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LINEAR RECURRENCE RELATIONS ASSOCIATED WITH MULTINOMIAL PASCAL TRIANGLES

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Dedicated to the memory of Professor Gyula I. Maurer (1927–2012)

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Abstract: We consider linear recurrence relations associated with the sum of elements lying on a finite ray crossing a multinomial Pascal triangle. In the classical Pascal's triangle the recurrence relations associated with the sum of diagonal elements lying along a finite ray have already been described. We also discuss an extended Lagrange's identity.

1. Introduction

In [1, 2] we described the recurrence relations associated with the sum of diagonal elements lying along a finite ray crossing Pascal's triangle. We shall consider similar linear recurrence relations in a more general triangle. We associate the elements $\binom{n}{k}_s$ $(n = 0, 1, 2, \dots;\ 0 \le k \le sn)$ of the s-multinomial (or Generalized) Pascal triangle with points of the lattice $\mathbb{Z}\times\mathbb{Z}$ by the map $(n, k) \to \binom{n}{k}_s$. Here, $\binom{n}{k}_s$ are the coefficients appearing in the multinomial $(1+x+x^2+\dots+x^{s-1})^n$. In the s-multinomial (or Generalized) Pascal triangle

$$(1.1)\qquad \binom{n}{k}_s = \binom{n-1}{k-s}_s + \binom{n-1}{k-s+1}_s + \dots + \binom{n-1}{k}_s$$

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with the convention $\binom{n}{k}_s = 0$ for $k > sn$ or $k < 0$ and

$$\sum_{k=0}^{sn} \binom{n}{k}_s = s^n \tag{1.2}$$

hold (see also [3, 4, 10]). If $s = 2$, the triangle is reduced to Pascal's triangle with binomial coefficients $\binom{n}{k} = \binom{n}{k}_2$. If $s = 3$, the triangle is called *Trinomial triangle* ([5, Ch. 29], [9, A027907], [4]) with Trinomial coefficients $\binom{n}{k}_3$ [1], illustrated as follows.

1											
1	1	1									
1	2	3	2	1							
1	3	6	7	6	3	1					
1	4	10	16	19	16	10	4	1			
1	5	15	30	45	51	45	30	15	5	1	
1	6	21	50	90	126	141	126	90	50	21	...
1	7	28	77	161	266	357	393	357	266	161	...
1	8	36	112	266	504	784	1016	1107	1016	784	...
1	9	45	...								
1	10	55	...								
1	11	...									
...											

Figure 1. Trinomial triangle

Let r, q and p be integers with $r > 0$, $r + q > 0$ and $1 \le p \le r - 1$. Set

$$T_{n+1}^{(r,q,p)} := \sum_{k=0}^{\left\lfloor \frac{sn-p}{r+sq} \right\rfloor} T^{(r,q,p)}(n,k)$$

with

$$T^{(r,q,p)}(n,k) = \binom{n-qk}{p+rk}_s a^{sn-p-(sq+r)k} b^{p+rk} .$$

The pair (r, q) stands for r steps east and q steps north and describes the direction of a *diagonal ray* in a multinomial Pascal triangle. The variable p defines the *order* in the *intermediate ray*, which is the ray between two rays of the direction (r, q) if such a ray exists. The variables a and b play the role to weigh the sums: a is the weight in the vertical

[1] In some literature, $\binom{n}{k} = \binom{n}{k}_1$ denotes binomial coefficients, $\binom{n}{k}_2$ trinomial coefficients, $\binom{n}{k}_3$ quadrinomial coefficients and so on.

direction and b is in the horizontal direction. The case $s=1$ is defined in [1, 2]. The same quantities $T_n^{(r,q,p)}$ with $p=0$ and $a=b=1$ are considered and analyzed in [4]. For instance, the sequence $\{T_n^{(1,3,0)}\}_{n\geq 1} = 1,1,1,1,2,3,4,6,9,13,18,26,38,\ldots$ corresponding to the rays with direction $(r,q)=(1,3)$ can be obtained by following the arrows in the trinomial triangle as represented in Fig. 2. Then, we can find that $T_{n+1} := T_{n+1}^{(1,3,0)} = \sum_{k=0}^{\lfloor \frac{2n}{7} \rfloor} \binom{n-3k}{k}_2 a^{2n-7k} b^k$ satisfies the relation $T_n = a^2T_{n-1} + abT_{n-4} + b^2T_{n-7}$ $(n \geq 2)$ with $T_1=1$, $T_0=T_{-1}=\cdots=T_{-5}=0$. The example depicted in Fig. 2 is the case where $a=b=1$.

