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GROWTH OF SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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Abstract: In this paper, we investigate the order and the hyper-order of solutions of the higher order linear differential equation

$$f^{(k)}+A_{k-1}e^{P_{k-1}(z)}f^{(k-1)}+\cdots+A_1e^{P_1(z)}f'+A_0e^{P_0(z)}f=0,$$

where $P_j(z)$ $(j=0,1,\cdots,k-1)$ are nonconstant polynomials such that $\deg P_j=n \geqslant 1$ and $A_j(z)$ $(j=0,1,\cdots,k-1)$ are meromorphic functions of finite order such that $\max\{\rho(A_j):j=0,1,\cdots,k-1\}<n$. Under some conditions, we prove that every meromorphic solution $f\not\equiv 0$ of the above equation is of infinite order. Then, we obtain an estimation of the hyper-order and the exponent of convergence of zeros of the solutions. Finally, we give an estimation of the exponent of convergence of zeros of the function $f-\varphi$, where $\varphi\not\equiv 0$ is a transcendental meromorphic function of finite order, while the meromorphic solution f of respective differential equation is of infinite order.

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1. Introduction and statement of results

In this paper, we use the standard notations of Nevanlinna's value distribution theory (see [17], [25]). In addition, we will use $\lambda(f)$ to denote the exponent of convergence of the zero-sequence of f, $\rho(f)$ to denote the order of growth of f. To express the rate of growth of meromorphic solutions of infinite order, we recall the following concept.

Definition 1.1 [19, 28]**.** Let $f(z)$ be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\longrightarrow+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

For the second order linear differential equation

$$f'' + e^{-z} f' + A(z) f = 0, \tag{1.1}$$

where $A(z)$ is an entire function of finite order, it is well-known that each solution f of eq. (1.1) is an entire function, and that if f_1, f_2 are two linearly independent solutions of (1.1), then by [11], there is at least one of f_1, f_2 must have infinite order. Hence, "most" solutions of (1.1) will have infinite order.

A natural question arises: What conditions on $A(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Several authors, such as, Frei [12], Ozawa [26], Gundersen [14], Langley [22], Amemiya and Ozawa [1] have studied this problem. They proved that when $A(z)$ is a nonconstant polynomial or $A(z)$ is a transcendental entire function with order $\rho(A) \neq 1$, then every solution $f \not\equiv 0$ of (1.1) has infinite order. In [8], Chen considered eq. (1.1) in the case when $\rho(A) = 1$ and obtained different results concerning the growth of its solutions.

Consider the second order linear differential equation

$$f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = 0, \tag{1.2}$$

where $P(z)$, $Q(z)$ are nonconstant polynomials, $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$ are entire functions such that $\rho(A_1) < \deg P(z)$, $\rho(A_0) < \deg Q(z)$. Gundersen showed in [16, p. 419] that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of (1.2) is of infinite order. If $\deg P(z) = \deg Q(z)$, then (1.2) may have nonconstant solutions of finite order. For instance $f(z) = e^z + 2$ satisfies $f'' + \frac{1}{2}e^z f' - \frac{1}{2}e^z f = 0$.

Chen [9] and Kwon [19] have investigated eq. (1.2) in the case when $\deg P(z) = \deg Q(z)$. Later, Chen and Shon [10] , Belaïdi [3, 4], Belaïdi and Abbas [5] extended the results of [9] and [19] for higher order linear differential equations. After this work Xiao and Chen [27] improved the results of [10] for a class of higher order linear differential equations and obtained the following result.

Theorem A [27]. *Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be entire functions with $\rho(A_j) < 1$ $(j = 0, 1, \cdots, k-1)$, a_j $(j = 0, 1, \cdots, k-1)$ be complex numbers (if $A_j \equiv 0$ we define $a_j = 0$, otherwise $a_j \neq 0$). Suppose that there exist $\{a_{i_1}, a_{i_2}, \cdots, a_{i_m}\} \subset \{a_0, a_1, \cdots, a_{k-1}\}$ such that $\arg a_{i_j}$ $(j = 1, 2, \cdots, m)$ are different from each other. Suppose further that for each $a_l \in \{a_0, a_1, \cdots, a_{k-1}\} - \{a_{i_1}, a_{i_2}, \cdots, a_{i_m}\}$ and $a_l \neq 0$, there exists some $a_{i_j} \in \{a_{i_1}, a_{i_2}, \cdots, a_{i_m}\}$ such that $a_l = c_l^{(i_j)} a_{i_j}$, where $0 < c_l^{(i_j)} < 1$, $l \in \{0, 1, \cdots, k-1\}$, $j = 1, 2, \cdots, m$. Then every transcendental solution of the equation*

$$f^{(k)} + A_{k-1}e^{a_{k-1}z}f^{(k-1)} + \cdots + A_1e^{a_1z}f' + A_0e^{a_0z}f = 0 \tag{1.3}$$

is of infinite order. Furthermore, if $a_0 = a_{i_{j_0}}$ or $a_0 = c_0^{(i_{j_0})} a_{i_{j_0}}$ $(0 < c_0^{(i_{j_0})} < 1)$ and $0 < c_0^{(i_{j_0})} \neq c_s^{(i_{j_0})}$, $s \in \{1, 2, \cdots, k-1\}$, where $a_{i_{j_0}} \in \{a_{i_1}, a_{i_2}, \cdots, a_{i_m}\}$, then every solution $f \not\equiv 0$ of (1.3) is of infinite order.

