FUNCTIONAL EQUATIONS ON THE $SU(2)$-HYPERGROUP

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Received: November 2011
MSC 2010: 20 N 20, 60 F 99
Keywords: Functional equation, polynomial hypergroup.
Abstract: We consider classical functional equations on a special hypergroup which is related to continuous unitary irreducible representations of the special linear group in two dimensions.

1. Introduction

Functional equations on hypergroups have been treated in [6], [7]. In this paper we study functional equations on a special hypergroup, which is related to the set of continuous unitary irreducible representations of the group $G = SU(2)$, the special linear group in two dimensions. We show how to determine all exponentials, additive functions and generalized moment function sequences on this hypergroup. Moment functions on other types of hypergroups have been described in [3], [4] and [5]. The
definition of the underlying hypergroup is taken from [1].

If $G$ is a compact topological group then its dual object $\hat{G}$ consists of equivalence classes of continuous irreducible representations of $G$. For any two classes $U, V$ of this type their tensor product can be decomposed into its irreducible components $U_1, U_2, \ldots, U_n$ with the respective multiplicities $m_1, m_2, \ldots, m_n$ (see [2]). We define convolution on $\hat{G}$ by

\[
\delta_U \ast \delta_V = \sum_{i=1}^{n} \frac{m_i d(U_i)}{d(U) d(V)} \delta_{U_i}
\]

where $d(U)$ denotes the dimension of $U$ and $\delta_U$ is the Dirac measure concentrated at $U$. Then $\hat{G}$ with this convolution and with the discrete topology is a commutative hypergroup.

In the special case of $G = SU(2)$ the dual object $\hat{G}$ can be identified with the set $\mathbb{N}$ of natural numbers as it is indicated in [1]: the set of equivalence classes of continuous unitary irreducible representations of $SU(2)$ is given by $\{T^{(0)}, T^{(1)}, T^{(2)}, \ldots\}$, where $T^{(n)}$ has dimension $n + 1$, and we identify this set with $\mathbb{N}$.

For every $m, n$ in $\mathbb{N}$ the tensor product of $T^{(m)}$ and $T^{(n)}$ is unitary equivalent to

\[
T^{(|m-n|)} \bigoplus T^{(|m-n|+2)} \bigoplus \ldots \bigoplus T^{(m+n)}.
\]

The convolution is given by

\[
\delta_m \ast \delta_n = \sum_{k=|m-n|}^{m+n} \frac{k+1}{(m+1)(n+1)} \delta_k,
\]

where the prime denotes that every second term appears in the sum, only. With this convolution $\mathbb{N}$ becomes a discrete commutative hypergroup, and since all the $T^{(n)}$ are self-conjugate, the hypergroup is in fact Hermitian. We call this hypergroup the $SU(2)$-hypergroup.

2. Exponential functions on the $SU(2)$-hypergroup

In this section we describe the exponential functions on the $SU(2)$-hypergroup. We recall that the function $M : \mathbb{N} \to \mathbb{C}$ is an exponential if
and only if it satisfies

\begin{equation}
M(m)M(n) = M(m \ast n) = \sum_{k = |m - n|}^{m+n} \frac{k + 1}{(m + 1)(n + 1)} M(k)
\end{equation}

for all natural numbers \( m, n \).

**Theorem 1.** The function \( M : \mathbb{N} \rightarrow \mathbb{C} \) is an exponential on the \( SU(2) \)-hypergroup if and only if there exists a complex number \( \lambda \) such that

\begin{equation}
M(n) = \frac{\sinh[(n + 1)\lambda]}{(n + 1) \sinh \lambda}
\end{equation}

holds for each natural number \( n \). (Here \( \lambda = 0 \) corresponds to the exponential \( M = 1 \).)

**Proof.** Let \( M : \mathbb{N} \rightarrow \mathbb{C} \) be a solution of (2.1) and let \( f(n) = (n + 1)M(n) \) for each \( n \) in \( \mathbb{N} \). Then we have

\[
f(m)f(n) = \sum_{k = |m - n|}^{m+n} 'f(k)
\]

for each \( m, n \) in \( \mathbb{N} \). With \( m = 1 \) it follows that \( f \) satisfies the following second order homogeneous linear difference equation

\begin{equation}
f(n + 2) - f(1)f(n + 1) + f(n) = 0
\end{equation}

for each \( n \) in \( \mathbb{N} \) with \( f(0) = 1 \).

