LINEAR RECURRENCE RELATIONS ASSOCIATED WITH MULTINOMIAL PASCAL TRIANGLES

Takao Komatsu
Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan

Dedicated to the memory of Professor Gyula I. Maurer (1927–2012)

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Abstract: We consider linear recurrence relations associated with the sum of elements lying on a finite ray crossing a multinomial Pascal triangle. In the classical Pascal’s triangle the recurrence relations associated with the sum of diagonal elements lying along a finite ray have already been described. We also discuss an extended Lagrange’s identity.

1. Introduction

In [1, 2] we described the recurrence relations associated with the sum of diagonal elements lying along a finite ray crossing Pascal’s triangle. We shall consider similar linear recurrence relations in a more general triangle. We associate the elements \( \binom{n}{k}_s \) \((n = 0, 1, \ldots; 0 \leq k \leq sn)\) of the s-multinomial (or Generalized) Pascal triangle with points of the lattice \( \mathbb{Z} \times \mathbb{Z} \) by the map \((n, k) \rightarrow \binom{n}{k}_s\). Here, \( \binom{n}{k}_s \) are the coefficients appearing in the multinomial \((1+x+x^2+\cdots+x^{s-1})^n\). In the s-multinomial (or Generalized) Pascal triangle

\[
\binom{n}{k}_s = \binom{n-1}{k-1}_s + \binom{n-1}{k-s+1}_s + \cdots + \binom{n-1}{k}_s
\]

E-mail address: komatsu@cc.hirosaki-u.ac.jp
with the convention \( \binom{n}{k}_s = 0 \) for \( k > sn \) or \( k < 0 \) and

\[
\sum_{k=0}^{sn} \binom{n}{k}_s = s^n
\]

(1.2)

hold (see also [3, 4, 10]). If \( s = 2 \), the triangle is reduced to Pascal’s triangle with binomial coefficients \( \binom{n}{k}_2 \). If \( s = 3 \), the triangle is called Trinomial triangle ([5, Ch. 29], [9, A027907], [4]) with Trinomial coefficients \( \binom{n}{k}_3 \), illustrated as follows.

\[
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 2 & 1 & 1 & 1 & 1 & 1 \\
1 & 3 & 6 & 7 & 6 & 3 & 1 & 1 & 1 \\
1 & 4 & 10 & 16 & 16 & 10 & 4 & 1 & 1 \\
1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 1 \\
1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & \ldots \\
1 & 7 & 28 & 77 & 161 & 266 & 357 & 393 & 357 & 266 & 161 & \ldots \\
1 & 8 & 36 & 112 & 266 & 504 & 784 & 1016 & 1107 & 1016 & 784 & \ldots \\
1 & 9 & 45 & \ldots \\
1 & 10 & 55 & \ldots \\
1 & 11 & \ldots \\
\ldots
\end{array}
\]

Figure 1. Trinomial triangle

Let \( r, q \) and \( p \) be integers with \( r > 0, r + q > 0 \) and \( 1 \leq p \leq r - 1 \). Set

\[
T^{(r,q,p)}_{n+1} := \sum_{k=0}^{\left[ \frac{sn-p}{r+q} \right]} T^{(r,q,p)}(n,k)
\]

with

\[
T^{(r,q,p)}(n,k) = \binom{n-qk}{p+rk}_s a^{sn-p-(sq+r)k} b^{p+rk}
\]

The pair \((r, q)\) stands for \( r \) steps east and \( q \) steps north and describes the direction of a diagonal ray in a multinomial Pascal triangle. The variable \( p \) defines the order in the intermediate ray, which is the ray between two rays of the direction \((r, q)\) if such a ray exists. The variables \( a \) and \( b \) play the role to weigh the sums: \( a \) is the weight in the vertical

\[^1\]In some literature, \( \binom{n}{k}_s = \binom{n}{k} \) denotes binomial coefficients, \( \binom{n}{k}_2 \) trinomial coefficients, \( \binom{n}{k}_3 \) quadrinomial coefficients and so on.
direction and $b$ is in the horizontal direction. The case $s = 1$ is defined in [1, 2]. The same quantities $T_n^{(r,q,p)}$ with $p = 0$ and $a = b = 1$ are considered and analyzed in [4]. For instance, the sequence $\{T_n^{(1,3,0)}\}_{n \geq 1} = 1, 1, 1, 1, 2, 3, 4, 6, 9, 13, 18, 26, 38, \ldots$ corresponding to the rays with direction $(r,q) = (1,3)$ can be obtained by following the arrows in the trinomial triangle as represented in Fig. 2. Then, we can find that

$$T_{n+1} := T_n^{(1,3,0)} = \sum_{k=0}^{[2n]} \binom{n-3k}{k} a 2^{n-7k} b^k$$

satisfies the relation $T_n = a^2 T_{n-1} + ab T_{n-4} + b^2 T_{n-7} (n \geq 2)$ with $T_1 = 1, T_0 = T_{-1} = \cdots = T_{-5} = 0$. The example depicted in Fig. 2 is the case where $a = b = 1$.