The main purpose of this paper is to extend the results of Th. A to some higher order linear differential equations. In fact, we will prove the following results.

Theorem 1.1. *Let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ be meromorphic functions that have finitely many poles and $\rho = \max\{\rho(A_j) : j = 0, 1, \cdots, k-1\} < n$. Let $P_j(z) = a_{n,j}z^n + a_{n-1,j}z^{n-1} + \cdots + a_{1,j}z + a_{0,j}$ $(j = 0, 1, \cdots, k-1)$ be polynomials, where $a_{0,j}, \cdots, a_{n,j}$ $(j = 0, 1, \cdots, k-1)$ are complex numbers (if $A_j \equiv 0$ we define $a_{k,j} = 0$ $(k = 0, 1, \cdots, n)$, otherwise $a_{k,j} \neq 0$ $(k = 0, 1, \cdots, n)$). Suppose that there exist $\{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\} \subset$ $\subset \{a_{n,0}, a_{n,1}, \cdots, a_{n,k-1}\}$ such that $\arg a_{n,i_j}$ $(j = 1, 2, \cdots, m)$ are different from each other. Suppose further that for each*

$$a_{n,l} \in \{a_{n,0}, a_{n,1}, \cdots, a_{n,k-1}\} - \{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\} \quad \text{and} \quad a_{n,l} \neq 0,$$

there exists some $a_{n,i_j} \in \{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\}$ such that $a_{n,l} = c_{n,l}^{(i_j)} a_{n,i_j}$, where $0 < c_{n,l}^{(i_j)} < 1$, $l \in \{0, 1, \cdots, k-1\}$, $j = 1, 2, \cdots, m$. Then, every transcendental meromorphic solution of the equation

$$f^{(k)} + A_{k-1}e^{P_{k-1}(z)}f^{(k-1)} + \cdots + A_1e^{P_1(z)}f' + A_0e^{P_0(z)}f = 0 \tag{1.4}$$

is of infinite order. Furthermore, if $a_{n,0} = a_{n,i_{j_0}}$ *or* $a_{n,0} = c_{n,0}^{(i_{j_0})} a_{n,i_{j_0}}$ $(0 < c_{n,0}^{(i_{j_0})} < 1)$ *and* $0 < c_{n,0}^{(i_{j_0})} \neq c_{n,s}^{(i_{j_0})}$, $s \in \{1, 2, \cdots, k-1\}$, *where* $a_{n,i_{j_0}} \in$ $\in \{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\}$, *then every meromorphic solution* $f \not\equiv 0$ *of* (1.4) *is of infinite order.*

Theorem 1.2. *Under the hypotheses of Th. 1.1, the following statements hold:*

(i) *Every transcendental meromorphic solution* f *of* (1.4) *satisfies* $\lambda(f) \geqslant n$ *or* $\rho_2(f) = n$.

(ii) *Furthermore, if* $a_{n,0} = a_{n,i_{j_0}}$ *or* $a_{n,0} = c_{n,0}^{(i_{j_0})} a_{n,i_{j_0}}$ $(0 < c_{n,0}^{(i_{j_0})} < 1)$ *and* $0 < c_{n,0}^{(i_{j_0})} \neq c_{n,s}^{(i_{j_0})}$, $s \in \{1, 2, \cdots, k-1\}$, *where* $a_{n,i_{j_0}} \in$ $\in \{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\}$, *then every meromorphic solution* $f \not\equiv 0$ *of* (1.4) *satisfies* $\lambda(f) \geqslant n$ *or* $\rho_2(f) = n$.

Theorem 1.3. *Under the hypotheses of Th. 1.1, suppose further that* $\varphi(z) \not\equiv 0$ *is a meromorphic function with finitely many poles and* $\rho(\varphi) <$ $< \infty$, *then:*

(i) *If* $\varphi(z)$ *is transcendental, then every transcendental meromorphic solution* f *of* (1.4) *satisfies* $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \infty$.

(ii) *Furthermore, if* $a_{n,0} = a_{n,i_{j_0}}$ *or* $a_{n,0} = c_{n,0}^{(i_{j_0})} a_{n,i_{j_0}}$ $(0 < c_{n,0}^{(i_{j_0})} < 1)$ *and* $0 < c_{n,0}^{(i_{j_0})} \neq c_{n,s}^{(i_{j_0})}$, $s \in \{1, 2, \cdots, k-1\}$, *where* $a_{n,i_{j_0}} \in$ $\in \{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\}$, *then every meromorphic solution* $f \not\equiv 0$ *of* (1.4) *satisfies* $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \infty$.

