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SEMI-OPEN, SEMI-CLOSED SETS AND SEMI-CONTINUITY OF FUNCTIONS

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Abstract: We state a condition under which the well-known Levine's Th. 15 of [7] is reversible. A topology τ_w determined by a given topology τ on X is introduced in order to generalize the Hamlett's main result of [5].

1. Preliminaries

Throughout the present paper (X, τ), (Y, σ), and (Z, γ) mean topological spaces on which no separation axioms are assumed unless explicitly stated. The closure and the interior of a subset S in (X, τ) are denoted by $\mathrm{cl}\,(S)$ and $\mathrm{int}\,(S)$ respectively. A subset S of (X, τ) is said to be ***semi-open*** [7] (resp. ***semi-closed*** [2, Th. 1.1]) if there exists an open set O with $O \subset S \subset \mathrm{cl}\,(O)$ (resp. if there exists a closed F with $\mathrm{int}\,(F) \subset S \subset F$). The family of all semi-open (resp. semi-closed; closed) subsets of (X, τ) is denoted as $\mathrm{SO}\,(X, \tau)$ (resp. $\mathrm{SC}\,(X, \tau)$; $\mathrm{c}(\tau)$). Obviously, $F \in \mathrm{SC}\,(X, \tau)$ if and only if $X \setminus F \in \mathrm{SO}\,(X, \tau)$. It is well-known [7] that $\bigcup_{t\in T} S_t \in \mathrm{SO}\,(X,\tau)$ for every collection $\{S_t : t \in T\} \subset \mathrm{SO}\,(X,\tau)$. In [7, Th. 7] Levine proved that if $A \in \mathrm{SO}\,(X,\tau)$, then $A = G \cup N$ for a certain $G \in \tau$ and a certain nowhere dense N. Dlaska et al. made a deeper remark [3, Sec.1, p.1163]: $A \in \mathrm{SO}\,(X,\tau)$ if and only if $A = G_A \cup N_A$ with G_A being a suitable open set and a nowhere dense $N_A \subset \mathrm{Fr}\,(G_A)$ ($\mathrm{Fr}\,(S)$ stands for the boundary of S).

A space (X,τ) is said to be ***extremally disconnected*** if $\mathrm{cl}\,(S) \in \tau$ for every $S \in \tau$.

2. Two semi-continuous functions

In 1963 Levine has shown [7, Th. 15], that if $h\colon (X,\tau) \to (Y,\sigma)\times\times(Z,\gamma)$ defined by $h(x) = (f(x), g(x))$, where $f\colon (X,\tau) \to (Y,\sigma)$ and $g\colon (X,\tau) \to (Z,\gamma)$, is semi-continuous, then also f and g are both semi-continuous. [7, Ex. 10] shows that the converse to this theorem fails to be true in general. In our note we propose a condition under which the converse holds.

The remark of Dlaska et al. [3] concerning representation of semi-open sets is reformulated as follows.

Lemma 1. *Let (X,τ) be a topological space. Then, $A \in \mathrm{SO}\,(X,\tau)$ if and only if $A = \mathrm{int}\,(A) \cup N$ for a certain $N \subset \mathrm{Fr}\,(\mathrm{int}\,(A))$.*

Proof. Obvious. ♢

Lemma 2. *Let (X,τ) be a topological space. For each $S \subset X$ and $G \in \tau$ we have*

$$G \cap \mathrm{Fr}\,(S) \subset \mathrm{Fr}\,(G \cap S).$$

Proof. We calculate as follows:

$$\begin{aligned} G \cap \mathrm{Fr}\,(S) &= \big(G \cap \mathrm{cl}\,(S)\big) \setminus (G \cap \mathrm{int}\,(S)) \subset \\ &\subset \mathrm{cl}\,(G \cap S) \setminus \mathrm{int}\,(G \cap S) = \mathrm{Fr}\,(G \cap S). \qquad \lozenge \end{aligned}$$

Theorem 1. *Let $f\colon (X,\tau) \to (Y,\sigma)$, $g\colon (X,\tau) \to (Z,\gamma)$ be both semi-continuous on (X,τ). If for each $U \in \sigma$ and $V \in \gamma$ we have*

$$\mathrm{Fr}\,\big(\mathrm{int}\,\big(f^{-1}(U)\big)\big) \cap \mathrm{Fr}\,\big(\mathrm{int}\,\big(g^{-1}(V)\big)\big) = \emptyset,$$

then the function $h\colon (X,\tau) \to (Y\times Z, \sigma\times\gamma)$ defined as $h(x) = (f(x), g(x))$ for $x \in X$, is semi-continuous on (X,τ).

