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ON THE L^1-NORM OF THE WEIGHTED MAXIMAL FUNCTION OF KERNELS WITH RESPECT TO THE VILENKIN-LIKE SPACE

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Abstract: The aim of this paper is to investigate the integral of the weighted maximal function of the Dirichlet and Fejér kernels with respect to the so-called Vilenkin-like system. We give – in the bounded case – a necessary and sufficient condition for that the weighted maximal functions belong to L^1 in this general space.

1. Introduction

1.1. Historical notes

It is a well known result that for the Walsh–Paley system we have $\sup_{n\in\mathbb{N}} |D_n(x)| < \infty$ for each $x \neq 0$. (For Walsh–Kaczmarz system it does not hold.) (See [14], for bounded Vilenkin-like system see Cor. 6.) This property is a useful fact for proving convergence theorems of the Fourier series in Walsh–Paley system. But what can we say for the

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norm of maximal functions? It is easy to obtain that the L^1 norm of $\sup_{n\in\mathbb{N}} |D_n(x)|$ with respect to the Walsh–Paley and Walsh–Kaczmarz systems (and also for bounded Vilenkin-like system) is infinite. (See e.g. [14], for bounded Vilenkin-like system see Cor. 9.) "What happens if we apply some weight function α?" asked G. Gát in [3].

In his paper he gave a necessary and sufficient condition for that the weighted maximal functions belong to L^1 in the case of Walsh–Paley and Walsh–Kaczmarz systems. K. Nagy (see [8]) proved similar statements for Fejér kernels for both systems and for (C, α) kernels with respect to Walsh–Paley system. I. Mező and P. Simon in their common work (see [9]) verified the necessary and sufficient condition with respect to bounded Vilenkin systems. They proved that the analogous statement with the Walsh–Paley case is not true for arbitrary unbounded Vilenkin system. In their paper they found a different necessary and sufficient condition for any Vilenkin systems.

In this article the author deals with the case of bounded Vilenkin space.

1.2. Notations and definitions

Let $m := (m_0, m_1, \dots)$ denote by a sequence of positive integers not less than 2. Denote by G_{m_j} a set, where the number of the elements is m_j $(j \in \mathbb{N})$. Define the measure on G_{m_j} as follows

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in G_{m_k}, k \in \mathbb{N}).$$

Let G_m be the complete direct product of the sets G_{m_j} (without any operation on it), with the product of the topologies and measures (denoted by μ). This product measure is a regular Borel one on G_m with $\mu(G_m) = 1$. If the sequence m is bounded, then G_m is called by bounded Vilenkin space, otherwise it is unbounded one. The elements of G_m can be represented by sequences $x := (x_0, x_1, \dots)$ $(x_j \in G_{m_j})$. It is easy to give a neighbourhood base of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m | y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

for $x \in G_m$, $0 < n \in \mathbb{N}$. Define the well-known generalized number system in the usual way. If $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in \mathbb{N})$, then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in G_{m_j}$ $(j \in \mathbb{N})$, and only a finite number of n_js differ from zero. Let

$$|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$$

(that is, $M_{|n|} \leq n < M_{|n|+1}$) if $0 < n \in \mathbb{N}$, and $|0| := 0$. Let $n^{(k)} = = \sum_{j=k}^{\infty} n_k M_k$.

Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) ($1 \leq p \leq \infty$), $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)$ ($x \in G_m$, $n \in \mathbb{N}$) and E_n the conditional expectation operator with respect to $\mathcal{A}_n$, ($n \in \mathbb{N}$).

Now we introduce an orthonormal system on G_m, which will be called Vilenkin-like system. This system was defined by G. Gát in his paper [4]. The complex valued functions $r_k^n : G_m \to \mathbb{C}$ ($k, n \in \mathbb{N}$) are called generalized Rademacher functions, if they have the following four properties.

(i) r_k^n ($k, n \in \mathbb{N}$) is $\mathcal{A}_{k+1}$ measurable (i.e. $r_k^n(x)$ depends only on $x_0, \ldots, x_k$ ($x \in G_m$)) and $r_k^0 = 1$.

(ii) If M_k is a divisor of n, l and $n^{(k+1)} = l^{(k+1)}$ ($k, l, n \in \mathbb{N}$), then

$$E_k(r_k^n \bar{r}_k^l) = \begin{cases} 1 & \text{if } n_k = l_k, \\ 0 & \text{if } n_k \neq l_k \end{cases}$$

($\bar{z}$ is the complex conjugate of z).