Remark 1.1. Clearly, the method used in linear differential equations with entire coefficients can not deal with the case of meromorphic coefficients. The present paper may be understood as an extension and improvement of the results of the papers of Xiao and Chen [27], Belaïdi [4], Belaïdi and Abbas [5] from entire solutions to meromorphic solutions.

2. Lemmas for the proof of theorems

By using the same proof as in the proof of Lemma 3.1 in [21], we easily obtain the following lemma.

Lemma 2.1. *Let* $f(z)$ *be a meromorphic function having finitely many poles all lie in* $\{z : |z| < r_0\}$, *and suppose that* $\left|f^{(k)}(z)\right|$ *is unbounded* *on a ray* $\arg z = \theta$, *then there exists a sequence* $z_n = r_n e^{i\theta}$ *tending to infinity such* $f^{(k)}(z_n) \to \infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \leqslant \frac{1}{(k-j)!}(1+o(1))|z_n|^{k-j} \quad (j=0,\cdots,k-1). \tag{2.1}$$

Lemma 2.2 [15, p. 89]**.** *Let* $f(z)$ *be a transcendental meromorphic function of finite order* ρ. *Let* $\Gamma = \{(k_1,j_1),(k_2,j_2),\cdots,(k_m,j_m)\}$ *denote a set of distinct pairs of integers satisfying* $k_i > j_i \geqslant 0$ $(i=1,2,\cdots,m)$ *and let* $\varepsilon > 0$ *be a given constant. Then, there exists a set* $E_1 \subset [0,2\pi)$ *that has linear measure zero such that if* $\psi_0 \in [0,2\pi) - E_1$, *then there is a constant* $R_0 = R_0(\psi_0) > 1$ *such that for all* z *satisfying* $\arg z = \psi_0$ *and* $|z| \geqslant R_0$ *and for all* $(k,j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.2}$$

Lemma 2.3 ([6], [24, p. 254])**.** *Let* $P(z) = a_n z^n + \cdots + a_0$, $(a_n = \alpha + i\beta \neq 0)$ *be a polynomial with degree* $n \geqslant 1$ *and* $A(z) \not\equiv 0$ *be a meromorphic function with* $\rho(A) < n$. *Set* $f(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P,\theta) = \alpha\cos n\theta - \beta\sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_2 \subset [0,2\pi)$ *that has linear measure zero, such that if* $\theta \in [0,2\pi) - (E_2 \cup E_3)$, *where* $E_3 = \{\theta \in [0,2\pi) : \delta(P,\theta) = 0\}$ *is a finite set, then for sufficiently large* $|z| = r$, *we have*

(i) *if* $\delta(P,\theta) > 0$, *then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} \leqslant |f(z)| \leqslant \exp\{(1+\varepsilon)\delta(P,\theta)r^n\}, \tag{2.3}$$

(ii) *if* $\delta(P,\theta) < 0$, *then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} \leqslant |f(z)| \leqslant \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}. \tag{2.4}$$

Lemma 2.4 [10]**.** *Let* $f(z)$ *be an entire function with* $\rho(f) = \rho < \infty$. *Suppose that there exists a set* $E_4 \subset [0,2\pi)$ *that has linear measure zero, such that for any ray* $\arg z = \theta_0 \in [0,2\pi) - E_4$, $\left|f\left(re^{i\theta_0}\right)\right| \leqslant Mr^k$, *where* $M = M(\theta_0) > 0$ *is a constant and* $k\ (>0)$ *is a constant independent of* θ_0. *Then* $f(z)$ *is a polynomial with* $\deg f \leqslant k$.

Lemma 2.5 [13, p. 30]**.** *Let* $P_1,\ P_2,\cdots,P_n$ $(n \geqslant 1)$ *be nonconstant polynomials with the degree in order* $d_1,d_2,\cdots,d_n$, *respectively. Suppose that when* $i \neq j$, *then* $\deg(P_i - P_j) = \max\{d_i,\ d_j\}$. *Let* $A(z) =$

$= \sum_{j=1}^{n} B_j(z)e^{P_j(z)}$, *where* $B_j(z) \not\equiv 0$ *are meromorphic functions satisfying* $\rho(B_j) < d_j$. *Then*

$$\rho(A) = \max_{1 \leqslant\ j \leqslant\ n} \{d_j\}. \tag{2.5}$$

Lemma 2.6 [17, 25]. *Let f be a meromorphic function, and let $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f), \tag{2.6}$$

where $S(r,f) = O(\log T(r,f) + \log r)$, *possibly outside a set* $E_5 \subset [0, +\infty)$ *with a finite linear measure* $m(E_5) < +\infty$. *If f is of finite order of growth, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r). \tag{2.7}$$

Lemma 2.7 [7]. *Let* $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ *be finite order meromorphic functions. If f is a meromorphic solution with $\rho(f) = +\infty$ of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1\ f' + A_0 f = F, \tag{2.8}$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

Lemma 2.8 [23]. *Suppose that $k \geqslant 2$ and $A_0, A_1, \cdots, A_{k-1}$ are meromorphic functions that have finitely many poles. Let* $\rho = \max\{\rho(A_j) : j = 0, 1, \cdots, k-1\}$ *and let $f(z)$ be a transcendental meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1\ f' + A_0 f = 0. \tag{2.9}$$

Then $\rho_2(f) \leqslant \rho$.