Proof. Let $U \times V$ be any basic open subset of the product $(Y\times Z, \sigma\times\gamma)$. By semi-continuity of f and g we infer that $f^{-1}(U) = \mathrm{int}\,\big(f^{-1}(U)\big) \cup N_U$

and $g^{-1}(V) = \operatorname{int}\left(g^{-1}(V)\right) \cup N_V$, where $N_U \subset \operatorname{Fr}\left(\operatorname{int}\left(f^{-1}(U)\right)\right)$, $N_V \subset \operatorname{Fr}\left(\operatorname{int}\left(g^{-1}(V)\right)\right)$ (see Lemma 1). Clearly, we have

$$\begin{aligned} h^{-1}(U \times V) = {} & f^{-1}(U) \cap g^{-1}(V) = \\ & = \operatorname{int}\left(f^{-1}(U) \cap g^{-1}(V)\right) \cup \left(\operatorname{int}\left(f^{-1}(U)\right) \cap N_V\right) \cup \\ & \cup \left(\operatorname{int}\left(g^{-1}(V)\right) \cap N_U\right) \cup (N_U \cap N_V) \subset \operatorname{int}\left(f^{-1}(U) \cap g^{-1}(V)\right) \cup \\ & \cup \operatorname{Fr}\left(\operatorname{int}\left(f^{-1}(U) \cap g^{-1}(V)\right)\right) \cup (N_U \cap N_V) \end{aligned}$$

by Lemma 2. Thus with the assumption one gets

$$\begin{aligned} h^{-1}(U \times V) \subset {} & \operatorname{int}\left(f^{-1}(U) \cap g^{-1}(V)\right) \cup \operatorname{Fr}\left(\operatorname{int}\left(f^{-1}(U) \cap g^{-1}(V)\right)\right) = \\ & = \operatorname{cl}\left(\operatorname{int}\left(f^{-1}(U) \cap g^{-1}(V)\right)\right) = \operatorname{cl}\left(\operatorname{int}\left(h^{-1}(U \times V)\right)\right), \end{aligned}$$

whence h is semi-continuous. $\Diamond$

With the aid of Lemma 1 one can easily obtain the following corollary.

Corollary 1. *Let $f\colon (X,\tau) \to (Y,\sigma)$ and $g\colon (X,\tau) \to (Z,\gamma)$ be any functions. Then, $h = (f,g)$ is semi-continuous if and only if for each $U \in \sigma$ and $V \in \gamma$ we have $h^{-1}(U \times V) = \operatorname{int}\left(f^{-1}(U) \cap g^{-1}(V)\right) \cup N_{U,V}$, where $N_{U,V} \subset \operatorname{Fr}\left(\operatorname{int}\left(f^{-1}(U) \cap g^{-1}(V)\right)\right)$.*

A classical theorem concerning continuous functions (see for instance [4, Th. 1.5]), was generalized by Hamlett [5] as follows: *Let (X,τ) be arbitrary, (Y,σ) be Hausdorff, and $f,g\colon (X,\tau) \to (Y,\sigma)$, where f is continuous and g is semi-continuous. Then*

(1) $\{x \in X : f(x) = g(x)\} \in \operatorname{SC}(X,\tau)$,

(2) *if $D \subset X$ is dense and $f \restriction D = g \restriction D$, then $f = g$ on X.*

[5, Ex. 2.2] shows that for the case 'f and g are both semi-continuous', (1) and (2) do not hold, in general.

The reader is advised to compare the following lemma to Lemma 1.

