, let us prove the following useful consequence of Th. 1 (see Fig. 3).

Corollary 1. *Let us suppose that the circles $(\mathcal{C}_a), (\mathcal{C}_b)$ and $(\mathcal{C}_c)$ have the property they intersect themselves in the points M and N. Let us choose the points $A \in (\mathcal{C}_a)$, $B \in (\mathcal{C}_b)$ and $C \in (\mathcal{C}_c)$ so that the points A, B, C are not collinear and do not coincide with one of the point of intersections of the circles. We consider the following intersection points: $AB \cap (\mathcal{C}_b) = D$, $AB \cap (\mathcal{C}_a) = E$, $BC \cap (\mathcal{C}_c) = F$, $BC \cap (\mathcal{C}_b) = G$, $CA \cap (\mathcal{C}_a) = H$, $CA \cap (\mathcal{C}_c) = I$ and $FE \cap GH = U$, $GH \cap DI = V$, $DI \cap FE = W$. Then:*

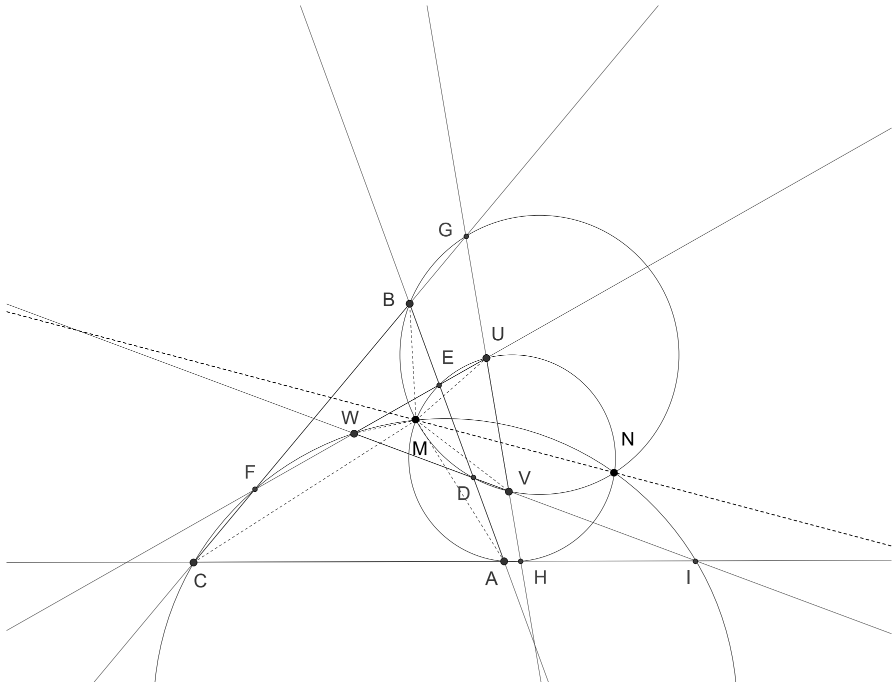

Figure 3

(a) $U \in \left(\mathcal{C}_a\right)$, $V \in \left(\mathcal{C}_b\right)$ *and* $W \in \left(\mathcal{C}_c\right)$.

(b) *The triangles ABC and UVW are oppositely similar, with homologous vertices A and U, B and V, C and W.*

(c) *Let us denote M_1, N_1 the isogonal conjugates with respect to the triangle ABC, of the points M, respectively N. Then the homologous of the points M_1 and N_1 with respect to the triangle UVW are the points M, respectively N.*

Proof. Applying the Converse of Th. 1, we obtain that the statements of the points (a) and (b) are true. In order to prove the point (c), it suffices to take a look, for example, at the circle $\left(\mathcal{C}_a\right)$, where we notice that $\angle VUM = \angle HUM = \angle CAM$. Analogously, we get $\angle UWM = \angle BCM$ and $\angle WVM = \angle ABM$. This concludes the proof. $\Diamond$

4.1. An application to Schröder point

Let ABC be a triangle with the incenter I. The incircle of triangle ABC touches BC, CA and AB at A', B' and C'. The triangle $A'B'C'$ is