Lemma 2.9 [18, Th. 12.4]. *Let f be an entire function with $\rho(f) = \infty$. Then f can be represented in the form $f(z) = g(z)e^{h(z)}$, where $g(z)$ and $h(z)$ are entire functions such that*

$$\rho_2(f) = \max\left\{\rho_2(g), \rho_2(e^h)\right\}, \tag{2.10}$$

$$\rho_2(g) = \limsup_{r\longrightarrow+\infty} \frac{\log\log N\left(r, \frac{1}{g}\right)}{\log r} = \limsup_{r\longrightarrow+\infty} \frac{\log\log N\left(r, \frac{1}{f}\right)}{\log r}. \tag{2.11}$$

Lemma 2.10 [2]. *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \le h(r)$ outside of an exceptional set $E_6 \subset (0, +\infty)$ of finite linear measure. Then for any $\lambda > 1$, there exists $r_1 > 0$ such that $g(r) \le h(\lambda r)$ for all $r > r_1$.*

3. Proof of Theorem 1.1

Suppose that f is a transcendental meromorphic solution of (1.4). Then z_0 is a pole of f only if z_0 is a pole of one of $A_j(z)$ $(j=0,1,\cdots,k–1)$. Since $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ have only finitely many poles, then f has finitely many poles. We assert that any transcendental meromorphic solution of eq. (1.4) must have infinite order. Suppose that f is a transcendental meromorphic solution of eq. (1.4) satisfying $\rho(f) = = \alpha < \infty$. By Lemma 2.2, there exists a subset $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\theta \in [0, 2\pi) - E_1$, then there exists a constant $R_0 = R_0(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geqslant R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leqslant |z|^{k\alpha} \quad (0 \leqslant i < j \leqslant k). \tag{3.1}$$

We note

$$E_2 = \left\{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0,\ j = 0, 1, \cdots, k-1\right\} \cup \\ \cup \left\{\theta \in [0, 2\pi) : \delta(P_{i_j}, \theta) = \delta(P_{i_d}, \theta),\ 1 \leqslant d < j \leqslant m\right\}.$$

Then E_2 is a finite set. By Lemma 2.3, there are exceptional sets $H_j \subset \subset [0, 2\pi)$ $(j = 0, 1, \cdots, k-1)$, each of them has a linear measure zero, such that for all $z = re^{i\theta}$, $\theta \in [0, 2\pi) - (E_1 \cup E_2 \cup E_3)$ ($E_3 = \bigcup_{j=0}^{k-1} H_j$ is a set with linear measure zero) we have $\delta(P_j, \theta) \neq 0$ $(j = 0, 1, \cdots, k-1)$ and $\delta(P_{i_j}, \theta) \neq \delta(P_{i_d}, \theta)$ $(1 \leqslant d < j \leqslant m)$. Let

$$\delta_t = \delta(P_{i_t}, \theta) = \max\{\delta(P_{i_j}, \theta) : j = 1, 2, \cdots, m\}.$$

Then $\delta_t > 0$ or $\delta_t < 0$ since $\delta(P_j, \theta) \neq 0$. We consider two cases.

Case (i): $\delta_t > 0$. Let $\delta = \max\left\{0, \delta(P_{i_j}, \theta) : j \in \{1,\ 2, \cdots, m\} - \{t\}\right\}$. Then $0 \leqslant \delta < \delta_t$. Set $\delta = c'\delta_t$. Then $0 \leqslant c' < 1$. By Lemma 2.3, for any

$\varepsilon_1 (0 < \varepsilon_1 < \frac{1}{2})$, we have as $r \longrightarrow +\infty$

$$\left|A_{i_t}(z)e^{P_{i_t}(z)}\right| \geqslant \exp\left\{(1-\varepsilon_1)\delta_t r^n\right\}. \tag{3.2}$$

