Mathematica Pannonica
23/1 (2012), 75–84

PEDAL CURVES OF CONICS IN PSEUDO-EUCLIDEAN PLANE

Ana **Sliepčević**

University of Zagreb, Faculty of Civil Engineering, 10000 Zagreb, Croatia

Mirela **Katić Žlepalo**

Polytechnics of Zagreb, Department of Civil Engineering, Zagreb, Croatia

Dedicated to the memory of Professor Gyula I. Maurer (1927–2012)

Received: April 2011

MSC 2010: 51 A 05, 51 M 15

Keywords: Pseudo-Euclidean plane, pedal curves, entirely circular quartic, entirely circular cubic.

Abstract: We construct pedal curves of conics in the projective model of the pseudo-Euclidean plane (further in text: PE-plane). Generally pedal curves are circular quartics, but in certain cases they are degenerated into circular cubics or even conics, as in the Euclidean plane. Since the absolute points are real and since there are more types of conics in the PE-plane, there are many types of pedal curves of conics that we can not derive in the Euclidean plane. Those are entirely circular quartics and cubics with different types of singularities in the absolute points or special cases when a pedal curve is degenerated into such type of a conic which does not exist in the Euclidean plane. In this article we show only cases specific to the PE-plane, so we do not construct cases that are analogous to the Euclidean plane.

1. Introduction

Given a curve k and a fixed point P (which we will call the pole of pedal transformation or simply the pole), a pedal point is the intersection of a tangent line of the curve k with the line from P perpendicular to the tangent line. The pedal curve of the curve k with respect to the fixed point P is the locus of all pedal points on all tangent lines of the given curve k. Obtaining the pedal curve for a given curve k is so-called pedal transformation. Instead of dealing with the pedal curve of any curve k, in the present paper we will be dealing with pedal curves of conics. It is well known that a pedal curve of a conic in the Euclidean plane is generally a bicircular rational quartic [6] with singular points in the absolute points and the pole of pedal transformation. Depending on the type of a conic and its position to the pole, a singular point can be a node, a cusp or an isolated double point, where the absolute points are the conjugate-imaginary pair so they are always singular points of the same type. The pedal curve of a parabola degenerates into a circular cubic and the absolute line. If the pole is in a focus of a conic, the pedal curve degenerates into the isotropic lines of that focus and a circle [6].

The absolute points in the PE-plane are real, so there are 9 types of conics (Fig. 1) [3], [5] and more different types of pedal curves than in the Euclidean plane. Some of pedal curves in the PE-plane are analogue to the Euclidean case and we will not construct them here, but there are many types of pedal curves that do not exist in the Euclidean plane. In this article we will construct such curves and show their characteristics.

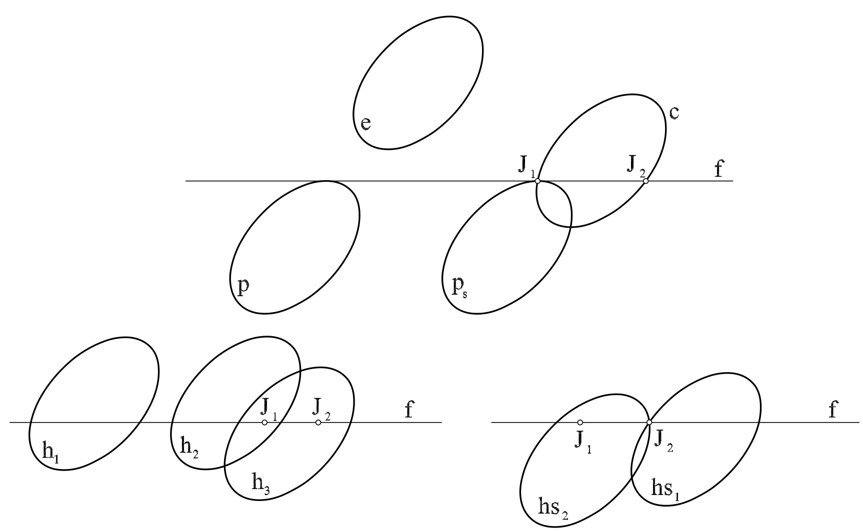

Figure 1

In the present paper, the curves are constructed in the projective model of the PE-plane with the absolute figure (f, J_1, J_2) at finity. This gives the clearer view of a type of singular points for a certain curve. Besides that, this model is suitable because of the fact that we can represent any type of the conic of the PE-plane with the Euclidean circle without loss of generality, which of course simplifies constructions.