In this paper, we describe a general recurrence relation, which is satisfied by $T_{n+1}^{(r,q,p)}$ with $r = 1$ and $p = 0$ in the multinomial Pascal triangle.

![Figure 2](image-url)

Figure 2. The sequence $\{T_n^{(1,3,0)}\}_{n \geq 1}$ with $a = b = 1$ in the trinomial triangle
2. Main result

The main theorem states a more general situation in the multinomial (s-nomial) Pascal triangle.

**Theorem 1.** Let $s \geq 1$. Then for $q \geq 1$

\[
T_{n+1} := T_{n+1}^{(1,q,0)} = \sum_{k=0}^{\left\lfloor \frac{2n}{s+1} \right\rfloor} \binom{n-k}{k} a^{sn-(sq+1)k} b^k
\]

satisfies the relation

\[
T_{n+1} = a^s T_n + a^{s-1} b T_{n-q} + \cdots + ab^{s-1} T_{n-(s-1)q} + b^s T_{n-sq} \quad (n \geq 1)
\]

with

\[
T_1 = 1 \quad \text{and} \quad T_0 = T_{-1} = \cdots = T_{1-sq} = 0.
\]

**Example.** If $q = 1$, the sequence $\{T_n\}_{n \geq 1}$ means the weighted sum of $(r, q) = (1, 1)$ direction:

\[
\begin{align*}
T_1 &= 1, \\
T_2 &= a^2, \\
T_3 &= a^4 + ab, \\
T_4 &= a^6 + 2a^3 b + b^2,
\end{align*}
\]

\[
T_{n+1} := T_{n+1}^{(1,1,0)}(n, k) = \sum_{k=0}^{\left\lfloor \frac{2n}{3} \right\rfloor} \binom{n-k}{k} a^{2n-3k} b^k
\]

satisfies the relation

\[
T_{n+1} = a^2 T_n + ab T_{n-1} + b^2 T_{n-2} \quad (n \geq 1) \quad \text{with} \quad T_1 = 1, \quad T_0 = T_{-1} = 0.
\]

**Remark.** If $q = 1$ and $s = 1$, we have the nice well-known identity

\[
F_{n+1} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k}
\]

for Fibonacci numbers $F_n$ ([9, A000045]). If $q = 1$ and $s = 2$, then we have the identity for Tribonacci numbers, satisfying $T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad (n \geq 4)$ with $T_1 = T_2 = T_3 = 1$ ([4] [9, A000073]). If $q = 2$ and $s = 2$, then $T_n$ corresponds to the number of ordered partitions of $n$ into 1’s, 3’s and 5’s ([9, A060961]).
Figure 3. The sequence \( \{T_n^{(1,1,0)}\}_{n \geq 1} \) in the trinomial triangle
Proof of Theorem 1.
\(a^s T_n + a^{s-1} b T_{n-q} + \cdots + a b^{s-1} T_{n-(s-1)q} + b^s T_{n-sq} =\)

\[
= \sum_{k=0}^{\left\lfloor \frac{sn}{sq+1} \right\rfloor} \binom{n-qk-1}{k} s^{sn-(sq+1)k} b^k + \]

\[
+ \sum_{k=0}^{\left\lfloor \frac{s(n-q)}{sq+1} \right\rfloor} \binom{n-q(k+1)-1}{k} s^{sn-(sq+1)(k+1)} b^{k+1} + \]

\[
+ \cdots + \]

\[
+ \sum_{k=0}^{\left\lfloor \frac{s(n-sq-1)}{sq+1} \right\rfloor} \binom{n-q(k+s)-1}{k} s^{sn-(sq+1)(k+s)} b^{k+s} = \]

\[
= \sum_{k=0}^{\left\lfloor \frac{s}{sq+1} \right\rfloor} \binom{n-qk-1}{k} s^{sn-(sq+1)k} b^k + \sum_{k=1}^{\left\lfloor \frac{s(n-sq-1)}{sq+1} \right\rfloor} \binom{n-qk-1}{k-1} s^{sn-(sq+1)k} b^k + \]

\[
+ \cdots + \]

\[
+ \sum_{k=s-1}^{\left\lfloor \frac{s}{sq+1} \right\rfloor} \binom{n-qk-1}{k-s+1} s^{sn-(sq+1)k} b^k + \sum_{k=s}^{\left\lfloor \frac{s}{sq+1} \right\rfloor} \binom{n-qk-1}{k-s} s^{sn-(sq+1)k} b^k. \]