For $A_l e^{P_l(z)}$ $(l \neq i_t)$, we have $a_{n,l} = c_{n,l}^{(i_t)} a_{n,i_t}$ $(0 < c_{n,l}^{(i_t)} < 1)$ or $a_{n,l} = a_{n,i_j}$ $(j \neq t)$ or $a_{n,l} = c_{n,l}^{(i_j)} a_{i_j}$ $(j \neq t)$ $(0 < c_{n,l}^{(i_j)} < 1)$. Hence $\delta(P_l, \theta) = c_{n,l}^{(i_t)} \delta(P_{i_t}, \theta) = c_{n,l}^{(i_t)} \delta_t$ or $\delta(P_l, \theta) = \delta(P_{i_j}, \theta) \leqslant \delta$ or $\delta(P_l, \theta) = c_{n,l}^{(i_j)} \delta(P_{i_j}, \theta) \leqslant c_{n,l}^{(i_j)} \delta$. Let $c = \max\{c_{n,l}^{(i_t)}, c', c_{n,l}^{(i_j)} c'\}$, we have $0 \leqslant c < 1$. By Lemma 2.3, when $r \longrightarrow +\infty$, we have

$$\left|A_l e^{P_l(z)}\right| \leqslant \exp\{(1+\varepsilon_1) c_{n,l}^{(i_t)} \delta_t r^n \},$$

or

$$\left|A_l e^{P_l(z)}\right| \leqslant \exp\{(1+\varepsilon_1)\delta r^n\},$$

or

$$\left|A_l e^{P_l(z)}\right| \leqslant \exp\{(1+\varepsilon_1) c_{n,l}^{(i_t)} \delta r^n\}.$$

Thus

$$\left|A_l e^{P_l(z)}\right| \leqslant \exp\left\{(1+\varepsilon_1) c \delta_t r^n\right\} \quad (l \in \{0, 1, \cdots, k-1\} - \{t\}). \tag{3.3}$$

Now we affirm that $\left|f^{(i_t)}(z)\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(i_t)}(z)\right|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.1, there exists an infinite sequence of points $z_q = r_q e^{i\theta}$ such that as $q \longrightarrow +\infty$, we have $r_q \longrightarrow +\infty$, $f^{(i_t)}(z_q) \longrightarrow \infty$ and

$$\left|\frac{f^{(j)}(z_q)}{f^{(i_t)}(z_q)}\right| \leqslant \frac{1}{(i_t - j)!} \left|z_q\right|^{i_t - j} (1 + o(1)) \quad (j = 0, 1, \cdots, i_t). \tag{3.4}$$

Since $f^{(i_t)} \not\equiv 0$, by (1.4), (3.1), (3.3) and (3.4) as $z_q \longrightarrow \infty$ we have

$$\begin{aligned} \left|A_{i_t}(z_q)e^{P_{i_t}(z_q)}\right| \leqslant & \left|\frac{f^{(k)}(z_q)}{f^{(i_t)}(z_q)}\right| + \left|A_{k-1}(z_q)e^{P_{k-1}(z_q)}\right| \left|\frac{f^{(k-1)}(z_q)}{f^{(i_t)}(z_q)}\right| + \cdots + \\ & + \left|A_{i_t+1}(z_q)e^{P_{i_t+1}(z_q)}\right| \left|\frac{f^{(i_t+1)}(z_q)}{f^{(i_t)}(z_q)}\right| + \\ & + \left|A_{i_t-1}(z_q)e^{P_{i_t-1}(z_q)}\right| \left|\frac{f^{(i_t-1)}(z_q)}{f^{(i_t)}(z_q)}\right| + \cdots + \\ & + \left|A_1(z_q)e^{P_1(z_q)}\right| \left|\frac{f'(z_q)}{f^{(i_t)}(z_q)}\right| + \left|A_0(z_q)e^{P_0(z_q)}\right| \left|\frac{f(z_q)}{f^{(i_t)}(z_q)}\right| \leqslant \\ \leqslant & r_q^{(1+\alpha)k} \exp\left\{(1+\varepsilon_1) c \delta_t r_q^n\right\}. \end{aligned} \tag{3.5}$$

We can take $0<\varepsilon_1<\min\left\{\frac{1-c}{1+c},\frac{1}{2}\right\}$, then (3.5) is a contradiction to (3.2). Hence, $\left|f^{(i_t)}(z)\right|\leqslant M_1$ on $\arg z=\theta$. Therefore

$$|f(z)|\leqslant M_1 r^k \tag{3.6}$$

holds on $\arg z=\theta$.

Case (ii): $\delta_t<0$. Let $\delta=\max\left\{\delta(P_{i_j},\theta): j\in\{1,2,\cdots,m\}-\{t\}\right\}$. Then $\delta<\delta_t<0$. It follows that, there exists a constant c' such that $c'>1$ and $\delta=c'\delta_t$. We have for all $l\neq i_t$, the following cases for $\delta(P_l,\theta)$: $\delta(P_l,\theta)=c_{n,l}^{(i_t)}\delta(P_{i_t},\theta)=c_{n,l}^{(i_t)}\delta_t$ or $\delta(P_l,\theta)=\delta(P_{i_j},\theta)\leqslant\delta$ or $\delta(P_l,\theta)=$ $=c_{n,l}^{(i_j)}\delta(P_{i_j},\theta)\leqslant c_{n,l}^{(i_j)}\delta$. Set $c_1=\min\left\{c_{n,l}^{(i_t)},\ c',\ c_{n,l}^{(i_j)}c',\ 1\right\}$. Then $c_1>0$. By Lemma 2.3, for any ε_2 $(0<\varepsilon_2<\frac{1}{2})$, when $r\longrightarrow+\infty$ we have