In this paper, we describe a general recurrence relation, which is satisfied by $T_{n+1}^{(r,q,p)}$ with $r=1$ and $p=0$ in the multinomial Pascal triangle.

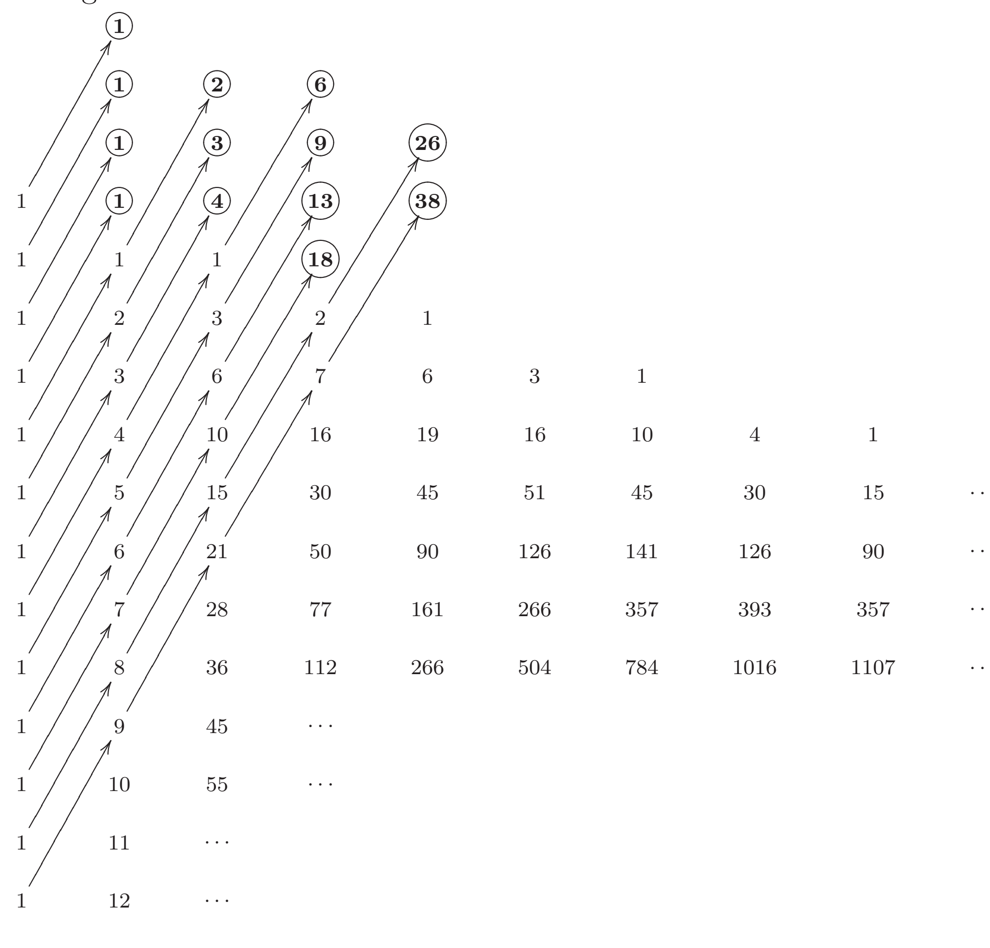

Figure 2. The sequence $\{T_n^{(1,3,0)}\}_{n\geq 1}$ with $a=b=1$ in the trinomial triangle

2. Main result

The main theorem states a more general situation in the multinomial (s-nomial) Pascal triangle.