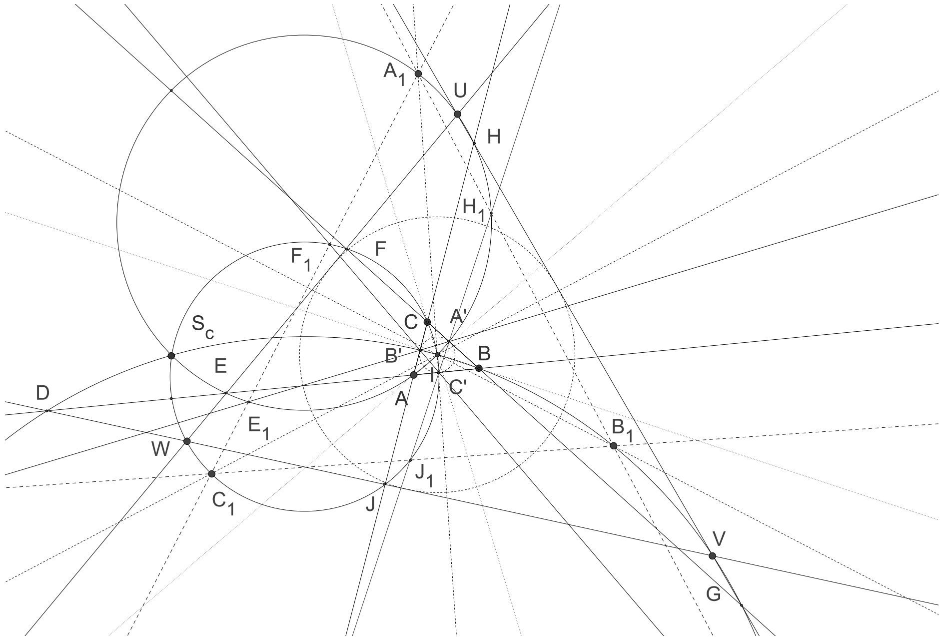

Figure 4

called the Gergonne triangle of ABC. The Schröder point S_c (see Fig. 4) is the second intersection point of the circles AIA', BIB' and CIC'.

Theorem 2. *Let us denote the circles AIA', BIB' and CIC' respectively with $(\mathcal{C}_a), (\mathcal{C}_b)$ and $(\mathcal{C}_c)$. We consider the following points of intersections: $AB \cap (\mathcal{C}_b) = D$, $AB \cap (\mathcal{C}_a) = E$, $BC \cap (\mathcal{C}_c) = F$, $BC \cap (\mathcal{C}_b) = G$, $CA \cap (\mathcal{C}_a) = H$, $CA \cap (\mathcal{C}_c) = J$ and $FE \cap GH = U$, $GH \cap DJ = V$, $DJ \cap FE = W$. It follows that:*

(a) *$U \in (\mathcal{C}_a)$, $V \in (\mathcal{C}_b)$ and $W \in (\mathcal{C}_c)$.*

(b) *The triangles ABC and UVW are oppositely similar, with homologous vertices A and U, B and V, C and W.*

(c) *I is the incenter of the triangle UVW.*

Theorem 3. *Let us keep the notations in Th. 2 and let us consider the following points of intersections: $A'B' \cap (\mathcal{C}_b) = D_1$, $A'B' \cap (\mathcal{C}_a) = E_1$, $B'C' \cap (\mathcal{C}_c) = F_1$, $B'C' \cap (\mathcal{C}_b) = G_1$, $C'A' \cap (\mathcal{C}_a) = H_1$, $C'A' \cap (\mathcal{C}_c) = J_1$ and $F_1E_1 \cap G_1H_1 = A_1$, $G_1H_1 \cap D_1J_1 = B_1$, $D_1J_1 \cap F_1E_1 = C_1$. It follows that:*

(a) *$A_1 \in (\mathcal{C}_a)$, $B_1 \in (\mathcal{C}_b)$ and $C_1 \in (\mathcal{C}_c)$.*

(b) *The triangles $A'B'C'$ and $A_1B_1C_1$ are oppositely similar, with homologous vertices A' and A_1, B' and B_1, C' and C_1.*

(c) *I is the orthocenter of the triangle $A_1B_1C_1$.*

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