$$\left|A_{i_t}e^{P_{i_t}(z)}\right|\leqslant\exp\left\{(1-\varepsilon_2)\delta_t r^n\right\},$$

and for $l\neq i_t$,

$$\left|A_l e^{P_l(z)}\right|\leqslant\exp\left\{(1-\varepsilon_2)c_{n,l}^{(i_t)}\delta_t r^n\right\},$$

or

$$\left|A_l e^{P_l(z)}\right|\leqslant\exp\left\{(1-\varepsilon_2)\delta r^n\right\},$$

or

$$\left|A_l e^{P_l(z)}\right|\leqslant\exp\left\{(1-\varepsilon_2)c_{n,l}^{(i_j)}\delta r^n\right\}.$$

Hence, for all $j=0,1,\cdots,k-1$

$$\left|A_j(z)e^{P_j(z)}\right|\leqslant\exp\left\{(1-\varepsilon_2)c_1\delta_t r^n\right\}. \tag{3.7}$$

If $\left|f^{(k)}(z)\right|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 2.1, there exists an infinite sequence of points $z_q'=r_q'e^{i\theta}$ such that as $q\longrightarrow+\infty$, we have $r_q'\longrightarrow+\infty$, $f^{(k)}(z_q')\longrightarrow\infty$ and

$$\left|\frac{f^{(j)}(z_q')}{f^{(k)}(z_q')}\right|\leqslant\frac{1}{(k-j)!}\left|z_q'\right|^{k-j}(1+o(1))\ \ (j=0,1,\cdots,k-1)\,. \tag{3.8}$$

Since $f^{(k)}\not\equiv 0$, by (1.4), (3.7) and (3.8), when $r_q'\longrightarrow+\infty$, we have

$$\begin{aligned}1\leqslant&\left|A_{k-1}(z_q')e^{P_{k-1}(z_q')}\right|\left|\frac{f^{(k-1)}(z_q')}{f^{(k)}(z_q')}\right|+\left|A_{k-2}(z_q')e^{P_{k-2}(z_q')}\right|\left|\frac{f^{(k-2)}(z_q')}{f^{(k)}(z_q')}\right|+\\&+\cdots+\left|A_1(z_q')e^{P_1(z_q')}\right|\left|\frac{f'(z_q')}{f^{(k)}(z_q')}\right|+\left|A_0(z_q')e^{P_0(z_q')}\right|\left|\frac{f(z_q')}{f^{(k)}(z_q')}\right|\leqslant\\&\leqslant(r_q')^k\exp\left\{(1-\varepsilon_2)c_1\delta_t(r_q')^n\right\}.\end{aligned} \tag{3.9}$$

Since $(r'_q)^k \exp\left\{(1-\varepsilon_2)(r'_q)^n c_1\delta_t\right\} \longrightarrow 0$ as $r'_q \longrightarrow +\infty$ we obtain a contradiction with the left term of the inequality (3.9). Hence $\left|f^{(k)}(z)\right| \leqslant M_2$ on $\arg z = \theta$ (M_2 is a positive constant). Thus

$$|f(z)| \leqslant M_2 r^k. \tag{3.10}$$

In the both cases (3.6) and (3.10), we have

$$|f(z)| \leqslant M r^k \tag{3.11}$$

holds on $\arg z = \theta$. Since $f(z)$ is a meromorphic function with finitely many poles, then we can write $f(z)$ on the form $f(z) = \frac{g(z)}{Q(z)}$ with $Q(z)$ is a polynomial and $g(z)$ is an entire function. We know that for all $z = re^{i\theta}$ and $r \geqslant r_0$, there is a natural number s ($s \geqslant \deg Q$) such that

$$|Q(z)| \leqslant r^s. \tag{3.12}$$

From (3.11) and (3.12), we have $|g(z)| \leqslant M r^{\beta}$ ($\beta = s + k$) for all $r \geqslant r_0$ and $\theta \in [0, 2\pi) - (E_1 \cup E_2 \cup E_3)$. By applying Lemma 2.4, we find that $g(z)$ is a polynomial of degree $\deg g \leqslant \beta$. Then $f(z)$ is a rational function, this contradicts the assumption that $f(z)$ is a transcendental function. Hence $\rho(f) = \infty$.

