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SUBORDINATION RESULTS FOR SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY A DIFFERENTIAL OPERATOR RELATED TO CONIC DOMAINS

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Abstract: Recently, Al-Oboudi et al. [F.M. Al-Oboudi and K.A. Al-Amoudi, Subordination results for classes of analytic functions related to conic domains defined by a fractional operator, *J. Math. Anal. Appl.* **354** (2009), 412–420] have studied subordination results for some subclasses of analytic functions by using a fractional differential operator. In this paper we generalize some of the previous results particularly results of Al-Oboudi et al. [4], which yield sharp distortion and rotation theorems and Koebe domain. We use the properties of convolution in geometric functions theory to obtain our main results.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in Δ. If f and g are analytic in Δ, we say that f is subordinate to g, written symbolically as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta)$$

if there exists a Schwarz function $w(z)$, is analytic in Δ (with $w(0) = 0$ and $|w(z)| < 1$ in Δ) such that $f(z) = g(w(z))$, $z \in \Delta$. In particular, if the function $g(z)$ is univalent in Δ, then we have the following equivalence:

$$f(z) \prec g(z)(z \in \Delta) \;\Leftrightarrow\; \quad f(0) = g(0) \text{ and } f(\Delta) \subseteq g(\Delta).$$

A function $f \in \mathcal{A}$ is said to be in the class of uniformly convex functions of order γ and type β, denoted by $\beta - UCV(\gamma)$ (see [5]) if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta\left|\frac{zf''(z)}{f'(z)}\right| + \gamma, \tag{1.2}$$

where $\beta \geq 0$, $-1 \leq \gamma < 1$, $\beta + \gamma \geq 0$ and it is said to be in the corresponding class denoted by $\beta - SP(\gamma)$ if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma, \tag{1.3}$$

where $\beta \geq 0$, $-1 \leq \gamma < 1$ and $\beta + \gamma \geq 0$.

These classes generalize various other classes which are worthy to mention here. For example the class $\beta{-}UCV(0) = \beta{-}UCV$ is the known class of $\beta-$ *uniformly convex* functions (see [12]). Using the Alexander type relation, we can obtain the class $\beta - SP(\gamma)$ in the following way:

$$f \in \beta - SP(\gamma) \Leftrightarrow \frac{1}{z}\int_0^z f(t)dt \in \beta - UCV(\gamma) \quad \text{or}$$

$$f \in \beta - UCV(\gamma) \Leftrightarrow zf' \in \beta - SP(\gamma).$$

The class $1 - UCV(0) = UCV$ of uniformly convex functions was defined by Goodman (see [11]) while the class $1 - SP(0) = SP$ was considered by Ronning in [24]. The classes $1-UCV(\gamma)=UCV(\gamma)$ and $1-SP(\gamma)=SP(\gamma)$ were investigated by Ronning in [23]. Furthermore, the classes $\beta-UCV(0)=\beta-UCV$ and $\beta-SP(0)=\beta-SP$, respectively, were defined by Kanas and Wisniowska in [12] and [13].

Geometric interpretation. It is known that $f\in\beta-UCV(\gamma)$ and $g\in\beta-SP(\gamma)$ if and only if $1+\frac{zf''(z)}{f'(z)}$ and $\frac{zg'(z)}{g(z)}$, respectively, takes all the values in the conic domain $\mathcal{R}_{\beta,\gamma}$ which is included in the right half plane $\Re(w)>\frac{\beta+\gamma}{1+\beta}$ and is given by

$$\mathcal{R}_{\beta,\gamma}:=\Big\{w=u+iv\in\mathbb{C}:u>\beta\sqrt{(u-1)^2+v^2}+\gamma,\quad \beta\geq 0 \text{ and } \gamma\in[-1,1)\Big\}. \tag{1.4}$$

Let $\hat{P}_{\beta,\gamma}=1+P_1z+\dots$ denote the function which maps the unit disk conformally onto the domain $\mathcal{R}_{\beta,\gamma}$ given in (1.4). Let $\partial\mathcal{R}_{\beta,\gamma}$ be a curve defined by the equality

$$\partial\mathcal{R}_{\beta,\gamma}:=\Big\{w=u+iv\in\mathbb{C}:u^2=\Big(\beta\sqrt{(u-1)^2+v^2}+\gamma\Big)^2,\quad \beta\geq 0 \text{ and } \gamma\in[-1,1)\Big\}. \tag{1.5}$$

After some elementary calculations, we saw that for $\beta\neq 0$, $\partial\mathcal{R}_{\beta,\gamma}$ represent conic curves symmetric about the real axis. Thus the region $\mathcal{R}_{\beta,\gamma}$ is an elliptic domain for $\beta>1$, a parabolic domain for $\beta=1$, a hyperbolic domain for $0<\beta<1$ and the right half plane $\Re(w)>\gamma$, for $\beta=0$.

The functions $\hat{P}_{\beta,\gamma}$ play the role of extremal functions of the classes $\mathcal{P}(\hat{P}_{\beta,\gamma})$ were obtained in [12] (also see [1], [3] and for place Taylor series expansion of $\hat{P}_{\beta,\gamma}$, [15], [24]) as follows:

$$\hat{P}_{\beta,\gamma}(z)=\begin{cases}\frac{1+(1-2\gamma)z}{1-z}, & \beta=0,\\ 1+\frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \beta=1,\\ \frac{1-\gamma}{1-\beta^2}\cos\left\{\frac{2}{\pi}(\arccos\beta)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{\beta^2-\gamma}{1-\beta^2}, & 0<\beta<1,\\ \frac{1-\gamma}{\beta^2-1}\sin\left(\frac{\pi}{2\mathcal{K}(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}dx\right)+\frac{\beta^2-\gamma}{\beta^2-1}, & \beta>1,\end{cases} \tag{1.6}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{t}z}$, $t \in (0,1)$, $z \in \Delta$ and t is chosen such that $\beta = \cosh \frac{\pi \mathcal{K}'(t)}{4\mathcal{K}(t)}$, $\mathcal{K}(t)$ is *Legendre's complete elliptic integral* of the first kind and $\mathcal{K}'(t)$ is the complementary integral of $\mathcal{K}(t)$.