Suppose that \( f(1) = 2 \). Then from (2.3) we infer that \( f(n) = n + 1 \) and \( M = 1 \) which corresponds to the case \( \lambda = 0 \) in (2.2). Otherwise \( f(1) \neq 2 \) and let \( \lambda \neq 0 \) be a complex number with \( f(1) = 2 \cosh \lambda \). Then we have that

\[
f(n) = \alpha e^{n\lambda} + \beta e^{-n\lambda}
\]

holds for any \( n \) in \( \mathbb{N} \) with some complex numbers \( \alpha, \beta \) satisfying \( \alpha + \beta = 1 \).

It is easy to see that in this case

\[
f(n) = \frac{\sinh[(n + 1)\lambda]}{\sinh \lambda}
\]

holds for each \( n \) in \( \mathbb{N} \). Finally, we have

\[
M(n) = \frac{\sinh[(n + 1)\lambda]}{(n + 1) \sinh \lambda}.
\]

Conversely, it is easy to check that any function \( M \) of the given form is an exponential on the \( SU(2) \)-hypergroup, hence the theorem is proved. \( \diamond \)
3. Additive functions on the $SU(2)$-hypergroup

Now we describe the additive functions on the $SU(2)$-hypergroup. We recall that the function $A : \mathbb{N} \to \mathbb{C}$ is an additive function if and only if it satisfies

$$A(m) + A(n) = A(m \ast n) = \sum_{k=|m-n|}^{m+n} \frac{k+1}{(m+1)(n+1)} A(k)$$

for all natural numbers $m, n$.

**Theorem 2.** The function $A : \mathbb{N} \to \mathbb{C}$ is an additive function on the $SU(2)$-hypergroup if and only if there exists a complex number $c$ such that

$$A(n) = \frac{c}{3} n(n+2)$$

holds for each natural number $n$.

**Proof.** Let $A : \mathbb{N} \to \mathbb{C}$ be a solution of (3.1) and let $f(n) = (n+1)A(n)$ for each $n$ in $\mathbb{N}$. Then we have

$$(n+1)f(m) + (m+1)f(n) = \sum_{k=|m-n|}^{m+n} f(k)$$

for each $m, n$ in $\mathbb{N}$. With $m = 1$ it follows that $f$ satisfies the following second order homogeneous linear difference equation

$$f(n+2) - 2f(n+1) + f(n) = 2c(n+2)$$

for each $n$ in $\mathbb{N}$ with $f(0) = 0$ and $f(1) = 2c$. As the second difference of $f$ is linear it follows that $f$ is a cubic polynomial and simple computation gives that $A$ has the desired form.

Conversely, it is easy to check that any function $A$ of the given form is an additive function on the $SU(2)$-hypergroup, hence the theorem is proved. ♦

4. Generalized moment functions on the $SU(2)$-hypergroup

Finally we describe the generalized moment functions on the $SU(2)$-hypergroup. Let $N$ be a nonnegative integer. We recall that the functions
\( \varphi_0, \varphi_1, \ldots, \varphi_N : \mathbb{N} \to \mathbb{C} \) form a generalized moment function sequence if and only if they satisfy

\[
(4.1) \quad \varphi_k(m*n) = \sum_{j=0}^{k} \binom{k}{j} \varphi_j(m) \varphi_{k-j}(n)
\]

for all natural numbers \( m, n \) and for \( k = 0, 1, \ldots, N \).

Making use of the results in Sec. 2 we introduce the function

\[
(4.2) \quad \Phi(n, \lambda) = \frac{\sinh[(n+1)\lambda]}{(n+1) \sinh \lambda}
\]

for each \( n \) in \( \mathbb{N} \) and \( \lambda \neq 0 \) in \( \mathbb{C} \), while \( \Phi(n, 0) = 1 \) for each \( n \) in \( \mathbb{N} \).

The function \( \Phi : \mathbb{N} \times \mathbb{C} \to \mathbb{C} \) is called an exponential family for the SU(2)-hypergroup: each exponential on this hypergroup has the form \( n \mapsto \Phi(n, \lambda) \) with some unique \( \lambda \) in \( \mathbb{C} \), and, conversely, the function \( n \mapsto \Phi(n, \lambda) \) is an exponential on the SU(2)-hypergroup for every complex \( \lambda \).