Lemma 3. *For any space (X,τ), $B \in \operatorname{SC}(X,\tau)$ if and only if there exist $F \in \operatorname{c}(\tau)$ and $M \subset X$ with*

(1) *$B = \operatorname{int}(F) \cup M$ and*

(2) *$M \subset \operatorname{Fr}(F)$.*

Proof. ($\Rightarrow$). Let $B \in \operatorname{SC}(X,\tau)$. Then $\operatorname{int}(F) \subset B \subset F$ for a certain set $F \in \operatorname{c}(\tau)$. Clearly, $B = \operatorname{int}(F) \cup M$ and $M = B \setminus \operatorname{int}(F) \subset \operatorname{Fr}(F)$, where $\operatorname{Fr}(F)$ is a nowhere dense subset of X.

($\Leftarrow$). Obvious. $\Diamond$

Remark 1. It should be noticed that the boundary of each semi-open and each semi-closed subset S of (X,τ), is nowhere dense in (X,τ). Indeed, by [8, Lemma 2] and its dual we have

$$\operatorname{int}\Big[\operatorname{cl}\Big(\operatorname{cl}(\operatorname{int}(S))\setminus\operatorname{int}(S)\Big)\Big]=\operatorname{int}\Big[\operatorname{cl}\Big(\operatorname{cl}(S)\setminus\operatorname{int}(\operatorname{cl}(S))\Big)\Big]=\emptyset.$$

Lemma 4. *Let (X,τ) be any topological space and (Y,σ) be a $\mathcal{T}_1$-space. Let $f,g\colon(X,\tau)\to(Y,\sigma)$ be both semi-continuous. Then the set $\{x\in$ $\in X:\ f(x)=g(x)\}$ is of the form $\bigcap_\alpha(G_\alpha\cup N_\alpha)$, where $\{G_\alpha\}_\alpha\subset\tau$ and each N_α is a certain nowhere dense subset of (X,τ).*

Proof. Consider the set $A=X\backslash\{x\in X:\ f(x)=g(x)\}$ and an arbitrary $x\in A$. We have $f(x)\neq g(x)$. Since (Y,σ) is $\mathcal{T}_1$, then $\{f(x)\},\{g(x)\}\in$ $\in\mathrm{c}(\sigma)$. By hypothesis we obtain

$$f^{-1}\big(\{f(x)\}\big),g^{-1}\big(\{g(x)\}\big)\in\mathrm{SC}\,(X,\tau).$$

Let for each $x\in A$, $U_x=f^{-1}\big(\{f(x)\}\big)\cap g^{-1}\big(\{g(x)\}\big)$. Obviously, for each $z\in U_x$ we have $f(z)\neq g(z)$, thus $z\in A$. Consequently $\bigcup_{x\in A}U_x=A$. We calculate now as follows:

$$R=\{x\in X\colon f(x)=g(x)\}=X\backslash A=X\backslash\bigcup_{x\in A}U_x=X\backslash\left(\bigcup_{x\in A}A_x\cup\bigcup_{x\in A}L_x\right),$$

where for each $x\in A$, $U_x=A_x\cup L_x$ with $A_x\in\tau$, L_x is nowhere dense in (X,τ), since $U_x\in\mathrm{SC}\,(X,\tau)$ (see Lemma 3). We have (denote $A'=\bigcup_{x\in A}A_x$) that $R=(X\setminus A')\cap\bigcap_{x\in A}(X\setminus L_x)$, where $X\setminus A'\in\mathrm{c}(\tau)$ and $X\setminus L_x\in\mathrm{SO}\,(X,\tau)$ for each $x\in A$ (since $L_x\in\mathrm{SC}\,(X,\tau)$; see [2, Th. 1.3]). Clearly $X\setminus A'=G_0\cup N_0$ for a certain $G_0\in\tau$ and a nowhere dense $N_0\subset X$. Similarly, for each $x\in A$, $X\setminus L_x=G_x\cup N_x$, where $G_x\in\tau$ and N_x is nowhere dense in (X,τ) [7, Th. 7]. So, it means that

$$R=(G_0\cup N_0)\cap\bigcap_{x\in A}(G_x\cup N_x).$$

The proof is complete. $\Diamond$

Lemma 5. *Let (X,τ) be any topological space. Let $\hat{\tau}_w$ denote the family of all subsets of X of the form $X\setminus\bigcap_{\alpha\in A}(G_\alpha\cup N_\alpha)$, where A is arbitrary, $G_\alpha\in\tau$ for each $\alpha\in A$, and each N_α is nowhere dense in (X,τ). Then $\hat{\tau}_w$ is a basis for a certain topology, designed as τ_w, on X.*