For two analytic functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

their *Hadamard product (or convolution)* is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k \quad (z \in \Delta).$$

Note that $f * g \in \mathcal{A}$. Define the incomplete Beta function $\varphi(a,c)$, for $a \in \mathbb{R}$; $c \neq 0, -1, -2, \dots$ by

$$\varphi(a,c;z) := z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad (z \in \Delta), \tag{1.7}$$

where $(\kappa)_n$ is the Pochhammer symbol (or the *shifted factorial*) in terms of the Gamma function, given by

$$(\kappa)_n := \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)} = \begin{cases} 1 & n = 0, \\ \kappa(\kappa+1)\dots(\kappa+n-1) & n \in \mathbb{N} := \{1,2,\dots\}. \end{cases}$$

The Carlson–Shaffer operator (see [6]) $\mathcal{L}(a,c)$ is defined in terms of Hadamard product by

$$\mathcal{L}(a,c)f(z) = \varphi(a,c;z) * f(z), \quad z \in \Delta, \ f \in \mathcal{A}. \tag{1.8a}$$

Note that $\mathcal{L}(a,a)$ is the identity operator and $\mathcal{L}(a,c) = \mathcal{L}(a,b)\mathcal{L}(b,c)$, $(b, c \neq 0, -1, -2, \dots)$.

We also need the following definition of *fractional derivative.*

Definition 1.1 (see [19])**.** The fractional derivative of order α is defined, for a function $f(z)$, *by* $D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta \ \ (0 \leq \alpha < 1)$, where f is an analytic function in a simply connected domain of the $z-$plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Using D_z^α Owa and Srivastava (see [20]) introduced and studied the operator $\Omega^\alpha : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral, as follows

$$\begin{aligned}\Omega^\alpha f(z) &= \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z) = \qquad (\alpha \in [0,1)) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k = \\ &= \varphi(2, 2-\alpha; z) * f(z) = \\ &= \mathcal{L}(2, 2-\alpha) f(z).\end{aligned} \tag{1.9}$$

Note that $\Omega_z^0 f(z) = f(z)$.

The *linear multiplier fractional differential operator* $D_{\lambda,\mu}^{n,\alpha} f : \mathcal{A} \to \mathcal{A}$ was defined by Orhan et al. in [18] as follows

$$\begin{aligned} &D_{\lambda,\mu}^{0,\alpha} f(z) = f(z) \\ &D_{\lambda,\mu}^{1,\alpha} f(z) = D_{\lambda,\mu}^{\alpha} f(z) = \lambda\mu z^2 [\Omega^\alpha f(z)]'' + \\ &\qquad\qquad + (\lambda-\mu) z [\Omega^\alpha f(z)]' + (1-\lambda+\mu)[\Omega^\alpha f(z)] \\ &\qquad \vdots \\ &D_{\lambda,\mu}^{n,\alpha} f(z) = D_{\lambda,\mu}^{\alpha}\left(D_{\lambda,\mu}^{n-1,\alpha} f(z)\right) \end{aligned} \tag{1.10}$$

where $\lambda \geq \mu \geq 0,\ 0 \leq \alpha < 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If f is given by (1.1) then from the definitions of the $D_{\lambda,\mu}^{n,\alpha}$ and Ω^α it is easy to see that

$$D_{\lambda,\mu}^{n,\alpha} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,n}(\lambda,\mu,\alpha) a_k z^k \tag{1.11}$$

where

$$\Psi_{k,n}(\lambda,\mu,\alpha) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left(1+(\lambda\mu k+\lambda-\mu)(k-1)\right)\right]^n. \tag{1.12}$$

From (1.9) and (1.12), the operator $D_{\lambda,\mu}^{n,\alpha} f(z)$ can be written, in terms of convolution as

(1.13)

$$D_{\lambda,\mu}^{n,\alpha} f(z) = \underbrace{\left[(\varphi(2,2-\alpha;z) * g_{\lambda,\mu}(z)) * \cdots * (\varphi(2,2-\alpha;z) * g_{\lambda,\mu}(z))\right]}_{n\text{-times}} * f(z)$$

where

$$g_{\lambda,\mu}(z) = \frac{z^3(1-\lambda+\mu)+z^2(\lambda-\mu+2\lambda\mu-2)+z}{(1-z)^3} =$$
$$= z + \sum_{k=2}^{\infty} \left(1+(\lambda\mu k+\lambda-\mu)(k-1)\right) z^k.$$

It should be remarked that the operator $D_{\lambda,\mu}^{n,\alpha}$ is a generalization of many other linear operators considered earlier. In particular, for $f \in \mathcal{A}$ we have the following:

a. $D_{1,0}^{n,0} f(z) \equiv D^n f(z)$, the operator introduced by Sălăgean (see [28]).
b. $D_{\lambda,0}^{n,0} f(z) \equiv D_\lambda^n f(z)$, the operator defined and studied by Al-Oboudi (see [2]).
c. $D_{0,0}^{1,\alpha} f(z) \equiv \Omega^\alpha f(z)$, the fractional derivative operator introduced by Owa and Srivastava (see [20]).
d. $D_{\lambda,\mu}^{n,0} f(z) \equiv D_{\lambda,\mu}^n f(z)$, the operator worked by Răducanu and Orhan (see [22]) also Deniz and Orhan (see [7]).
e. $D_{\lambda,0}^{n,\alpha} f(z) \equiv D_\lambda^{n,\alpha} f(z)$, the operator investigated by Al-Oboudi and Al-Amoudi (see [3]).
f. $D_{\lambda,0}^{1,\alpha} f(z) \equiv D_\lambda^\alpha f(z)$, the operator introduced by Noor et al. (see [17]).