Theorem 1. *Let $s \geq 1$. Then for $q \geq 1$*

$$T_{n+1} := T_{n+1}^{(1,q,0)} = \sum_{k=0}^{\lfloor \frac{sn}{sq+1} \rfloor} \binom{n-qk}{k}_s a^{sn-(sq+1)k} b^k$$

satisfies the relation

$$T_{n+1} = a^s T_n + a^{s-1} b T_{n-q} + \cdots + a b^{s-1} T_{n-(s-1)q} + b^s T_{n-sq} \quad (n \geq 1)$$

with

$$T_1 = 1 \quad \text{and} \quad T_0 = T_{-1} = \cdots = T_{1-sq} = 0\,.$$

Example. If $q = 1$, the sequence $\{T_n\}_{n\geq 1}$ means the weighted sum of $(r,q) = (1,1)$ direction:

$T_1 = 1,$ $T_5 = a^8 + 3a^5 b + 3a^2 b^2,$

$T_2 = a^2,$ $T_6 = a^{10} + 4a^7 b + 6a^4 b^2 + 2ab^3,$

$T_3 = a^4 + ab,$ $T_7 = a^{12} + 5a^9 b + 10a^6 b^2 + 7a^3 b^3 + b^4.$

$T_4 = a^6 + 2a^3 b + b^2,$

$$T_{n+1} := T_{n+1}^{(1,1,0)}(n,k) = \sum_{k=0}^{\lfloor \frac{2n}{3} \rfloor} \binom{n-k}{k}_2 a^{2n-3k} b^k$$

satisfies the relation

$T_{n+1} = a^2 T_n + ab T_{n-1} + b^2 T_{n-2} \quad (n \geq 1)$ with $T_1 = 1, \quad T_0 = T_{-1} = 0\,.$

Remark. If $q = 1$ and $s = 1$, we have the nice well-known identity

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}$$

for Fibonacci numbers F_n ([9, A000045]). If $q = 1$ and $s = 2$, then we have the identity for Tribonacci numbers, satisfying $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ ($n \geq 4$) with $T_1 = T_2 = T_3 = 1$ ([4] [9, A000073]). If $q = 2$ and $s = 2$, then T_n corresponds to the number of ordered partitions of n into 1's, 3's and 5's ([9, A060961]).

	$\mathbf{T_1}$								
1	$\mathbf{T_2}$	$\mathbf{T_3}$	$\mathbf{T_4}$						
$1a^2$	$1ab$	$1b^2$	$\mathbf{T_5}$	$\mathbf{T_6}$	$\mathbf{T_7}$				
$1a^4$	$2a^3b$	$3a^2b^2$	$2ab^3$	$1b^4$	$\mathbf{T_8}$	$\mathbf{T_9}$	$\mathbf{T_{10}}$		
$1a^6$	$3a^5b$	$6a^4b^2$	$7a^3b^3$	$6a^2b^4$	$3ab^5$	$1b^6$	$\mathbf{T_{11}}$	$\mathbf{T_{12}}$	
$1a^8$	$4a^7b$	$10a^6b^2$	$16a^5b^3$	$19a^4b^4$	$16a^3b^5$	$10a^2b^6$	$4ab^7$	1	
$1a^{10}$	$5a^9b$	$15a^8b^2$	$30a^7b^3$	$45a^6b^4$	$51a^5b^5$	$45a^4b^6$	30	15	$\cdots$
$1a^{12}$	$6a^{11}b$	$21a^{10}b^2$	$50a^9b^3$	$90a^8b^4$	$126a^7b^5$	141	126	90	$\cdots$
$1a^{14}$	$7a^{13}b$	$28a^{12}b^2$	$77a^{11}b^3$	$161a^{10}b^4$	266	357	393	357	$\cdots$
$1a^{16}$	$8a^{15}b$	$36a^{14}b^2$	$112a^{13}b^3$	266	504	784	1016	1107	$\cdots$
$1a^{18}$	$9a^{17}b$	$45a^{16}b^2$	$\cdots$						
$1a^{20}$	$10a^{19}b$	55	$\cdots$						
$1a^{22}$	11	$\cdots$							
$\cdots$									

Figure 3. The sequence $\{T_n^{(1,1,0)}\}_{n\geq 1}$ in the trinomial triangle

Proof of Theorem 1.