2. Perpendicular lines in the PE-plane

In order to construct a pedal curve, it is necessary to explain the construction of perpendicular lines in the PE-plane. It is well known that two lines in the Euclidean plane are perpendicular if they intersect the absolute line in a pair of points associated in the circular involution on the absolute line [4]. In that plane the mentioned involution is elliptical with the pair of conjugate-imaginary double absolute points. The real absolute points J_1, J_2 on the absolute line f in the PE-plane as the double points define a hyperbolic circular involution and the lines a, a' are perpendicular if they intersect the absolute line in a pair of points associated in this hyperbolic involution. The construction (Fig. 2) is based on the properties of the complete quadrangle and the following known theorem about involutions:

Theorem 1. *The double elements together with a pair of corresponding elements of an involution form a harmonic range* [1], [2].

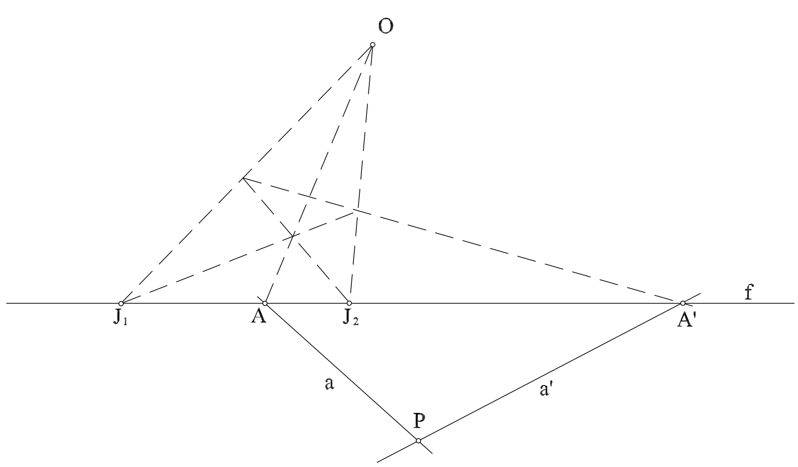

Figure 2

3. Entirely circular curves in the PE-plane

Definition 1. A curve k_n of the n-th order in the PE-plane is called circular if it passes through at least one absolute point. If a curve has no other intersections with the absolute line but the absolute points, it is called entirely circular.

Theorem 2. *A pedal curve of a conic in the PE-plane is entirely circular rational quartic.*

Proof. The proof is analogue to the proof of the theorem about pedal curves of conics in the Euclidean plane [1]. The pedal curve is the result of associating the pencil of lines (P) of the 1st order with the vertex in the pole and the pencil of lines (k_2) of the 2nd order containing all tangent lines of the given conic. A pair of tangent lines of a conic is associated to every line in the pencil (P) and one line in the pencil (P) is associated to every tangent line of the conic (k_2). From the Chasles' relation [6] we can conclude that the product of such an association is the curve of the 4th order. Since the isotropic tangent line and its perpendicular line through P pass through the same absolute point, it is clear that the absolute points are double points of the pedal curve. Two tangent lines of a conic pass through the pole P so it is a double point of the pedal curve too.

4. Pedal curves analogue to the ones in the Euclidean plane

With the general position of the pole, the pedal curve of an ellipse, a hyperbola of type h_1, h_3 and a circle c is analogue to the corresponding bicircular quartic in the Euclidean plane. The pedal curve of a parabola degenerates into a circular cubic and the absolute line, which is also analogue to the Euclidean plane. The pedal curve has nodes in the absolute points in the case of an ellipse and a hyperbola of type h_1, isolated double points in the case of a hyperbola of type h_3 and cusps in the case of a circle c. In the case of a parabola p, the pedal curve intersects the absolute line in the point F'. In the circular involution on the absolute line f, this point is associated to the point F where the absolute line touches the parabola p. If the pole is in a focus of any of these conics, the pedal curve is degenerated analogously to such degenerations in the Euclidean plane. All of these cases are not shown in this article. The focus of a conic is defined analogously as in the Euclidean plane – as the intersection of isotropic tangent lines of that conic.