Using the operator $D_{\lambda,\mu}^{n,\alpha}$, authors defined in [8] and [18] the classes $\beta - UCV_{\lambda,\mu}^{n,\alpha}(\gamma)$ and $\beta - SP_{\lambda,\mu}^{n,\alpha}(\gamma)$ as follows

Definition 1.2. For $\lambda \geq \mu \geq 0$, $0 \leq \alpha < 1$, $\beta \geq 0$, $-1 \leq \gamma < 1$ and$\beta + \gamma \geq 0$ a function $f \in \mathcal{A}$ is said to be in the class $\beta - UCV_{\lambda,\mu}^{n,\alpha}(\gamma)$ if it satisfies the following condition:

$$\Re\left\{1 + \frac{z(D_{\lambda,\mu}^{n,\alpha} f(z))''}{(D_{\lambda,\mu}^{n,\alpha} f(z))'}\right\} > \beta \left|\frac{z(D_{\lambda,\mu}^{n,\alpha} f(z))''}{(D_{\lambda,\mu}^{n,\alpha} f(z))'}\right| + \gamma \qquad (z \in \Delta). \tag{1.14}$$

Definition 1.3. For $\lambda \geq \mu \geq 0$, $0 \leq \alpha < 1$, $\beta \geq 0$, $-1 \leq \gamma < 1$ and $\beta + \gamma \geq 0$ a function $f \in \mathcal{A}$ is said to be in the class $\beta - SP_{\lambda,\mu}^{n,\alpha}(\gamma)$ if it satisfies the following condition:

$$\Re\left\{\frac{z(D_{\lambda,\mu}^{n,\alpha} f(z))'}{D_{\lambda,\mu}^{n,\alpha} f(z)}\right\} > \beta \left|\frac{z(D_{\lambda,\mu}^{n,\alpha} f(z))'}{D_{\lambda,\mu}^{n,\alpha} f(z)} - 1\right| + \gamma \qquad (z \in \Delta). \tag{1.15}$$

Note that $f \in \beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ if and only if $D^{n,\alpha}_{\lambda,\mu}f \in \beta - SP(\gamma)$. Using the Alexander type relation, it is clear that

$$f \in \beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma) \Leftrightarrow zf' \in \beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma), \tag{1.16}$$

and also

$$\beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma).$$

Geometric interpretation. From (1.14) and (1.15), $f \in \beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $f \in \beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ if and only if $p(z) = 1 + \frac{z(D^{n,\alpha}_{\lambda,\mu}f(z))''}{(D^{n,\alpha}_{\lambda,\mu}f(z))'}$ and $q(z) = \frac{z(D^{n,\alpha}_{\lambda,\mu}f(z))'}{D^{n,\alpha}_{\lambda,\mu}f(z)}$ take all the values in the domain $\mathcal{R}_{\beta,\gamma}$ given in (1.4) which is included in the right half plane. Thus, we can rewrite the conditions (1.14) and (1.15) in the form

$$p \prec \hat{P}_{\beta,\gamma}, \quad q \prec \hat{P}_{\beta,\gamma} \qquad (z \in \Delta), \tag{1.17}$$

where the function $\hat{P}_{\beta,\gamma}$ given by (1.6).

By virtue of (1.14), (1.15) and the properties of domain $\mathcal{R}_{\beta,\gamma}$,we have, respectively

$$\Re\left\{1 + \frac{z(D^{n,\alpha}_{\lambda,\mu}f(z))''}{(D^{n,\alpha}_{\lambda,\mu}f(z))'}\right\} > \frac{\beta+\gamma}{1+\beta} > 0 \tag{1.18}$$

and

$$\Re\left\{\frac{z(D^{n,\alpha}_{\lambda,\mu}f(z))'}{D^{n,\alpha}_{\lambda,\mu}f(z)}\right\} > \frac{\beta+\gamma}{1+\beta} > 0, \tag{1.19}$$

which means that

$$f \in \beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma) \Rightarrow D^{n,\alpha}_{\lambda,\mu}f \in CV\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq CV \tag{1.20}$$

and

$$f \in \beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma) \Rightarrow D^{n,\alpha}_{\lambda,\mu}f \in ST\left(\frac{\beta+\gamma}{1+\beta}\right) \subseteq ST \tag{1.21}$$

where $CV(\gamma)$, $ST(\gamma)$, CV, ST denote the well-known classes of γ-convex, γ-starlike, convex and starlike functions, respectively.

We note that by specializing the parameters $n, \alpha, \lambda, \mu, \beta$ and γ, the subclass $\beta{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ reduces to several well-known subclasses of analytic functions. Detailed information can be find in [8] and [18].

For special values of parameters $n, \alpha, \lambda, \mu, \beta$ and γ, from the general class $\beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $\beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma)$, we refer to the following classes in [8]:

- $\beta{-}SP^{n,0}_{\lambda,\mu}(\gamma) \equiv \beta - SP^{n}_{\lambda,\mu}(\gamma)$ and $\beta{-}UCV^{n,0}_{\lambda,\mu}(\gamma) \equiv \beta{-}UCV^{n}_{\lambda,\mu}(\gamma)$,
- $0 - SP^{n,\alpha}_{\lambda,\mu}(\gamma) \equiv ST^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $0 - UCV^{n,\alpha}_{\lambda,\mu}(\gamma) \equiv CV^{n,\alpha}_{\lambda,\mu}(\gamma)$,
- $1 - SP^{n,\alpha}_{\lambda,\mu}(0) \equiv SP^{n,\alpha}_{\lambda,\mu}$ and $1 - UCV^{n,\alpha}_{\lambda,\mu}(0) \equiv UCV^{n,\alpha}_{\lambda,\mu}$,

which have not been studied.