(iii) If M_k is a divisor of n (that is, $n = n_k M_k + n_{k+1} M_{k+1} + \cdots + n_{|n|} M_{|n|}$), then

$$\sum_{n_k=0}^{m_k-1} |r_k^n(x)|^2 = m_k$$

for all $x \in G_m$.

(iv) There exists a $\delta > 1$, for which $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$ for all $k, n \in \mathbb{N}$.

Now define the Vilenkin-like system $\psi := (\psi_n : n \in \mathbb{N})$ as follows

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} \quad (n \in \mathbb{N}).$$

(Since $r_k^0 = 1$, then $\psi_n = \prod_{k=0}^{|n|} r_k^{n^{(k)}}$.) The Vilenkin-like system ψ is orthonormal (see e.g. [4]).

1.3. Examples

Let us see some known examples to the Vilenkin-like system.

(1) The Walsh–Paley and Vilenkin systems. For more on these see e.g. [1], [14].

(2) The group of 2-adic (m-adic) integers [7], [13], [15].

(3) The product system of coordinate functions of unitary irreducible representation of non commutative Vilenkin groups (in this case the group G_m is the Cartesian product of any finite groups) [6].

(4) A system in the field of number theory. The so-called $\psi\alpha$ Vilenkin-like system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions [5], [10].

(5) The UDMD product system (introduced by F. Schipp on the Walsh–Paley group) [12], [13].

(6) The universal contractive projections system (UCP) (introduced by F. Schipp) [11].

For more on these examples and their proofs see e.g. [4].

1.4. Further definitions

Finally, we introduce some definitions of the Fourier-analysis in the usual way. We define the Dirichlet and Fejér kernels

$$D_n(y,x) := \sum_{k=0}^{n-1} \psi_k(y)\overline{\psi}_k(x) \qquad (0 < n \in \mathbb{N},\ D_0 := 0),$$

$$K_n(y,x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(y,x) \qquad (n \in \mathbb{N},\ K_0 := 0).$$

Let sequence $\alpha_n \to \mathbb{R}^+$ be monotone increasing, and define the weighted maximal function of the Dirichlet and Fejér kernels in the following way:

$$D_\alpha(y,x) := \sup_{n\in\mathbb{N}} \frac{|D_n(y,x)|}{\alpha_{|n|}}, \qquad K_\alpha(y,x) := \sup_{n\in\mathbb{N}} \frac{|K_n(y,x)|}{\alpha_{|n|}} \quad (x, y \in G_m).$$

1.5. Auxiliary results

Lemma 1 [3]. *Let $x, y \in G_m,\ n \in \mathbb{N}$. Then*

$$D_{M_n}(y,x) = \begin{cases} M_n & \text{if } y \in I_n(x), \\ 0 & \text{if } y \notin I_n(x). \end{cases}$$

Lemma 2 [2]. *Let $x \in I_A(y)\setminus I_{A+1}(y)$, where $x, y \in G_m,\ A \in \mathbb{N}$. Then*

$$D_n(y,x) = \sum_{i=0}^{A} M_i \left(\sum_{j=0}^{n_i-1} r_i^{n^{(i+1)}+jM_i}(y) \bar{r}_i^{n^{(i+1)}+jM_i}(x) \right) \psi_{n^{(i+1)}}(y) \bar{\psi}_{n^{(i+1)}}(x).$$

2. Results

From now let m be a bounded sequence, that is let G_m be a bounded Vilenkin space. Constants denoted by c_m, C_m depend only on the finite number $\sup_{n\in\mathbb{N}} m_n$.

Corollary 3. *Let $x \in I_A(y)\setminus I_{A+1}(y)$, where $x, y \in G_m,\ A \in \mathbb{N}$. Then there is a constant C_m such that*

$$|D_n(y,x)| \le C_m \sum_{i=0}^{A} M_i \left| \psi_{n^{(i+1)}}(y) \bar{\psi}_{n^{(i+1)}}(x) \right| .$$

Proof. Because of $\sum\limits_{n_k=0}^{m_k-1} |r_k^n(x)|^2 = m_k$ we obtain $|r_k^n(x)| \le \sqrt{m_k}$, so

$$\left| \sum_{j=0}^{n_i-1} r_i^{n^{(i+1)}+jM_i}(y) \bar{r}_i^{n^{(i+1)}+jM_i}(x) \right| \le n_i m_i < m_i^2 < C_m.$$