$$a^sT_n+a^{s-1}bT_{n-q}+\cdots+ab^{s-1}T_{n-(s-1)q}+b^sT_{n-sq}=$$

$$=\sum_{k=0}^{\left\lfloor\frac{s(n-1)}{sq+1}\right\rfloor}\binom{n-qk-1}{k}_s a^{sn-(sq+1)k}b^k+$$

$$+\sum_{k=0}^{\left\lfloor\frac{s(n-q-1)}{sq+1}\right\rfloor}\binom{n-q(k+1)-1}{k}_s a^{sn-(sq+1)(k+1)}b^{k+1}+$$

$$+\cdots+$$

$$+\sum_{k=0}^{\left\lfloor\frac{s(n-(s-1)q-1)}{sq+1}\right\rfloor}\binom{n-q(k+s-1)-1}{k}_s a^{sn-(sq+1)(k+s-1)}b^{k+s-1}+$$

$$+\sum_{k=0}^{\left\lfloor\frac{s(n-sq-1)}{sq+1}\right\rfloor}\binom{n-q(k+s)-1}{k}_s a^{sn-(sq+1)(k+s)}b^{k+s}=$$

$$=\sum_{k=0}^{\left\lfloor\frac{sn-s}{sq+1}\right\rfloor}\binom{n-qk-1}{k}_s a^{sn-(sq+1)k}b^k+\sum_{k=1}^{\left\lfloor\frac{sn-s+1}{sq+1}\right\rfloor}\binom{n-qk-1}{k-1}_s a^{sn-(sq+1)k}b^k+$$

$$+\cdots+$$

$$+\sum_{k=s-1}^{\left\lfloor\frac{sn-1}{sq+1}\right\rfloor}\binom{n-qk-1}{k-s+1}_s a^{sn-(sq+1)k}b^k+\sum_{k=s}^{\left\lfloor\frac{sn}{sq+1}\right\rfloor}\binom{n-qk-1}{k-s}_s a^{sn-(sq+1)k}b^k.$$

Notice that

$$\binom{n-1}{0}_s=1=\binom{n}{0}_s\quad\text{for}\quad k=0\,,$$

$$\binom{n-q-1}{0}_s+\binom{n-q-1}{1}_s=\binom{n-q}{1}_s\quad\text{for}\quad k=1\,,$$

$$\ldots$$

$$\binom{n-q(s-1)-1}{0}_s+\binom{n-q(s-1)-1}{1}_s+\cdots+\binom{n-q(s-1)-1}{s-1}_s=$$

$$=\binom{n-q(s-1)}{s-1}_s\quad\text{for}\quad k=s-1\,,$$

and for $s\le k\le\left\lfloor\frac{sn-s}{sq+1}\right\rfloor$

$$\binom{n-qk-1}{k-s}_s + \binom{n-qk-1}{k-s+1}_s + \cdots + \binom{n-qk-1}{k-1}_s + \binom{n-qk-1}{k}_s = \\ = \binom{n-qk}{k}_s .$$

In addition, if

$$\left\lfloor \frac{sn-i-1}{sq+1} \right\rfloor < \left\lfloor \frac{sn-i}{sq+1} \right\rfloor$$

or

$$\frac{sn-i-1}{sq+1} < \left\lfloor \frac{sn-i-1}{sq+1} \right\rfloor + 1 \leq \left\lfloor \frac{sn-s+1}{sq+1} \right\rfloor$$

for some integer i with $0 \leq i \leq s-1$, then by

$$s\left(n - q\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - 1\right) < \left\lfloor \frac{sn-i}{sq+1} \right\rfloor$$

we have

$$\binom{n-q\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - 1}{\left\lfloor \frac{sn-i}{sq+1} \right\rfloor}_s = \cdots = \binom{n-q\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - 1}{\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - s + i + 1}_s = 0\,,$$

so,

$$\binom{n-q\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - 1}{\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - s}_s + \binom{n-q\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - 1}{\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - s + 1}_s + \cdots + \binom{n-q\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - 1}{\left\lfloor \frac{sn-i}{sq+1} \right\rfloor - s + i}_s = \\ = \binom{n-q\left\lfloor \frac{sn-i}{sq+1} \right\rfloor}{\left\lfloor \frac{sn-i}{sq+1} \right\rfloor}_s .$$

Therefore,

$$a^s T_n + a^{s-1} b T_{n-q} + \cdots + a b^{s-1} T_{n-(s-1)q} + b^s T_{n-sq} = \\ = \sum_{k=0}^{\left\lfloor \frac{sn}{sq+1} \right\rfloor} \binom{n-qk}{k}_s a^{sn-(sq+1)k} b^k = T_{n+1} . \qquad \Diamond$$

The case $q = 0$ corresponds to horizontal lines in the triangle. This case can be stated as follows.