In [8], we proved following inclusion relations

$$\beta - SP^{n+1,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - SP(\gamma)$$

and

$$\beta - UCV^{n+1,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq \beta - UVC(\gamma).$$

By (1.18) and (1.19), respectively, we note that $\beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq$ $\subseteq CV^{n,\alpha}_{\lambda,\mu}(\frac{\beta+\gamma}{1+\beta})$ and $\beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma) \subseteq ST^{n,\alpha}_{\lambda,\mu}(\frac{\beta+\gamma}{1+\beta})$.

In [8], basic properties of the classes$\beta{-}UCV^{n,\alpha}_{\lambda,\mu}(\gamma)$ and $\beta{-}SP^{n,\alpha}_{\lambda,\mu}(\gamma)$ are studied, such as inclusion relations and coefficient bounds. In this paper, several interesting subordination results are derived for these classes, which yield sharp distortion, rotation theorems and Koebe domain. Consequences of the main results and their relevance to known results are also pointed out.

2. Subordination theorems and consequence

In order to derive our main results, we need the following lemmas.

Lemma 2.1 (see [26]). *Let f and g be convex univalent functions in Δ. Then so is $f * g$.*

Lemma 2.2 (see [27]). *Let F and G be convex univalent functions in Δ. Also, let $f \prec F$ and $g \prec G$. Then $f * g \prec F * G$.*

Lemma 2.3 (see [25]). *If $\Re(c) \geq 0$ or $c = 0$, then the function*

$$h(c; z) = \sum_{k=1}^{\infty} \frac{(1+c)}{(k+c)} z^k \tag{2.1}$$

is convex univalent.

Lemma 2.4. *For $\lambda \geq \mu \geq 0$, the function $\frac{h_{\lambda,\mu}(z)}{z}$ is convex univalent, where*

$$h_{\lambda,\mu}(z) = z + \sum_{k=2}^{\infty} \frac{1}{1 + (\lambda\mu k + \lambda - \mu)(k-1)} z^k. \tag{2.2}$$

Proof. For special case of $\lambda = \mu = 0$,we obtain the equality $\frac{h_{0,0}(z)}{z} = \frac{1}{1-z}$, which is convex univalent. Let $\lambda \geq \mu > 0$.

Firstly, we will show that the series $\tilde{h}_{\lambda,\mu}(z)$ can be written as convolution of $h(c_1; z)$ and $h(c_2; z)$ as follows:

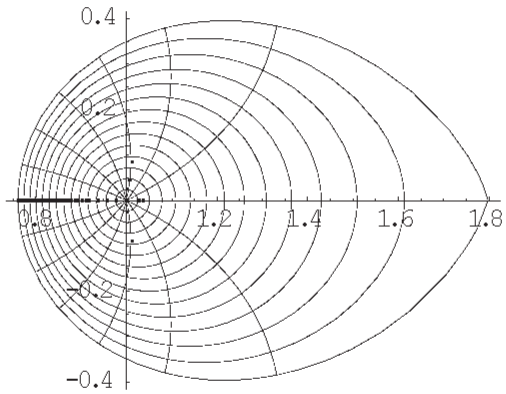
$$\tilde{h}_{\lambda,\mu}(z) = (1 + 2\lambda\mu + \lambda - \mu) \left[\frac{h_{\lambda,\mu}(z)}{z} - 1 \right] = h(c_1; z) * h(c_2; z). \tag{2.3}$$

Then, if we prove $h(c_1; z)$ and $h(c_2; z)$ are convex univalent functions so, we can say that the function $\frac{h_{\lambda,\mu}(z)}{z}$ is also convex univalent where the functions $h(c_1; z)$ and $h(c_2; z)$ are defined by (2.1). To show this, we can rewrite the equality (2.3) as follows:

$$\begin{aligned}
\tilde{h}_{\lambda,\mu}(z) &= (1 + 2\lambda\mu + \lambda - \mu) \left[\frac{h_{\lambda,\mu}(z)}{z} - 1 \right] = \\
&= \sum_{k=1}^{\infty} \frac{1 + 2\lambda\mu + \lambda - \mu}{1 + (\lambda\mu(k+1) + \lambda - \mu)k} z^k = \sum_{k=1}^{\infty} \frac{1 + \left(\frac{\lambda\mu + \lambda - \mu}{\lambda\mu}\right) + \frac{1}{\lambda\mu}}{k^2 + \left(\frac{\lambda\mu + \lambda - \mu}{\lambda\mu}\right)k + \frac{1}{\lambda\mu}} z^k = \\
&= \sum_{k=1}^{\infty} \frac{1 + (c_1 + c_2) + c_1 c_2}{k^2 + (c_1 + c_2)k + c_1 c_2} z^k = \sum_{k=1}^{\infty} \frac{(1 + c_1)}{(k + c_1)} z^k * \sum_{k=1}^{\infty} \frac{(1 + c_2)}{(k + c_2)} z^k = \\
&= h(c_1; z) * h(c_2; z).
\end{aligned}$$

From above equalities we obtain $c_1 + c_2 = \frac{\lambda\mu + \lambda - \mu}{\lambda\mu}$and $c_1 c_2 = \frac{1}{\lambda\mu}$. When we solve c_1 and c_2, we can see easily that $\Re(c_1) = \Re(c_2) > 0$. Therefore, from Lemma 2.3, $h(c_1; z)$ and $h(c_2; z)$ are convex univalent functions and from Lemma 2.1, the function $\frac{h_{\lambda,\mu}(z)}{z}$ is convex univalent.

