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A NOTE ON THE NUMBER OF ABELIAN GROUPS OF A GIVEN ORDER

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Abstract: We point out an asymptotic formula for the power moments of the function $a(n)$, representing the number of non-isomorphic Abelian groups of order n. For the quadratic moment this improves an earlier result due to L. Zhang, M. Lü and W. Zhai.

Let $a(n)$ denote the number of non-isomorphic Abelian groups of order n. The arithmetic function a is multiplicative and for every prime power p^ν ($\nu \geq 1$), $a(p^\nu) = P(\nu)$ is the number of unrestricted partitions of ν. Thus, for every prime p, $a(p) = 1$, $a(p^2) = 2$, $a(p^3) = 3$, $a(p^4) = 5$, $a(p^5) = 7$, etc. Asymptotic properties of the function a were investigated by several authors. See, e.g., [1, Ch. 14], [2, Ch. 7] for historical surveys.

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It is known that

$$\sum_{n\le x} a(n) = A_1 x + A_2 x^{1/2} + A_3 x^{1/3} + R(x),$$

where $A_j := \prod_{k=1,k\ne j}^{\infty} \zeta(k/j)$ $(j = 1, 2, 3)$, ζ denoting the Riemann zeta function, and the best result for the error term is $R(x) \ll x^{1/4+\varepsilon}$ for every $\varepsilon > 0$, proved by O. Robert and P. Sargos [6]. The asymptotic behavior of the sum $\sum_{n\le x} 1/a(n)$ was investigated by W. G. Nowak [5].

An asymptotic formula for the quadratic moment of the function a, i.e., for $\sum_{n\le x}(a(n))^2$ was given by L. Zhang, M. Lü and W. Zhai [8].

In the present note we point out the following result for the r-th power moment of the function a.

For $\mathbf{j} = (j_1, \ldots, j_t) \in \mathbb{N}^t$ with $1 \le j_1 \le \ldots \le j_t$ consider the generalized divisor function $d(\mathbf{j}; n) := \sum_{d_1^{j_1}\cdots d_t^{j_t}=n} 1$ and let $\Delta(\mathbf{j}; x)$ stand for the remainder term in the related asymptotic formula, i.e.,

$$\sum_{n\le x} d(\mathbf{j}; n) = H(\mathbf{j}; x) + \Delta(\mathbf{j}; x),$$

where $H(\mathbf{j}; x)$ is the main term, cf. [2, Ch. 6]. Furthermore, let $\Delta_r(x) :=$ $:= \Delta((1, \underbrace{2, 2, \ldots, 2}_{2^r-1}); x)$.

Theorem 1. *Let $r \ge 2$ be a fixed integer. Assume that $\Delta_r(x) \ll$ $\ll x^{\alpha_r}(\log x)^{\beta_r}$, with $1/3 < \alpha_r < 1/2$. Then*

$$\sum_{n\le x}(a(n))^r = C_r x + x^{1/2} Q_{2^r-2}(\log x) + R_r(x),$$

where

$$C_r := \prod_p \left(1 + \sum_{\nu=2}^{\infty} \frac{(P(\nu))^r - (P(\nu-1))^r}{p^\nu}\right),$$

Q_{2^r-2} is a polynomial of degree $2^r - 2$ and $R_r(x) \ll x^{\alpha_r}(\log x)^{\beta_r}$ (is the same).

According to a recent result of E. Krätzel [3], $\Delta_2(x) \ll x^{45/127}(\log x)^5$, where $45/127 \approx 0{,}3543 \in (1/3, 1/2)$, hence the same is the remainder term for $\sum_{n\le x}(a(n))^2$. This improves $R_2(x) \ll x^{96/245+\varepsilon}$ with $96/245 \approx$ $\approx 0{,}3918$, obtained in [8] by reducing the error term to the Piltz divisor problem concerning $d_3(n)$.

If $r \ge 3$, then $\Delta_r(x) \ll x^{u_r+\varepsilon}$ for every $\varepsilon > 0$, where $u_r := \frac{2^{r+1}-1}{2^{r+2}+1} \in$ $\in (1/3, 1/2)$. See [2, Th. 6.10]. Therefore $R_r(x) \ll x^{u_r+\varepsilon}$ holds as well.