Proof. One easily checks that $\emptyset,X\in\hat{\tau}_w$. Consider arbitrary $V_1=$ $=X\setminus\bigcap_{\alpha\in A_1}(G_\alpha\cup N_\alpha)\in\hat{\tau}_w$ and $V_2=X\setminus\bigcap_{\beta\in A_2}(G_\beta\cup N_\beta)\in\hat{\tau}_w$. We have (use [4, Th. 4.2(1)])

$$V_1\cap V_2=X\setminus\left[\left(\bigcap_{\alpha\in A_1}(G_\alpha\cup N_\alpha)\right)\cup\left(\bigcap_{\beta\in A_2}(G_\beta\cup N_\beta)\right)\right]=$$

$$= X \setminus \bigcap_{(\alpha,\beta)\in A_1\times A_2} [(G_\alpha \cup G_\beta) \cup (N_\alpha \cup N_\beta)].$$

Thus $V_1 \cap V_2 \in \hat{\tau}_w$. $\Diamond$

Theorem 2. *Let (X,τ) be any topological space and let (Y,σ) be a $\mathcal{T}_1$-space. If $f,g\colon (X,\tau) \to (Y,\sigma)$ are both semi-continuous, then*

(1) *The set $\{x \in X :\ f(x) = g(x)\}$ is closed in (X,τ_w).*

(2) *If $D \subset X$ is dense in (X,τ_w) and $f \restriction D = g \restriction D$, then $f = g$ on X.*

Proof. (1) follows from Lemma 4. To prove (2) apply (1) together with [4, Th. 4.13]. $\Diamond$

Recall that a subset A of a space (X,τ) is called ***simply open*** [1] if $A = G \cup N$, where $G \in \tau$ and N is nowhere dense.

Lemma 6. *Let (X,τ) be any topological space. Then, each simply open subset of (X,τ) is τ_w-clopen.*

Proof. Let $A = O \cup N$ for a certain $O \in \tau$ and nowhere dense N. Hence $X \setminus A = (X \setminus O) \cap (X \setminus N) = \big(\mathrm{int}\,(X \setminus O) \cup \mathrm{Fr}\,(X \setminus O)\big) \cap (G \cup M)$, where $G \in \tau$ and $M \subset \mathrm{Fr}\,(G)$ (each nowhere dense set is semi-closed and hence the complement to X of it is semi-open). Thus $X \setminus A = = \mathrm{int}\,\big((X \setminus O) \cap G\big) \cup L$ for a certain nowhere dense L (in (X,τ)). So, $X \setminus A \in \mathrm{c}(\tau_w)$. Finally, A is τ_w-open. $\Diamond$

The following statement is now obvious.

Corollary 2. *Each semi-closed (or semi-open) subset of (X,τ) is τ_w-open.*

Theorem 3. *Let (X,τ) be extremally disconnected and (Y,σ) be Hausdorff. If both $f,g\colon (X,\tau) \to (Y,\sigma)$ are semi-continuous, then*

(1) $\{x \in X :\ f(x) = g(x)\} \in \mathrm{SC}\,(X,\tau)$;

(2) *if $D \subset X$ is dense in (X,τ) and $f \restriction D = g \restriction D$ then $f = g$ on X.*

Proof. (1). The proof is analogous to the classical one. We use the fact that in extremally disconnected space (X,τ), $V_1 \cap V_2 \in \mathrm{SO}\,(X,\tau)$ for any $V_1, V_2 \in \mathrm{SO}\,(X,\tau)$ [6, Prop. and Rem.].

(2). Use [5, Th. 2.4]. $\Diamond$

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