By using *Mathematica program 7.0*, we can easily see that the function $\frac{h_{\lambda,\mu}(z)}{z}$ is convex.



(a) Graphic of the function $\frac{h_{1,1}(z)}{z}$ $\Diamond$

Theorem 2.1. *Let $M(z) = 1 + m_1 z + m_2 z^2 + \dots$ be a convex and univalent function in Δ and $f \in \mathcal{A}$. If*

$$\frac{D^{n,\alpha}_{\lambda,\mu} f(z)}{z} \prec M(z), \tag{2.4}$$

then we have

$$\frac{f(z)}{z} \prec \left(\frac{1}{z}\right)\{\phi^{\alpha}_{\lambda,\mu}(z) * zM(z)\},$$

where

$$\phi^{\alpha}_{\lambda,\mu}(z) = \underbrace{(\varphi(2-\alpha,2;z) * h_{\lambda,\mu}(z)) * \cdots * (\varphi(2-\alpha,2;z) * h_{\lambda,\mu}(z))}_{n-\text{times}}. \tag{2.5}$$

Proof. Let $h_{\lambda,\mu}$ be defined by (2.2). By using (1.13) and (2.5), we can see that

$$\frac{f(z)}{z} = \left[\frac{\varphi(2-\alpha,2;z) * h_{\lambda,\mu}(z)}{z} * \cdots * \frac{\varphi(2-\alpha,2;z) * h_{\lambda,\mu}(z)}{z}\right] * \frac{D^{n,\alpha}_{\lambda,\mu} f(z)}{z} \tag{2.6}$$

$$= \frac{\phi^{\alpha}_{\lambda,\mu}(z)}{z} * \frac{D^{n,\alpha}_{\lambda,\mu} f(z)}{z}.$$

In [9], it is shown that the function $\frac{\varphi(2-\alpha,2;z)}{z}$ is convex univalent in Δ. Also, by using Lemma 2.4, the function $\frac{h_{\lambda,\mu}(z)}{z}$ is convex univalent in Δ, and applying Lemma 2.1 n-times, we get that $\frac{\phi^{\alpha}_{\lambda,\mu}(z)}{z}$ is convex univalent.

From (2.4), (2.6) and using Lemma 2.2 with $g(z) = G(z) = \frac{\phi^{\alpha}_{\lambda,\mu}(z)}{z}$, we get

$$\frac{f(z)}{z} \prec \frac{\phi^{\alpha}_{\lambda,\mu}(z)}{z} * M(z) = \frac{1}{z}\{\phi^{\alpha}_{\lambda,\mu}(z) * zM(z)\}.$$

By considering the function $f(z) = \phi^{\alpha}_{\lambda,\mu}(z) * zM(z)$, we can show that the result is best possible. ♢

Remark 2.2. For special case of $\alpha = 0$, we have new subordination results for differential operator $D^{n}_{\lambda,\mu}f(z)$ in ([22] and [7]).

Let

$$\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z) = \left(\frac{1}{z}\right)\{\phi^{\alpha}_{\lambda,\mu}(z) * z\mathcal{G}'_{\beta,\gamma}(z)\}, \tag{2.7}$$

where

$$\mathcal{G}'_{\beta,\gamma}(z) = \exp\int_0^z \frac{\hat{P}_{\beta,\gamma}(\xi)-1}{\xi}d\xi, \tag{2.8}$$

and $\phi^{\alpha}_{\lambda,\mu}$, $\hat{P}_{\beta,\gamma}$ are defined by (2.5) and (1.6), respectively.

Using (2.7) and (2.8) we prove the next results.

Theorem 2.3. *Let $\beta+2\gamma \geq 1$. Then the function $\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)$ defined by* (2.7) *is convex univalent in* Δ.

Proof. Now, we observe that

$$\begin{aligned}
&\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z) = \\
&= \frac{\phi^{\alpha}_{\lambda,\mu}(z)}{z} * \mathcal{G}'_{\beta,\gamma}(z) = \\
&= \left[\frac{\varphi(2-\alpha,2;z) * h_{\lambda,\mu}(z)}{z} * \cdots * \frac{\varphi(2-\alpha,2;z) * h_{\lambda,\mu}(z)}{z}\right] * \mathcal{G}'_{\beta,\gamma}(z).
\end{aligned}$$

Also, we must remember that

$\frac{\varphi(2-\alpha,2;z)}{z} \rightarrow$ is convex univalent (in [9]),

$\frac{h_{\lambda,\mu}(z)}{z} \rightarrow$ is convex univalent (from Lemma 2.4),

$\mathcal{G}'_{\beta,\gamma}(z) \rightarrow$ is convex univalent (in [4], Th. 2.3).