Notice that
\[
\binom{n-1}{0} = 1 = \binom{n}{0} \quad \text{for} \quad k = 0, \]
\[
\binom{n-q-1}{0} + \binom{n-q-1}{1} = \binom{n-q}{1} \quad \text{for} \quad k = 1, \]
\[
\cdots \]
\[
\binom{n-q(s-1)-1}{0} + \binom{n-q(s-1)-1}{1} + \cdots + \binom{n-q(s-1)-1}{s-1} = \]
\[
= \binom{n-q(s-1)}{s-1} \quad \text{for} \quad k = s-1, \]

and for \(s \leq k \leq \left\lfloor \frac{sn-s}{sq+1} \right\rfloor\)
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\[
\binom{n-qk-1}{k-s} + \binom{n-qk-1}{k-s+1} + \cdots + \binom{n-qk-1}{k-1} + \binom{n-qk-1}{k} = \binom{n-qk}{k}. 
\]

In addition, if
\[
\left\lfloor \frac{sn-i-1}{sq+1} \right\rfloor < \frac{sn-i}{sq+1}
\]
or
\[
\frac{sn-i-1}{sq+1} < \left\lfloor \frac{sn-i-1}{sq+1} \right\rfloor + 1 \leq \frac{sn-s+1}{sq+1}
\]
for some integer \(i\) with \(0 \leq i \leq s-1\), then by
\[
s \left( n - q \left\lfloor \frac{sn-i}{sq+1} \right\rfloor \right) < \frac{sn-i}{sq+1}
\]
we have
\[
\binom{n-q \left\lfloor \frac{sn-i}{sq+1} \right\rfloor - 1}{s} = \cdots = \binom{n-q \left\lfloor \frac{sn-i}{sq+1} \right\rfloor - s + i + 1}{s} = 0,
\]
so,
\[
\binom{n-q \left\lfloor \frac{sn-i}{sq+1} \right\rfloor - 1}{s} + \binom{n-q \left\lfloor \frac{sn-i}{sq+1} \right\rfloor - s + 1}{s} + \cdots + \binom{n-q \left\lfloor \frac{sn-i}{sq+1} \right\rfloor - s + i}{s} = \binom{n-q \left\lfloor \frac{sn-i}{sq+1} \right\rfloor}{s}.
\]

Therefore,
\[
a^sT_n + a^{s-1}bT_n-q + \cdots + ab^{s-1}T_{n-(s-1)q} + b^sT_{n-sq} = \\
= \sum_{k=0}^{\left\lfloor \frac{sn}{sq+1} \right\rfloor} \binom{n-qk}{k} a^{sn-(sq+1)k} b^k = T_{n+1}. \quad \diamond
\]

The case \(q = 0\) corresponds to horizontal lines in the triangle. This case can be stated as follows.
Corollary 1.

\[ T_{n+1} := T_{n+1}^{(1,0,0)} = \sum_{k=0}^{sn} \binom{n}{k} a^{sn-k} b^k \quad (n \geq 0) \]

is equivalent to

\[ T_n = (a^s + a^{s-1} b + \cdots + ab^{s-1} + b^s)^{n-1} \quad (n \geq 1). \]

Remark. If \( a = b = 1 \) in Cor. 1, this case is reduced to (1.2).

3. An extended Lagrange’s identity

Suppose that each element in multinomial Pascal’s triangle is replaced by the square of the corresponding element. Then the \( n \)-th row sum of the resulting triangle is

\[ \binom{2n}{sn}_s \quad (n = 0, 1, 2, \ldots). \]

This is a special case of the following theorem.

Theorem 2. For \( 0 \leq l \leq 2sn \)

\[ \binom{2n}{l}_s = \sum_{i=0}^{l} \binom{n}{i}_s \binom{n}{l-i}_s = \sum_{i=0}^{l} \binom{n}{i}_s \binom{n}{sn-l+i}_s. \]

Proof. By the definition of the coefficients in generalized Pascal’s triangles,

\[ (1 + x + x^2 + \cdots + x^s)^n = \sum_{i=0}^{sn} \binom{n}{i}_s x^i. \]

Hence,

\[ (1 + x + x^2 + \cdots + x^s)^{2n} = \sum_{l=0}^{2sn} \binom{2n}{l}_s x^l. \]

On the other hand,

\[ (1 + x + x^2 + \cdots + x^s)^{2n} = (1 + x + x^2 + \cdots + x^s)^n (1 + x + x^2 + \cdots + x^s)^n = \]

\[ = \sum_{i=0}^{sn} \sum_{j=0}^{sn} \binom{n}{i}_s \binom{n}{j}_s x^{i+j} = \]

\[ = \sum_{l=0}^{2sn} \sum_{i=0}^{l} \binom{n}{i}_s \binom{n}{l-i}_s x^l. \]