**Theorem 3.** Let \( K \) denote the SU(2)-hypergroup and \( \Phi \) the exponential family given by (4.2). The functions \( \varphi_0, \varphi_1, \ldots, \varphi_N : K \to \mathbb{C} \) form a generalized moment sequence of order \( N \) on \( K \) if and only if there exist complex numbers \( c_j \) for \( j = 1, 2, \ldots, N \) such that

\[
(4.3) \quad \varphi_k(n) = \frac{d^k}{dt^k} \Phi(n, f(t))(0)
\]

holds for each \( n \) in \( \mathbb{N} \) and for \( k = 0, 1, \ldots, N \), where

\[
(4.4) \quad f(t) = \sum_{j=0}^{N} \frac{c_j t^j}{j!}
\]

for each \( t \) in \( \mathbb{C} \).

**Proof.** First we note that, by (1.3), we have for \( n \geq 1 \)

\[
(4.4) \quad \delta_n \ast \delta_1 = \sum_{k=n-1}^{n+1} \frac{k+1}{2(n+1)} \delta_k = \frac{n}{2(n+1)} \delta_{n-1} + \frac{n+2}{2(n+1)} \delta_{n+1},
\]

hence, by 3.2.1 Prop. in [1], \( K \) is a polynomial hypergroup, that is, there exists a sequence \( (P_n)_{n \in \mathbb{N}} \) of polynomials such that \( \deg P_n = n \) for
for each $x$ in $\mathbb{R}$ and $m, n$ in $\mathbb{N}$ with some nonnegative numbers $c(m, n, k)$, further we have

\[ \delta_m \ast \delta_n = \sum_{k=0}^{\infty} c(m, n, k) \delta_k \]

for each $m, n$ in $\mathbb{N}$. Here we shall determine this sequence of polynomials.

Our basic observation is that the function $\lambda \mapsto \Phi(n, \lambda)$ is a polynomial of $\cosh \lambda$ of degree $n$ for each $n$ in $\mathbb{N}$. We apply mathematical induction. For $n = 0$ and $n = 1$ we have by (4.2)

\[ \Phi(0, \lambda) = \frac{\sinh \lambda}{\sinh \lambda} = 1, \]
\[ \Phi(1, \lambda) = \frac{\sinh(2\lambda)}{2\sinh \lambda} = \cosh \lambda. \]

Suppose that for $k = 0, 1, \ldots, n$ there exists a polynomial $P_k$ of degree $k$ such that

\[ \Phi(k, \lambda) = P_k(\cosh \lambda) \]

holds. Clearly $P_0(x) = 1$ and $P_1(x) = x$. Then, by eq. (4.4), we have

\[ P_n(\cosh \lambda) \cosh \lambda = \frac{n}{2(n+1)} P_{n-1}(\cosh \lambda) + \frac{n+2}{2(n+1)} \Phi(n+1, \lambda), \]

that is

\[ \Phi(n+1, \lambda) = \frac{2(n+1)}{n+2} P_n(\cosh \lambda) \cosh \lambda - \frac{n}{n+2} P_{n-1}(\cosh \lambda), \]

and here the right-hand side is a polynomial of degree $n+1$ in $\cosh \lambda$: \[ P_{n+1}(x) = \frac{2(n+1)}{n+2} xP_n(x) - \frac{n}{n+2} P_{n-1}(x), \]

hence

\[ \Phi(n+1, \lambda) = P_{n+1}(\cosh \lambda), \]
which was to be proved.

Finally, we have for all $m, n$ in $\mathbb{N}$ and $\lambda$ in $\mathbb{C}$

$$P_n(cosh \lambda)P_m(cosh \lambda) = \Phi(n, \lambda)\Phi(m, \lambda) = \Phi(n \ast m, \lambda) =$$

$$= \sum_{k = |m-n|}^{m+n} \frac{k + 1}{(m+1)(n+1)} \Phi(k, \lambda) = \sum_{k = |m-n|}^{m+n} \frac{k + 1}{(m+1)(n+1)} P_k(cosh \lambda),$$

which implies

$$P_n(x)P_m(x) = \sum_{k = |m-n|}^{m+n} \frac{k + 1}{(m+1)(n+1)} P_k(x)$$

for each $x$ in $\mathbb{R}$ and $m, n$ in $\mathbb{N}$. This means that $K$ is the polynomial hypergroup associated to the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$. Then, by Th. 4 in [4], our statement follows. 

**Acknowledgement.** The research was supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. NK-81402.

**References**