Therefore, by Lemma 2.1, $\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)$ is convex univalent. ♢

Theorem 2.4. *Let $\beta+2\gamma \geq 1$ and let f be in the class $\beta - SP^{n;\alpha}_{\lambda,\mu}(\gamma)$. Then we have*

$$\frac{f(z)}{z} \prec \mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z),$$

where $\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)$ is defined by (2.7). *The result is best possible.*

Proof. Let $f \in \beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then by (1.17),

$$\frac{z(D^{n,\alpha}_{\lambda,\mu}f(z))'}{D^{n,\alpha}_{\lambda,\mu}f(z)} \prec \hat{P}_{\beta,\gamma}(z),$$

which implies

$$\frac{z(D^{n,\alpha}_{\lambda,\mu}f(z))'}{D^{n,\alpha}_{\lambda,\mu}f(z)} - 1 \prec \hat{P}_{\beta,\gamma}(z) - 1.$$

Note that $\hat{P}_{\beta,\gamma}(z) - 1$ is a univalent convex function in Δ. Using a result of Goluzin [10] (see also [21] p. 50]), we have

$$\operatorname{Log}\frac{D^{n,\alpha}_{\lambda,\mu}f(z)}{z} \prec \int_0^z \frac{\hat{P}_{\beta,\gamma}(\xi) - 1}{\xi}d\xi.$$

Thus, there exists a function $\omega \in \mathcal{A}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \Delta$ such that

$$\operatorname{Log}\frac{D^{n,\alpha}_{\lambda,\mu}f(z)}{z} = \int_0^z \frac{\hat{P}_{\beta,\gamma}(\omega(\xi)) - 1}{\omega(\xi)}d\omega(\xi),$$

which is equivalent to

$$\frac{D^{n,\alpha}_{\lambda,\mu}f(z)}{z} \prec \exp\left(\int_0^z \frac{\hat{P}_{\beta,\gamma}(\xi) - 1}{\xi}d\xi\right) = \mathcal{G}'_{\beta,\gamma}(z). \qquad \Diamond$$

Theorem 2.5. *Let $\beta + 2\gamma \geq 1$ and let f be in the class $\beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then*

$$\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;-r) \leq \left|\frac{f(z)}{z}\right| \leq \mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;r), \qquad |z| = r < 1 \tag{2.9}$$

and

$$\left|\operatorname{Arg}\frac{f(z_0)}{z_0}\right| \leq \max_{|z|=r}\{\operatorname{Arg}\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)\}, \qquad |z_0| < 1, \tag{2.10}$$

where $\mathcal{F}'_{\beta,\gamma}$ is defined by (2.7)*. Equality holds in* (2.9) *and* (2.10) *for some $z \neq 0$ if and only if f is a rotation of $z\mathcal{F}'_{\beta,\gamma}$.*

Proof. Let $f \in \beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then by Th. 2.4 and Lindelöf's principle of subordination, we get

(2.11)

$$\inf_{|z|\leq r}\Re\{\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)\} \leq \inf_{|z|\leq r}\Re\left\{\frac{f(z)}{z}\right\} \leq \sup_{|z|\leq r}\Re\left\{\frac{f(z)}{z}\right\} \leq \left|\frac{f(z)}{z}\right| \leq$$

$$\leq \sup\left|\frac{f(z)}{z}\right| \leq \sup_{|z|\leq r}\Re\{\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)\}.$$

Since $\mathcal{F}'_{\beta,\gamma}$ is convex univalent and has real coefficient, $\mathcal{F}'_{\beta,\gamma}(\Delta)$ is a convex domain symmetric with respect to real axis. Hence,

$$\inf_{|z|\le r} \Re\{\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)\} = \inf_{-r\le x\le r}\{\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;x)\} = \mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;-r),$$

$$\sup_{|z|\le r} \Re\{\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)\} = \sup_{-r\le x\le r}\{\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;x)\} = \mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;r).$$

Thus (2.11) gives the assertion (2.9) of Th. 2.5.

Similarly, from Th. 2.4, we get the rotation assertion (2.10). Equality holds true in (2.9) and (2.10) for some $z \neq 0,\ z_0 \neq 0$, respectively, if and only if f is a rotation of $f_{\beta,\gamma}(\alpha,\lambda,\mu;z) = z\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)$. $\Diamond$

Since coefficient bounds are sharp for $\beta = 0$, we get a better result for the function $f \in ST^{n,\alpha}_{\lambda,\mu}(\gamma)$ as follows:

Theorem 2.6. *Let f be in the class $ST^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then*

$$|f(z)| \le r\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;r), \qquad |z| = r < 1 \tag{2.12}$$

Furthermore, if $\frac{1}{2} \le \gamma < 1$, then

$$\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;-r) \le \Re\left\{\frac{f(z)}{z}\right\} \le \left|\frac{f(z)}{z}\right|. \tag{2.13}$$

Both estimates (2.12) and (2.13) are sharp if f is a rotation of $f_{0,\gamma}(\alpha,\lambda,\mu;z) = z\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;z)$, where

$$\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;z) = \frac{1}{z}\left(\phi^{\alpha}_{\lambda,\mu}(z) * \frac{z}{(1-z)^{2(1-\gamma)}}\right), \tag{2.14}$$

and $\phi^{\alpha}_{\lambda,\mu}$ is defined by (2.5).

Proof. By direct calculations, we find the following equality

$$z\mathcal{G}'_{0,\gamma}(z) = \frac{z}{(1-z)^{2(1-\gamma)}} = z + \sum_{k=2}^{\infty}\prod_{j=2}^{k}(j-2\gamma)\frac{z^k}{(k-1)!}.$$

Now let $f \in ST^{n,\alpha}_{\lambda,\mu}(\gamma)$. In [8], we proved that

$$|a_k| \le \frac{1}{\Psi_{k,n}(\lambda,\mu,\alpha)}\prod_{j=2}^{k}\frac{j-2\gamma}{(k-1)!},$$

where $\Psi_{k,n}(\lambda,\mu,\alpha)$is defined by (1.12) and we mentioned that the result is sharp. Then we have

$$|f(z)| \le |z| + \sum_{k=2}^{\infty} |a_k| \left|z^k\right| \le |z| + \sum_{k=2}^{\infty} \left(\prod_{j=2}^{k} \frac{j-2\gamma}{(k-1)!} \right) \frac{1}{\Psi_{k,n}(\lambda,\mu,\alpha)} \left|z^k\right| =$$

$$= \left(|z| + \sum_{k=2}^{\infty} \frac{1}{\Psi_{k,n}(\lambda,\mu,\alpha)} \left|z^k\right| \right) * \left(|z| + \sum_{k=2}^{\infty} \left(\prod_{j=2}^{k} \frac{j-2\gamma}{(k-1)!} \right) \left|z^k\right| \right) =$$

$$= r\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;r),$$

which yields (2.12). Next, suppose that $\frac{1}{2} \le \gamma < 1$. Then, by Th. 2.6 and inequality (2.11), we get (2.13). ◊