Using Lemma 1 and Lemma 2 we obtain the statement of Cor. 3. $\Diamond$

Corollary 4. *If $x, y \in G_m,\ n \in \mathbb{N}$ then there is a constant C_m such that*

$$|D_n(y,x)| \le C_m n.$$

Proof. Using $\|r_k^n\|_\infty \le \sqrt{m_k/\delta}$, easy to see that $\psi_{n^{(i+1)}}(x) \le \sqrt{\frac{M_{|n|}}{M_i}\delta^{i-|n|}}$. In this way from Cor. 3. we get

$$|D_n(y,x)| \le C_m \sum_{i=0}^{|n|} M_i \left| \psi_{n^{(i+1)}}(y) \bar{\psi}_{n^{(i+1)}}(x) \right| \le$$

$$\le C_m \sum_{i=0}^{|n|} M_i \frac{M_{|n|}}{M_i} \delta^{i-|n|} \le C_m M_{|n|} \le C_m n. \quad \Diamond$$

Corollary 5. *Let $x \in I_A(y)\backslash I_{A+1}(y)$, where $x, y \in G_m$, $A \in \mathbb{N}$. Then there is a constant C_m such that*

$$|D_n(y,x)| \leq C_m M_A.$$

Proof. Using $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$ and Cor. 3, if $y \in I_A(x)\backslash I_{A+1}(x)$, then

$$|D_n(y,x)| \leq C_m \sum_{i=0}^{A} M_i \left|\psi_{n^{(i+1)}}(y)\bar{\psi}_{n^{(i+1)}}(x)\right| \leq$$

$$\leq C_m \sum_{i=0}^{A} M_i \frac{M_A}{M_i}\delta^{i-A} \leq C_m M_A. \quad \lozenge$$

Corollary 6. *Let $x, y \in G_m$ and $x \neq y$. Then*

$$\sup_{n \in \mathbb{N}} |D_n(y,x)| < \infty.$$

Proof. If $x \neq y$, then there exists $A \in \mathbb{N}$, for which $x \in I_A(y)\backslash I_{A+1}(y)$. Now using Cor. 5 the statement of this corollary is obvious. $\lozenge$

Theorem 7. *Let $y \in G_m$. There exist positive constants c_m and C_m such that*

$$c_m \sum_{k=0}^{\infty} \frac{1}{\alpha_k} \leq \|R_\alpha(y,.)\|_1 \leq C_m \sum_{k=0}^{\infty} \frac{1}{\alpha_k},$$

where $R_\alpha = D_\alpha$ or $R_\alpha = K_\alpha$.

Proof. It is easy to see that

$$\frac{|K_n(y,x)|}{\alpha_{|n|}} \leq \frac{\frac{1}{n}\sum\limits_{i=0}^{n-1}|D_i(y,x)|}{\alpha_{|n|}} \leq$$

$$\leq \frac{1}{n}\sum_{i=1}^{n-1}\frac{|D_i(y,x)|}{\alpha_{|i|}} \leq \frac{1}{n}\sum_{i=1}^{n-1} D_\alpha(y,x) = D_\alpha(y,x),$$

where $x, y \in G_m$ and $n \in \mathbb{N}$ are arbitrary. From these inequalities it is obvious that

$$K_\alpha(y,x) = \sup_{n \in \mathbb{N}} \frac{|K_n(y,x)|}{\alpha_{|n|}} \leq D_\alpha(y,x)$$

for every $x, y \in G_m$. It means, for proving the theorem it is enough to verify existences of positive constants c_m, C_m that for any $y \in G_m$

$$\|D_\alpha(y,.)\|_1 \leq C_m \sum_{k=0}^{\infty}\frac{1}{\alpha_k} \quad \text{and} \quad c_m \sum_{k=0}^{\infty}\frac{1}{\alpha_k} \leq \|K_\alpha(y,.)\|_1.$$

Corollaries imply

$$\|D_\alpha(y,.)\|_1 = \sum_{A=0}^{\infty} \int\limits_{I_A(y)\setminus I_{A+1}(y)} D_\alpha(y,x)d\mu(x) \le$$

$$\le C_m \sum_{A=0}^{\infty} \int\limits_{I_A(y)\setminus I_{A+1}(y)} \sum_{k=0}^{A} \frac{M_k}{\alpha_k} d\mu(x) \le$$

$$\le C_m \sum_{A=0}^{\infty} \frac{1}{M_A} \sum_{k=0}^{A} \frac{M_k}{\alpha_k} =$$

$$= C_m \sum_{k=0}^{\infty} \frac{1}{\alpha_k} \sum_{A=k}^{\infty} \frac{M_k}{M_A} \le$$

$$\le C_m \sum_{k=0}^{\infty} \frac{1}{\alpha_k}.$$

It was the expected upper estimation for D_α.