Corollary 1.

$$T_{n+1} := T_{n+1}^{(1,0,0)} = \sum_{k=0}^{sn} \binom{n}{k}_s a^{sn-k} b^k \quad (n \ge 0)$$

is equivalent to

$$T_n = (a^s + a^{s-1}b + \cdots + ab^{s-1} + b^s)^{n-1} \quad (n \ge 1)\,.$$

Remark. If $a = b = 1$ in Cor. 1, this case is reduced to (1.2).

3. An extended Lagrange's identity

Suppose that each element in multinomial Pascal's triangle is replaced by the square of the corresponding element. Then the n-th row sum of the resulting triangle is

$$\binom{2n}{sn}_s \qquad (n = 0, 1, 2, \dots)\,.$$

This is a special case of the following theorem.

Theorem 2. *For* $0 \le l \le 2sn$

$$\binom{2n}{l}_s = \sum_{i=0}^{l} \binom{n}{i}_s \binom{n}{l-i}_s = \sum_{i=0}^{l} \binom{n}{i}_s \binom{n}{sn-l+i}_s\,.$$

Proof. By the definition of the coefficients in generalized Pascal's triangles,

$$(1 + x + x^2 + \cdots + x^s)^n = \sum_{i=0}^{sn} \binom{n}{i}_s x^i\,.$$

Hence,

$$(1 + x + x^2 + \cdots + x^s)^{2n} = \sum_{l=0}^{2sn} \binom{2n}{l}_s x^l\,.$$

On the other hand,

$$\begin{aligned}(1+x+x^2+\cdots+x^s)^{2n} &= (1+x+x^2+\cdots+x^s)^n(1+x+x^2+\cdots+x^s)^n = \\ &= \sum_{i=0}^{sn}\sum_{j=0}^{sn} \binom{n}{i}_s \binom{n}{j}_s x^{i+j} = \\ &= \sum_{l=0}^{2sn}\sum_{i=0}^{l} \binom{n}{i}_s \binom{n}{l-i}_s x^l\,.\end{aligned}$$

Equating the coefficients of x^l, we have the desired identity. $\Diamond$

By putting $l = sn$ in above theorem, we have

Corollary 2.

$$\binom{2n}{sn}_s = \sum_{i=0}^{sn} \left(\binom{n}{i}_s \right)^2 .$$

Remark. If $s = 1$, then

$$\binom{2n}{n}_s = \sum_{i=0}^{n} \binom{n}{i}^2 ,$$

which is Lagrange's identity ([5, Th. 5.1 and p. 130–131]). If $s = 2$, then we have the identity in the trinomial triangle:

$$\binom{2n}{2n}_2 = \sum_{i=0}^{2n} \left(\binom{n}{i}_s \right)^2 .$$

4. Riordan arrays

As stated in [7] a Riordan array is a pair $(d(t), h(t))$ where d and h are analytic functions and $d(0) \neq 0$. This pair then defines an infinite lower triangular array $\{d_{n,k}\}$, where

$$\sum_{n=0}^{\infty} d_{n,k} t^n = d(t)(t \cdot h(t))^k .$$

From this definition, $d(t)(t \cdot h(t))^k$ is the generating function of column k in the array. It is known that Pascal triangle $\{P_{n,k}\}_{n,k \geq 0}$ is represented by a Riordan array:

$$\sum_{n=0}^{\infty} P_{n,k} t^n = \frac{1}{1-t} \left(\frac{t}{1-t} \right)^k \quad (k \geq 0)$$

(e.g. [6, 11]). However, for the coefficients $\{d_{n,k}\}$ of some s-multinomial triangle, there are no two analytic functions $d(t)$ and $h(t)$, satisfying $\sum_{n=0}^{\infty} d_{n,k} t^n = d(t)(t \cdot h(t))^k$. For example, in the case of the trinomial triangle $\{d_{n,k}\}_{n,k \geq 0}$, we have $d_{n,0} = 1$ $(n \geq 0)$. Hence, $d(t) = 1/(1-t)$ because $d(t)$ is the generating function of column 0. So, there exists a function $f(t)$, satisfying

$$\sum_{n=0}^{\infty} d_{n,k} t^n = \frac{1}{1-t} (f(t))^k \quad (k \geq 0) .$$

By the second column with $d_{n,1} = n$ $(n \geq 0)$, we have

$$\frac{1}{1-t}f(t) = t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + \cdots = \frac{t}{(1-t)^2}\,.$$