Equating the coefficients of \( x^l \), we have the desired identity. ◇
By putting \( l = sn \) in above theorem, we have

**Corollary 2.**

\[
\binom{2n}{sn}_s = \sum_{i=0}^{sn} \left( \binom{n}{i}_s \right)^2.
\]

**Remark.** If \( s = 1 \), then

\[
\binom{2n}{n}_s = \sum_{i=0}^{n} \left( \binom{n}{i} \right)^2,
\]

which is Lagrange’s identity ([5, Th. 5.1 and p. 130–131]). If \( s = 2 \), then

we have the identity in the trinomial triangle:

\[
\binom{2n}{2n}_2 = \sum_{i=0}^{2n} \left( \binom{n}{i}_s \right)^2.
\]

### 4. Riordan arrays

As stated in [7] a Riordan array is a pair \((d(t), h(t))\) where \( d \) and \( h \) are analytic functions and \( d(0) \neq 0 \). This pair then defines an infinite lower triangular array \( \{d_{n,k}\} \), where

\[
\sum_{n=0}^{\infty} d_{n,k} t^n = d(t)(t \cdot h(t))^k.
\]

From this definition, \( d(t)(t \cdot h(t))^k \) is the generating function of column \( k \) in the array. It is known that Pascal triangle \( \{P_{n,k}\}_{n,k \geq 0} \) is represented by a Riordan array:

\[
\sum_{n=0}^{\infty} P_{n,k} t^n = \frac{1}{1 - t} \left( \frac{t}{1 - t} \right)^k \quad (k \geq 0)
\]

(e.g. [6, 11]). However, for the coefficients \( \{d_{n,k}\} \) of some \( s \)-multinomial triangle, there are no two analytic functions \( d(t) \) and \( h(t) \), satisfying

\[
\sum_{n=0}^{\infty} d_{n,k} t^n = d(t)(t \cdot h(t))^k.
\]

For example, in the case of the trinomial triangle \( \{d_{n,k}\}_{n,k \geq 0} \), we have \( d_{n,0} = 1 \) \((n \geq 0)\). Hence, \( d(t) = 1/(1 - t) \) because \( d(t) \) is the generating function of column 0. So, there exists a function \( f(t) \), satisfying

\[
\sum_{n=0}^{\infty} d_{n,k} t^n = \frac{1}{1 - t} (f(t))^k \quad (k \geq 0).
\]

By the second column with \( d_{n,1} = n \) \((n \geq 0)\), we have
\[
\frac{1}{1-t} f(t) = t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + \cdots = \frac{t}{(1-t)^2}.
\]

Thus,
\[
f(t) = \frac{t}{1-t}.
\]

But, by the third column with \(d_{n,2} = n(n+1)/2 \ (n \geq 0)\), we have
\[
\frac{1}{1-t} (f(t))^2 = t + 3t^2 + 6t^3 + 10t^4 + 15t^5 + \cdots = \frac{t}{(1-t)^3}.
\]

Thus,
\[
f(t) = \sqrt{\frac{t}{1-t}}.
\]

Furthermore, the relation
\[
\frac{1}{1-t} (f(t))^3 = \sum_{n=0}^{\infty} \frac{(n-1)n(n+4)}{2} t^n
\]
gives a still different function \(f(t)\).

We may also use the following result ([7, Th. 2.1], [8]) to see the non-existence of Riordan array.

**Lemma 1.** An array \(\{d_{n,k}\}_{n,k \geq 0}\) is a Riordan array with \(d(0) \neq 0\) and \(h(0) \neq 0\) if and only if there exists a sequence \(A = \{a_i\}_{i \geq 0}\) with \(a_0 \neq 0\) such that every element \(d_{n+1,k+1} \ (n,k \geq 0)\) can be expressed as a linear combination with coefficients in \(A\) of the elements in the preceding row, starting from the preceding column on, namely
\[
d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots.
\]

In the case of the trinomial triangle \(\{d_{n,k}\}_{n,k \geq 0}\), only the relation
\[
d_{n+1,k+1} = 1 \cdot d_{n,k-1} + 1 \cdot d_{n,k} + 1 \cdot d_{n,k+1}
\]
holds. So, there does not exist such a sequence \(A\).

5. **Future works**

More general rays where \(r \neq 1\) and/or \(p \neq 0\) may be treated similarly. For the moment, we have only a very special result, where \(r = 2\) with \(q = 1\), \(p = 0\) and \(a = b = 1\). Namely,
\[
T_{n+1} := T^{(2,1,0)}_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{2k},
\]
satisfies the relation
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\[ T_{n+1} = T_n + 2T_{n-1} + T_{n-2} - 1 \quad (n \geq 2) \]

with \( T_1 = T_2 = 1 \) and \( T_0 = 0 \).

A general result in the cases \( r \geq 2 \) will be considered in the future works.

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References