5. Pedal curves with no analogue curves in the Euclidean plane

5.1. Entirely circular quartics

An entirely circular rational quartic which does not exist in the Euclidean plane is the pedal curve of a hyperbola of type h_2 with the general position of the pole (Fig. 3). Since one of the absolute points is inside and the other outside of the hyperbola, this conic has one pair of the conjugate-imaginary isotropic tangent lines and one pair of the real isotropic tangent lines. Hence this pedal curve has the isolated double point in one absolute point and the node in the other absolute point.

A cusp in one absolute point and a node in another one appears in the case of the pedal curve of a special hyperbola hs_1 (Fig. 4). A cusp in one and an isolated double point in another absolute point is on the pedal curve of a special hyperbola of type hs_2 with the general position of the pole P (Fig. 5).

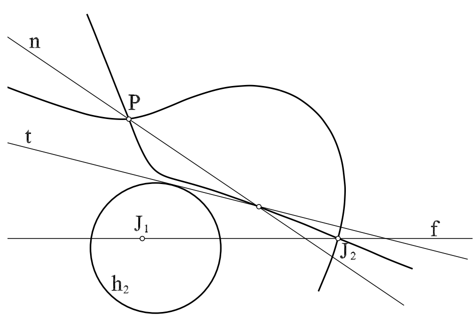

Figure 3

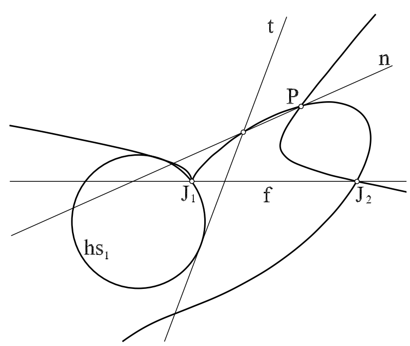

Figure 4

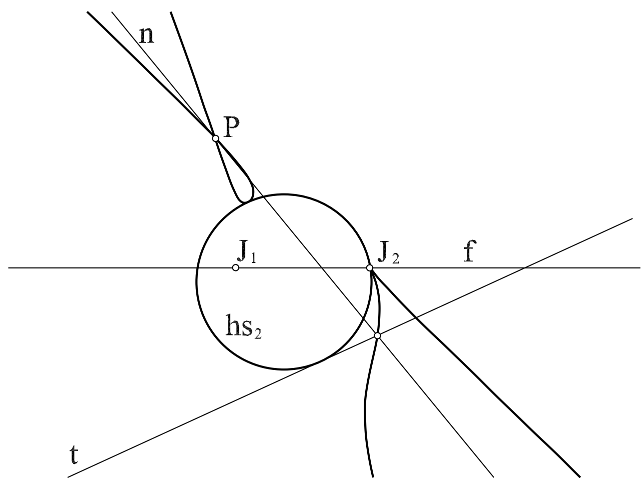

Figure 5

5.2. Entirely circular cubics

Since the absolute points in the PE-plane are real, the pole of the pedal transformation may be on the real isotropic tangent line of a conic. The results for such positions of the pole are very interesting and original pedal curves. The pedal curve in such cases degenerates into the isotropic tangent line that contains the pole and an entirely circular cubic. This isotropic tangent line passes through one absolute point and in that absolute point the cubic intersects the absolute line f. In another absolute point the cubic has a double point.

We construct such cubics as the pedal curves of an ellipse and two special hyperbolae (Figures 6, 7, 8). In the case of an ellipse e (or the hyperbola of type h_1), the double point in the absolute point is a node since two real tangent lines of the conic pass through J_2 (Fig. 6). In the case of a special hyperbola of type hs_1 we have a cusp in the absolute point J_1 (Fig. 7), since we have a double tangent line through that point. For a special hyperbola of type hs_2 the absolute point J_2 is an isolated double point of the cubic (Fig. 8) because two tangent lines through that point make a conjugate-imaginary pair. We observe in that same Fig. 8 that the conic and the pedal curve hyperosculate in another absolute point.