Let $x \in I_A(y)\setminus I_{A+1}(y)$, where $x, y \in G_m$. Let $A, k \in \mathbb{N}$ and let us suppose that $k \le M_A$. In this case $\psi_k(x)$ depends only on the first k coordinates of x, so $\psi_k(x)$ is constant on the set $I_A(y)$. It means that $\psi_k(x) = \psi_k(y)$ is realized in case of $x \in I_A(y)\setminus I_{A+1}(y)$, too. Based on conditions above, with help of Lemma 1 it yields

$$D_k(y,x) = \sum_{i=0}^{k-1} \psi_i(y)\bar{\psi}_i(x) = \sum_{i=0}^{k-1} \psi_i(x)\bar{\psi}_i(x) = \sum_{i=0}^{k-1} |\psi_i(x)|^2 \ge$$

$$\ge \sum_{i=0}^{M_{|k|}-1} |\psi_i(x)|^2 = M_{|k|}.$$

Thus – using Lemma 1 again – we have

$$K_{M_A}(y,x) = \frac{1}{M_A} \sum_{k=0}^{M_A-1} D_k(y,x) \ge \frac{1}{M_A} \sum_{k=0}^{M_A-1} M_{|k|} \ge c_m \frac{1}{M_A} \sum_{k=0}^{M_A-1} M_{|k|+1} \ge$$

$$\ge c_m \frac{1}{M_A} \sum_{k=0}^{M_A-1} k \ge c_m M_A.$$

Consequently,

$$\|K_\alpha(y,.)\|_1 = \sum_{A=0}^{\infty} \int\limits_{I_A(y)\setminus I_{A+1}(y)} K_\alpha(y,x)d\mu(x) \geq$$
$$\geq \sum_{A=0}^{\infty} \int\limits_{I_A(y)\setminus I_{A+1}(y)} \frac{K_{M_A}(y,x)}{\alpha_A}d\mu(x) \geq$$
$$\geq c_m \sum_{A=0}^{\infty} \frac{1}{M_{A+1}} \frac{M_A}{\alpha_A} \geq$$
$$\geq c_m \sum_{A=0}^{\infty} \frac{1}{\alpha_A}.$$

So the lower estimation for K_α is also proved. $\Diamond$

Corollary 8. *Let $y \in G_m$. $R_\alpha(y,.) \in L^1$ if and only if*
$$\sum_{k=0}^{\infty} \frac{1}{\alpha_k} < \infty,$$
where $R_\alpha = D_\alpha$ or $R_\alpha = K_\alpha$.

Proof. It comes from Th. 7 immediately. $\Diamond$

We mention that using some lower estimations from the proof of Th. 7 we can verify the infinity of $\| \sup_{n\in\mathbb{N}} |D_n(y,.)|\|_1$ easily.

Corollary 9. *Let $y \in G_m$. Then*
$$\left\| \sup_{n\in\mathbb{N}} |D_n(y,.)| \right\|_1 = \infty.$$

Proof. Let $x \in I_A(y)\setminus I_{A+1}(y)$, where $x,y \in G_m$. Let $A,n \in \mathbb{N}$ and let us suppose that $n \leq M_A$. We proved before that in this case
$$D_k(y,x) \geq M_{|n|} \geq M_{A-1}.$$
Therefore,
$$\left\| \sup_{n\in\mathbb{N}} |D_n(y,.)| \right\|_1 \geq \sum_{A=0}^{\infty} \int\limits_{I_A(y)\setminus I_{A+1}(y)} D_{M_A}(y,x)d\mu(x) \geq$$
$$\geq \sum_{A=0}^{\infty} \int\limits_{I_A(y)\setminus I_{A+1}(y)} M_{A-1}d\mu(x) \geq$$
$$\geq \sum_{A=0}^{\infty} \frac{M_{A-1}}{M_{A+1}} \geq$$
$$\geq c_m \sum_{A=0}^{\infty} 1 = \infty. \quad \Diamond$$

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