Corollary 2.7 (Koebe Domain)**.** *Let $\beta+2\gamma \ge 1$ and let f be in the class $\beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then for* $\left[0 < \lambda \le \frac{1+\sqrt{5}}{2} \text{ and } 0 < \mu \text{ and } \lambda - 1 \le \mu \le \frac{\lambda}{1+\lambda}\right]$ *or* $[0 = \mu = \lambda]$ *or* $[0 = \mu < \lambda]$,

$$\mathcal{K}\left(\beta - SP^{n,\alpha}_{\lambda,\mu}(\gamma)\right) = \left\{\omega : \ |\omega| \le \mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;-1) = \right.$$
$$\left. = -f_{0,\gamma}(\alpha,\lambda,\mu;-1)\right\} \subseteq f(\Delta).$$

The result is sharp for a rotation of $f_{\beta,\gamma}(\alpha,\lambda,\mu;z) = z\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)$.

By virtue of (1.16) and Th. 2.4 and Th. 2.5, we get the following results.

Corollary 2.8. *Let $\beta + 2\gamma \ge 1$ and let f be in the class $\beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then*

$$f'(z) \prec \mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z), \tag{2.15}$$

$$\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;-r) \le |f'(z)| \le \mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;r), \tag{2.16}$$

and

$$|\mathrm{Arg} f'(z_0)| \le \max_{|z|=r}\{\mathrm{Arg}\mathcal{F}'_{\beta,\gamma}(\alpha,\lambda,\mu;z)\}, \qquad |z_0| < 1, \tag{2.17}$$

where $\mathcal{F}'_{\beta,\gamma}$ is defined by (2.7). The result (2.15) is best possible and equality holds in (2.16) and (2.17) for some $z \ne 0$, $z_0 \ne 0$, respectively, if and only if f is a rotation of the function $\mathcal{F}_{\beta,\gamma}(\alpha,\lambda,\mu;z)$.

Corollary 2.9. *Let f be in the class $CV^{n,\alpha}_{\lambda,\mu}(\gamma)$. Then*

$$|f'(z)| \le \mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;r), \qquad |z| = r < 1 \tag{2.18}$$

Furthermore, if $\frac{1}{2} \le \gamma < 1$, then

$$\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;-r) \le \Re\{f'(z)\} \le |f'(z)|. \tag{2.19}$$

Both estimates (2.18) *and* (2.19) *are sharp if f is a rotation of* $\mathcal{F}_{0,\gamma}(\alpha,\lambda,\mu;z)$, *where* $\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;z)$ *is defined by* (2.14).

By using standard techniques and Cor. 2.8, we obtain the distortion theorem and Koebe domain of the class $\beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma)$.

Corollary 2.10 (Distortion bound and Koebe domain)**.** *Let* $\beta + 2\gamma \geq 1$ *and let f be in the class* $\beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma)$. *Then*

$$-\mathcal{F}_{\beta,\gamma}(\alpha,\lambda,\mu;-r) \leq |f(z)| \leq \mathcal{F}_{\beta,\gamma}(\alpha,\lambda,\mu;r),$$

and for $\left[0 < \lambda \leq \frac{1+\sqrt{5}}{2} \text{ and } 0 < \mu \text{ and } \lambda - 1 \leq \mu \leq \frac{\lambda}{1+\lambda}\right]$ *or* $[0 = \mu = \lambda]$ *or* $[0 = \mu < \lambda]$,

$$\mathcal{K}\left(\beta - UCV^{n,\alpha}_{\lambda,\mu}(\gamma)\right) = \{\omega : \ |\omega| \leq -\mathcal{F}_{\beta,\gamma}(\alpha,\lambda,\mu;-1)\} \subseteq f(\Delta).$$

The result is sharp for a rotation of $\mathcal{F}_{\beta,\gamma}(\alpha,\lambda,\mu;z)$.

Corollary 2.11. *Let f be in the class* $CV^{n,\alpha}_{\lambda,\mu}(\gamma)$. *Then*

$$|f(z)| \leq \mathcal{F}_{0,\gamma}(\alpha,\lambda,\mu;r), \qquad |z| = r. \tag{2.20}$$

Furthermore, if $\frac{1}{2} \leq \gamma < 1$, *then we have*

$$-\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;-r) \leq |f(z)|. \tag{2.21}$$

Both estimates (2.20) *and* (2.21) *are sharp if f is a rotation of* $\mathcal{F}_{0,\gamma}(\alpha,\lambda,\mu;z)$, *where* $\mathcal{F}'_{0,\gamma}(\alpha,\lambda,\mu;z)$ *is defined by* (2.14).

Remark 2.12. For special values of parameters n, α, γ, λ, μ and β, Th. 2.1–Th. 2.6 and Cor. 2.7–Cor. 2.11 reduce to some results obtained in [4], [12]–[16], [23], [29] and [30].