Thus,

$$f(t) = \frac{t}{1-t}\,.$$

But, by the third column with $d_{n,2} = n(n+1)/2$ $(n \ge 0)$, we have

$$\frac{1}{1-t}(f(t))^2 = t + 3t^2 + 6t^3 + 10t^4 + 15t^5 + \cdots = \frac{t}{(1-t)^3}\,.$$

Thus,

$$f(t) = \frac{\sqrt{t}}{1-t}\,.$$

Furthermore, the relation

$$\frac{1}{1-t}(f(t))^3 = \sum_{n=0}^{\infty}\frac{(n-1)n(n+4)}{2}t^n$$

gives a still different function $f(t)$.

We may also use the following result ([7, Th. 2.1], [8]) to see the non-existence of Riordan array.

Lemma 1. *An array $\{d_{n,k}\}_{n,k\ge 0}$ is a Riordan array with $d(0) \ne 0$ and $h(0) \ne 0$ if and only if there exists a sequence $A = \{a_i\}_{i\ge 0}$ with $a_0 \ne 0$ such that every element $d_{n+1,k+1}$ $(n, k \ge 0)$ can be expressed as a linear combination with coefficients in A of the elements in the preceding row, starting from the preceding column on, namely*

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots\,.$$

In the case of the trinomial triangle $\{d_{n,k}\}_{n,k\ge 0}$, only the relation

$$d_{n+1,k+1} = 1\cdot d_{n,k-1} + 1\cdot d_{n,k} + 1\cdot d_{n,k+1}$$

holds. So, there does not exist such a sequence A.

5. Future works

More general rays where $r \ne 1$ and/or $p \ne 0$ may be treated similarly. For the moment, we have only a very special result, where $r = 2$ with $q = 1$, $p = 0$ and $a = b = 1$. Namely,

$$T_{n+1} := T_{n+1}^{(2,1,0)} = \sum_{k=0}^{\lfloor\frac{2n}{4}\rfloor}\binom{n-k}{2k}_s$$

satisfies the relation

$$T_{n+1} = T_n + 2T_{n-1} + T_{n-2} - 1 \quad (n \geq 2)$$

with

$$T_1 = T_2 = 1 \quad \text{and} \quad T_0 = 0\,.$$

A general result in the cases $r \geq 2$ will be considered in the future works.

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References

[1] BELBACHIR, H., KOMATSU, T. and SZALAY, L.: Characterization of linear recurrence sequences associated to rays in Pascal's triangle, *AIP Conf. Proc.* **1264** (2010), 90–99.

[2] BELBACHIR, H., KOMATSU, T. and SZALAY, L.: Linear recurrences associated to rays in Pascal's triangle and combinatorial identities, *Math. Slovaca* (to appear).

[3] BICKNELL, M. and HOGGATT, V. E. JR.: Diagonal sums of the trinomial triangle, *Fibonacci Quart.* **12** (1974), 47–50.

[4] HOGGATT, V. E. JR. and BICKNELL, M.: Diagonal Sums of Generalized Pascal Triangles, *Fibonacci Quart.* **7** (1969), 341–358.

[5] KOSHY, T.: *Triangular arrays with applications*, Oxford Univ. Press, New York, 2011.

[6] LUZÓN, A., MERLINI, D., MORÓN, M. A. and SPRUGNOLI, R.: Identities induced by Riordan arrays, *Linear Algebra Appl.* **436** (2012), 631–647.

[7] MERLINI, D., ROGERS, D. G., SPRUGNOLI, R. and VERRI, M. C.: On some alternative characterizations of Riordan arrays, *Can. J. Math.* **49** (1997), 301–320.

[8] ROGERS, D. G.: Pascal triangles, Catalan numbers and renewal arrays, *Discrete Math.* **22** (1978), 301–310.

[9] SLOANE, N. J. A.: The On-Line Encyclopedia of Integer Sequences, Published electronically at http://oeis.org/, 2012.

[10] SMITH, C. and HOGGATT, V. E. JR.: Generating functions of central values in generalized Pascal triangles, *Fibonacci Quart.* **17** (1979), 58–67.

[11] SPRUGNOLI, R.: Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994), 267–290.