Furthermore, if $a_{n,i_{j_0}} = a_{n,0}$ with $a_{n,i_{j_0}} \in \{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\}$, then we assume that the solution $f(z)$ of eq. (1.4) is a rational function. Since we have $A_0 f e^{P_0} \not\equiv 0$ and we write

$$(A_s e^{P_s(z) - a_{n,s} z^n} f^{(s)} + A_t e^{P_t(z) - a_{n,t} z^n} f^{(t)}) e^{a_{n,s} z^n}$$

instead of

$$A_s f^{(s)} e^{P_s} + A_t f^{(t)} e^{P_t}$$

when $a_{n,s} = a_{n,t}$ ($a_{n,s}$, $a_{n,t} \in \{a_{n,1}, a_{n,2}, \cdots, a_{n,k-1}\}$). Consequently we can write eq. (1.4) in the form

$$A_0(z) f(z) e^{P_0(z)} + \sum_{j \neq 0} B_j(z) e^{a_{n,j} z^n} = 0, \tag{3.13}$$

where $B_j(z)$ are meromorphic functions with finitely many poles and of finite orders $\rho(B_j) < n$. We have $\arg a_{n,j} \neq \arg a_{n,0}$ or $\arg a_{n,j} = = \arg a_{n,0}$ but $|a_{n,j}| < |a_{n,0}|$, and that $a_{n,j} - a_{n,i} \neq 0$ when $i \neq j$ ($j \neq 0$). By Lemma 2.5, we find that the order of growth of the left side of eq. (3.13) is n, this contradicts the zero order of the right side of eq. (3.13).

Suppose that $a_{n,0} = c_{n,0}^{(i_{j_0})} a_{n,i_{j_0}}$ $(0 < c_{n,0}^{(i_{j_0})} < 1)$ and $0 < c_{n,0}^{(i_{j_0})} \neq \neq c_{n,s}^{(i_{j_0})}$, $s \in \{1, 2, \cdots, k-1\}$ and $a_{n,i_{j_0}} \in \{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\}$, we assume that the solution $f(z)$ of eq. (1.4) is a rational function. Thus, we have $A_0 f e^{P_0} \not\equiv 0$ and eq. (3.13) also holds. From the fact $\arg a_{n,j} \neq \arg a_{n,0}$ or $\arg a_{n,j} = \arg a_{n,0}$ but $0 < c_0^{(i_{j_0})} \neq c_s^{(i_{j_0})}$, $s \in \{1, 2, \cdots, k-1\}$ which means that $|a_{n,j}| \neq |a_{n,0}|$ and by Lemma 2.5 the order of growth of the left side of eq. (3.13) is n, this contradicts the zero order of the right side of eq. (3.13). Consequently, when $a_{n,i_{j_0}} = a_{n,0}$ or $a_{n,0} = c_{n,0}^{(i_{j_0})} a_{n,i_{j_0}}$ $(0 < c_{n,0}^{(i_{j_0})} < 1)$ and $0 < c_{n,0}^{(i_{j_0})} \neq c_{n,s}^{(i_{j_0})}$, $s \in \{1, 2, \cdots, k-1\}$ and $a_{n,i_{j_0}} \in \in \{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\}$, any meromorphic solution $f \not\equiv 0$ of eq. (1.4) is of infinite order.

4. Proof of Theorem 1.2

(i) Suppose that f is a transcendental meromorphic solution of (1.4). Since $A_j(z)$ $(j = 0, 1, \cdots, k-1)$ have only finitely many poles, then f has finitely many poles. Assume that f satisfies $\lambda(f) < n$. Then, we can write f in the form $f = \frac{\pi}{Q} e^h$, where π is an entire function with $\lambda(f) = \lambda(\pi) < n$, h is a transcendental entire function and Q is a polynomial. Put $g = \frac{f'}{f}$, then by using Lemma 2.6, we have

$$T(r, g) = T\left(r, \frac{f'}{f}\right) = m\left(r, \frac{f'}{f}\right) + N\left(r, \frac{f'}{f}\right)$$
$$= O(\log T(r, f)) + O(\log r) + N\left(r, \frac{f'}{f}\right)$$

holds for all r outside a set $E_5 \subset [0, +\infty)$ with a finite linear measure $m(E_5) < +\infty$. We know that

$$N(r, g) = N\left(r, \frac{f'}{f}\right) = \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) = O(\log r) + \overline{N}\left(r, \frac{1}{f}\right),$$

hence

$$T(r, g) \leqslant O(\log T(r, f)) + O(\log r) + \overline{N}\left(r, \frac{1}{f}\right), \quad r \notin E_5. \tag{4.1}$$

By Lemma 2.8, we have $\rho_2(f) \leqslant n$. It follows from (4.1), Lemma 2.10 and the fact $\lambda(f) < n$ that $\rho(g) \leqslant n$. We assert that $\rho(g) = n$. By