In the sense of circularity, such circular cubics do not exist in the

Euclidean plane.

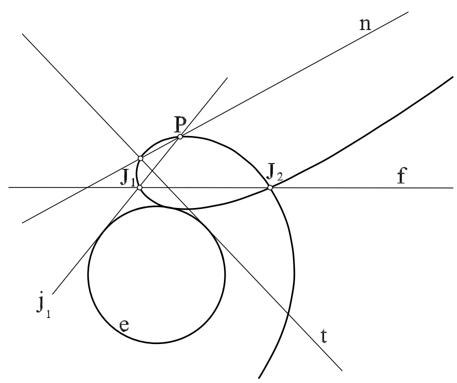

Figure 6

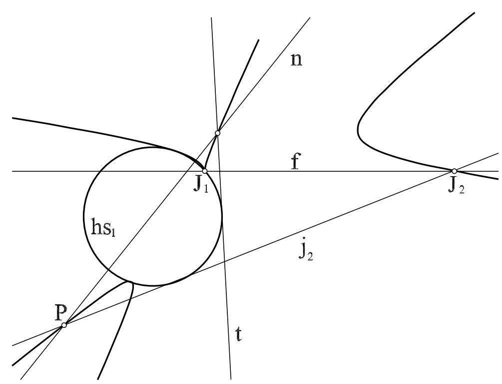

Figure 7

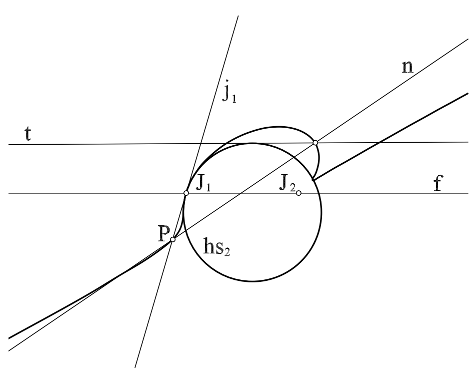

Figure 8

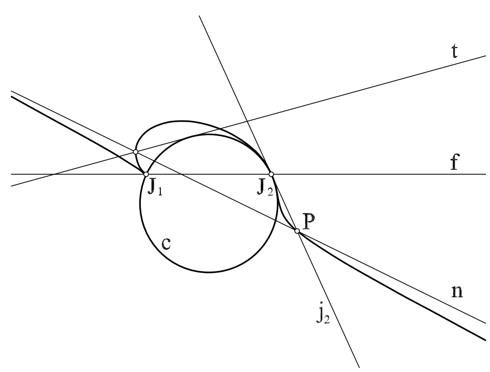

Figure 9

We get a very interesting pedal curve of a circle c when the pole is on one of the isotropic tangent lines (Fig. 9). This cubic is of the same type as the cubic in Fig. 7, but it's interesting because the pedal curve and the circle hyperosculate in the absolute point that lies on the mentioned isotropic line.

In Fig. 10 we show the pedal curve of a special parabola p_s with the general position of the pole. In the sense of circularity, this is the entirely circular cubic which does not exist in the Euclidean plane. Namely, the absolute line f is tangent both to the parabola p_s and the cubic.

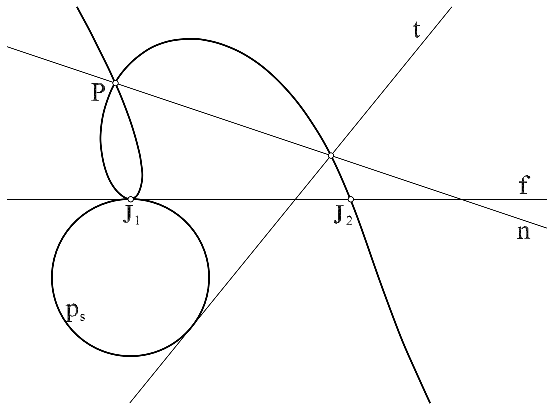

Figure 10

6. Special degenerations in the PE-plane

When the pole is in a focus of a conic, the pedal curve degenerates into a circle and a pair of isotropic lines as in the Euclidean plane. Here are the cases of degeneration which are specific for the PE-plane:

1. If the pole is on the isotropic tangent line of a parabola p, the pedal curve degenerates into that isotropic tangent line, the absolute line and a special hyperbola of type hs_2 (Fig. 11). This does not exist in the Euclidean plane since there is no such a hyperbola in that plane.