Th. 1 is a direct consequence of the next more general result, valid for a whole class of arithmetic functions. Let $\Delta_{k,\ell}(x) := \Delta((1, \underbrace{\ell, \ell, ..., \ell}_{k-1}); x)$.

Theorem 2. *Let f be a complex valued multiplicative arithmetic function. Assume that*

i) $f(p) = f(p^2) = \cdots = f(p^{\ell-1}) = 1$, $f(p^\ell) = k$ *for every prime p, where $\ell, k \geq 2$ are fixed integers,*

ii) $f(p^\nu) \ll 2^{\nu/(\ell+1)}$ $(\nu \to \infty)$ *uniformly for the primes p, i.e., there is a constant C such that $|f(p^\nu)| \leq C \cdot 2^{\nu/(\ell+1)}$ for every prime p and every sufficiently large ν.*

Then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s)\zeta^{k-1}(\ell s)V(s),$$

absolutely convergent for $\Re(s) > 1$, where the Dirichlet series $V(s)$ is absolutely convergent for $\Re(s) > 1/(\ell+1)$.

Furthermore, suppose that $\Delta_{k,\ell} \ll x^{\alpha_{k,\ell}}(\log x)^{\beta_{k,\ell}}$, with $1/(\ell+1) < \alpha_{k,\ell} < 1/\ell$. Then

$$\sum_{n \leq x} f(n) = C_f x + x^{1/\ell} P_{f,k-2}(\log x) + R_f(x),$$

where $P_{f,k-2}$ is a polynomial of degree $k-2$,

$$C_f := \prod_p \left(1 + \sum_{\nu=\ell}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu}\right),$$

and $R_f(x) \ll x^{\alpha_{k,\ell}}(\log x)^{\beta_{k,\ell}}$.

Note that for every $k, \ell \geq 2$, $\Delta_{k,\ell}(x) \ll x^{u_{k,\ell}+\varepsilon}$, where $u_{k,\ell} := \frac{2k-1}{3+(2k-1)\ell} \in (1/(\ell+1), 1/\ell)$. See [2, Th. 6.10]. Therefore $R_f(x) \ll x^{u_{k,\ell}+\varepsilon}$ is valid as well.

Proof. This is a variation of the theorem proved in [7]. Here the same proof works out, however the conditions are somewhat relaxed. Let $\mu_\ell(n) = \mu(m)$ or 0, according as $n = m^\ell$ or not, where μ is the Möbius function. Let $V(s) := \sum_{n=1}^{\infty} v(n)/n^s$. We obtain the desired Dirichlet series representation by taking $v = f * \mu * \underbrace{\mu_\ell * \cdots \mu_\ell}_{k-1}$ in terms of the Dirichlet convolution $*$.

Here v is multiplicative and easy computations show that $v(p^\nu) = 0$ for any $1 \leq \nu \leq \ell$. For $\nu \geq \ell + 1$,

$$v(p^{\nu}) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \left(f(p^{\nu-j\ell}) - f(p^{\nu-j\ell-1}) \right),$$

leading to the absolute convergence of $V(s)$ for $\Re(s) > 1/(\ell+1)$. Now the asymptotic formula follows from the representation

$$f(n) = \sum_{ab=n} d((1, \underbrace{\ell, \ell, \ldots, \ell}_{k-1}); a) v(b). \qquad \diamondsuit$$

Choosing $f(n) = (a(n))^r$, $k = 2^r$ and $\ell = 2$ we deduce Th. 1. Note that $P(\nu) < e^{\pi\sqrt{2\nu/3}}$ $(\nu \geq 1)$, see e.g., [4, p. 236], thus condition ii) is verified.

Th. 2 applies also for the r-th powers ($r \geq 2$ integer) of the exponential divisor function $\tau^{(e)}$ and the function $\phi^{(e)}$, where $\phi^{(e)}$ is multiplicative and $\phi^{(e)}(p^{\nu}) = \phi(\nu)$ for every prime power p^{ν} $(\nu \geq 1)$, ϕ denoting Euler's function. See [3, 7].

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