$g = \dfrac{f'}{f}$, we obtain (see [20, Lemma 2.3.7])

$$\frac{f^{(j)}}{f} = g^j + \frac{1}{2}j(j-1)g^{j-2}g' + G_{j-2}(g) \quad (j = 2, 3, \cdots, k), \tag{4.2}$$

where $G_{j-2}(g)$ is a differential polynomial of the meromorphic function g with constant coefficients and the degree no more than $j-2$. If $\rho(g) < n$, then by (4.2) we have

$$\rho\left(\frac{f^{(j)}}{f}\right) < n \quad (j = 2, 3, \cdots, k).$$

We have $A_{i_j}(z)\frac{f^{(i_j)}}{f} \not\equiv 0$ since f is transcendental and we can write the eq. (1.4) in the form

$$\sum_{j=1}^{m} A_{i_j}(z)\frac{f^{(i_j)}}{f}e^{P_{i_j}(z)} + \sum_{l} B_l(z)e^{a_{n,l}z^n} = 0, \tag{4.3}$$

where $B_l(z)$ are meromorphic functions with finitely many poles and of finite orders $\rho(B_l) < n$. We have $\arg a_{n,i_j}$ are different from each other in $\{a_{n,i_1}, a_{n,i_2}, \cdots, a_{n,i_m}\}$ and if $\arg a_{n,l} = \arg a_{n,i_j}$, then $|a_{n,l}| < \left|a_{n,i_j}\right|$. By Lemma 2.5, we find that the order of growth of the left side of eq. (4.3) is n, this contradicts the zero order of the right side of eq. (4.3). Hence $\rho(g) = n$.

Since $\rho_2(f) = \rho_2(\pi e^h)$, then by Lemma 2.9 we have $\rho_2(\pi e^h) = \rho(h) \leqslant n$. Suppose that $\rho(h) < n$. Then, it follows from $\dfrac{f'}{f} = \dfrac{\pi'}{\pi} - \dfrac{Q'}{Q} + h'$ that

$$\begin{aligned} T\left(r, \frac{f'}{f}\right) &\leqslant T\left(r, \frac{\pi'}{\pi}\right) + T\left(r, \frac{Q'}{Q}\right) + T(r, h') + O(1) = \\ &= m\left(r, \frac{\pi'}{\pi}\right) + \overline{N}\left(r, \frac{1}{\pi}\right) + O(\log r) + T(r, h') + O(1) = \\ &= O(\log r) + \overline{N}\left(r, \frac{1}{\pi}\right) + O(\log r) + T(r, h') + O(1) = \\ &= \overline{N}\left(r, \frac{1}{\pi}\right) + T(r, h') + O(\log r). \end{aligned} \tag{4.4}$$

By (4.4) and the fact $\lambda(\pi) < n$, we get $\rho(\frac{f'}{f}) = \rho(g) < n$, a contradiction to $\rho(g) = n$, hence $\rho(h) = n$, then $\rho_2(f) = n$.
(ii) By Th. 1.1, every meromorphic solution $f \not\equiv 0$ of eq. (1.4) is transcendental, by using the same reasoning as in (i), we obtain $\lambda(f) \geqslant n$ or $\rho_2(f) = n$.

5. Proof of Theorem 1.3

(i) Set $H_j(z) = A_j(z)e^{P_j(z)}$ $(j = 0, 1, \cdots, k-1)$. Assume that f is a transcendental meromorphic solution of eq. (1.4) and suppose that $\varphi(z) \not\equiv 0$ is a transcendental meromorphic function with finitely many poles and $\rho(\varphi) < \infty$. By Th. 1.1, we have $\rho(f) = \infty$. Set $g = f - \varphi$, then $\rho(g) = \infty$, substituing $f = g + \varphi$ into eq. (1.4) yields

$$(5.1)\qquad g^{(k)} + H_{k-1}g^{(k-1)} + \cdots + H_1 g' + H_0 g = = -\left(\varphi^{(k)} + H_{k-1}\varphi^{(k-1)} + \cdots + H_1\varphi' + H_0\varphi\right).$$

We assert that

$$\varphi^{(k)} + H_{k-1}\varphi^{(k-1)} + \cdots + H_1\varphi' + H_0\varphi \not\equiv 0$$

because if $\varphi^{(k)} + H_{k-1}\varphi^{(k-1)} + \cdots + H_1\varphi' + H_0\varphi = 0$, then φ is a transcendental meromorphic solution of eq. (1.4), by Th. 1.1 we have $\rho(\varphi) = \infty$ and this contradicts the fact $\rho(\varphi) < \infty$. Hence

$$(5.2)\qquad -\left(\varphi^{(k)} + H_{k-1}\varphi^{(k-1)} + \cdots + H_1\varphi' + H_0\varphi\right) \not\equiv 0.$$

By (5.1), (5.2) and Lemma 2.7, we obtain $\overline{\lambda}(g) = \lambda(g) = \rho(g) = \infty$. Therefore $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \infty$.

(ii) We have by Th. 1.1, every meromorphic solution $f \not\equiv 0$ of eq. (1.4) is transcendental, by using the same reasoning as in (i), we obtain $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \infty$.

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