2. In Fig. 12 we see the pedal curve of a special hyperbola of type hs_1 with the pole in its focus. The pedal curve degenerates into a pair of isotropic tangent lines through that focus and the hyperosculating circle of the given hyperbola in its absolute vertex J_1.

3. Besides a circle, a special parabola p_s is also an entirely circular conic, although it contains only one absolute point. We show its pedal curve with the pole on the isotropic tangent line in Fig. 13. The pedal curve degenerates into the absolute line f, the isotropic tangent line and a special parabola.

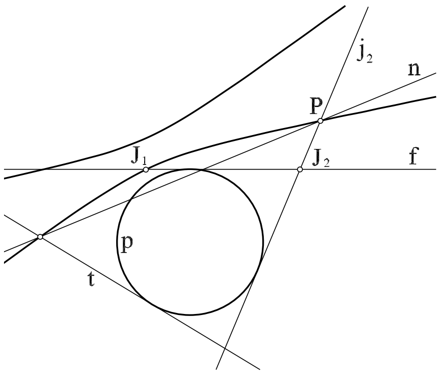

Figure 11

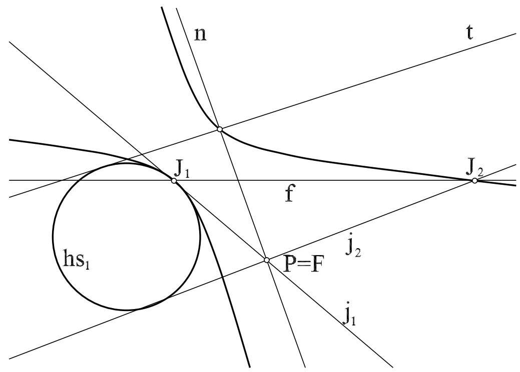

Figure 12

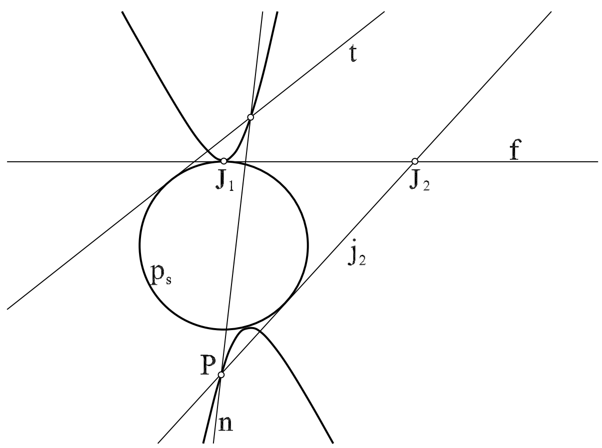

Figure 13

7. Conclusion

We can conclude that by using the pedal transformation we can construct six types of entirely circular quartics and four types of entirely circular cubics. Since those are not all the possible types of entirely circular quartics and cubics in the PE-plane, the authors intend to continue searching for other types of those curves and other possibilities how to obtain them using other types of transformations.

References

[1] MÜLLER, E. and KRAMES, J. L.: *Vorlesungen über darstellende Geometrie*, Bd. II, Leipzig und Wien, 1929.

[2] NIČE, V.: *Uvod u sintetičku geometriju*, Školska knjiga, Zagreb, 1956.

[3] REVERUK, N. V.: Krivie vtorogo porjadka v psevdoevklidovoi geometrii, *Uchenye zapiski Moskovskogo pedagogicheskogo instituta* **253** (1969), 160–177.

[4] SCHAAL, H.: Euklidische und pseudoeuklidische Sätze über Kreis und gleichseitige Hyperbel, *Elemente der Mathematik* **19** (1964), 53–56.

[5] SZIROVICZA, V. and KOVAČEVIĆ, N.: Inversion in Minkowskischer Geometrie, *Mathematica Pannonica* **21**/1 (2010), 89–113.

[6] WIELEITNER, H.: *Spezielle ebene Kurven*, Leipzig, 1908.