Remark 2.13. For special values of parameters n, α, γ, λ, μ and β in Th. 2.4–Th. 2.6 and Cor. 2.7–Cor. 2.11, we obtain some subordination results, distortion theorems, rotation theorems and Koebe domains for the classes $\beta - SP^{n}_{\lambda,\mu}(\gamma)$, $\beta - UCV^{n}_{\lambda,\mu}(\gamma)$, $ST^{n,\alpha}_{\lambda,\mu}(\gamma)$, $CV^{n,\alpha}_{\lambda,\mu}(\gamma)$, $SP^{n,\alpha}_{\lambda,\mu}$ and $UCV^{n,\alpha}_{\lambda,\mu}$.

References

[1] AGHALARY, R. and AZADI, G. H.: The Dziok–Srivastava operator and k-uniformly starlike functions, *J. Inequal. Pure Appl. Math.* **6** (2) (2005), 1–7, Article 52 (electronic).

[2] AL-OBOUDI, F. M.: On univalent functions defined by a generalized Sălăgean operator, *Int. J. Math. Math. Sci.* **27** (2004), 1429–1436.

[3] AL-OBOUDI, F. M. and AL-AMOUDI, K. A.: On classes of analytic functions related to conic domains, *J. Math. Anal. Appl.* **339** (2008), 655–667.

[4] AL-OBOUDI, F. M. and AL-AMOUDI, K. A.: Subordination results for classes of analytic functions related to conic domains defined by a fractional operator, *J. Math. Anal. Appl.* **354** (2009), 412–420.

[5] BHARTI, R., PARVATHAM, R. and SWAMINATHAN, A.: On subclasses of uniformly convex functions and corresponding class of starlike functions, *Tamkang J. Math.* **28** (1) (1997), 17–32.

[6] CARLSON, B. C. and SHAFFER, D. B.: Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.* **15** (4) (1984), 737–745.

[7] DENIZ, E. and ORHAN, H.: The Fekete–Szegő problem for a generalized subclass of analytic functions, *Kyungpook Math. J.* **50** (2010), 37–47.

[8] DENIZ, E., ORHAN, H. and SOKOL, J.: Classes of analytic functions defined by a differential operator related to conic domains (submitted).

[9] DINGGONG, Y.: The subclass of starlike functions of order α, *Chinese Ann. Math. Ser. A* **8** (6) (1987), 687–692.

[10] GOLUZIN, G. M.: On the majorization principle in function theory, *Dokl. Akad. Nauk SSSR* **42** (1935), 647–650 (in Russian).

[11] GOODMAN, A. W.: On uniformly convex functions, *Ann. Polon. Math.* **56** (1991), 87–92.

[12] KANAS, S. and WISNIOWSKA, A.: Conic regions and k-uniform convexity, *Comput. Appl. Math.* **105** (1999), 327–336.

[13] KANAS, S. and WISNIOWSKA, A.: Conic domains and starlike functions, *Rev. Roumaine Math. Pures Appl.* **45** (3) (2000), 647–657.

[14] LING, Y. and DING, S.: A class of analytic functions defined by fractional derivation, *J. Math. Anal. Appl.* **186** (2) (1994), 504–513.

[15] MA, W. and MINDA, D.: Uniformly convex functions, *Ann. Polon. Math.* **57** (1992), 165–175.

[16] MISHRA, A. K. and GOCHHAYAT, P.: Applications of the Owa–Srivastava operator to the class of k-uniformly convex functions, *Fract. Calc. Appl. Anal.* **9** (4) (2006), 323–331.

[17] NOOR, K. I., ARIF, M. and UL-HAQ, W.: On k-uniformly close-to-convex functions of complex order, *Appl. Math. Comp.* **215** (2) (2009), 629–635.

[18] ORHAN, H., DENIZ, E. and RADUCANU, D.: The Fekete–Szegő problem for subclasses of analytic functions defined by a differential operator related to conic domains, *Comput. Math. Appl.* **59** (1) (2010), 283–295.

[19] OWA, S.: On the distortion theorems, I, *Kyungpook Math. J.* **18** (1) (1978), 53–59.

[20] OWA, S. and SRIVASTAVA, H. M.: Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39** (5) (1987), 1057–1077.

[21] POMMERENKE, CH.: *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

[22] RADUCANU, D. and ORHAN, H.: Subclasses of analytic functions defined by a generalized differential operator, *Int. Journ. Math. Anal.* **4** (1-4) (2010), 1–16.

[23] RØNNING, F.: On starlike functions associated with parabolic regions, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **45** (14) (1991), 117–122.

[24] RØNNING, F.: Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.* **118** (1) (1993), 189–196.

[25] RUSCHEWEYH, ST.: New criteria for univalent functions, *Proc. Amer. Math. Soc.* **49** (1975), 109–115.

[26] RUSCHEWEYH, ST. and SHEIL-SMALL, T.: Hadamard products of Schlicht functions and the Polya–Schoenberg conjecture, *Comment. Math. Helv.* **48** (1973), 119–135.

[27] RUSCHEWEYH, ST. and STANKIEWICZ, J.: Subordination under convex univalent functions, *Bull. Pol. Acad. Sci. Math.* **33** (9–10) (1985), 499–502.

[28] SĂLĂGEAN, G. S.: Subclasses of univalent functions, in: *Complex Analysis, Fifth Romanian–Finnish Seminar, Part 1, City Bucharest, 1981*, Lecture Notes in Math. vol. 1013, Springer, Berlin, 1983, 362–372.

[29] SRIVASTAVA, H. M., MISHRA A. K. and DAS, M. K.: A nested class of analytic functions defined by fractional calculus, *Comm. Appl. Anal.* **2** (3) (1998), 321–332.

[30] SRIVASTAVA, H. M. and MISHRA, A. K.: Applications of fractional calculus to parabolic starlike and uniformly convex functions, *J. Comput. Math. Appl.* **39** (3/4) (2000), 57–69.