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ON GAUSS LEMNISCATE FUNCTIONS AND LEMNISCATIC MEAN II

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Dedicated to the memory of Professor Gyula I. Maurer (1927–2012)

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Abstract: This paper deals with inequalities for the Gauss lemniscate functions arcsl and arcslh and also with inequalities for another pair of lemniscate functions arctl and arctlh which have been introduced by the author in [6]. Simple computable lower and upper bounds for the quadruple of lemniscate functions are also obtained.

1. Introduction

Gauss' arc lemniscate sine and the hyperbolic arc lemniscate sine are defined, respectively, as follows

$$\operatorname{arcsl} x = \int_0^x \frac{dt}{\sqrt{1-t^4}}$$

($|x| \le 1$) and

$$\operatorname{arcslh} x = \int_0^x \frac{dt}{\sqrt{1+t^4}}$$

($x \in \mathbb{R}$) (see [2, (2.5)–(2.6)], [1, p. 259], [12, Ch. 1]). It is well known that the arc length s measured from the origin to a point with polar coordinates on the Bernoulli lemniscate $r^2 = \cos 2\theta$ is $s = \operatorname{arcsl} r$.

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Both lemniscate functions are elliptic integrals. It follows from [13, 19.14.4 and 19.14.7] that

$$\operatorname{arcsl} x = \frac{1}{\sqrt{2}} F\left(\phi, \frac{1}{\sqrt{2}}\right),$$

where $F(\phi, k)$ is the Legendre incomplete elliptic integral of the first kind and $\cos^2\phi = \frac{1-x^2}{1+x^2}$. Similarly, for $|x| \le 1$

$$\operatorname{arcslh} x = \frac{\operatorname{sign} x}{2} F\left(\phi, \frac{1}{\sqrt{2}}\right),$$

where now $\cos\phi = \frac{1-x^2}{1+x^2}$ (see [13, 19.14.3]).

This paper is a continuation of our earlier work [6] and is organized as follows. In Sec. 2 we recall definitions of another pair of lemniscate functions arctl and arctlh. Definitions of three R-hypergeometric functions are also included there. In the next section we provide information about several bivariate means which play an important role in the subsequent parts of this paper. The main results of this paper are presented in the remaining sections. In Sec. 4 we give several inequalities involving four functions under discussion. Computable lower and upper bounds for four lemniscate functions are obtained in the last section of this paper.

2. Lemniscate functions arctl and arctlh

To facilitate presentation we recall definitions of three R-hyperbolic functions R_B, R_F and R_C. Following [2, (3.14)]

$$R_B(x, y) = \frac{1}{4}\int_0^\infty (t+x)^{-3/4}(t+y)^{-1/2}\,dt \tag{2.1}$$

($x > 0$, $y \ge 0$). Also, we will need a completely symmetric elliptic integral of the first kind

$$R_F(x, y, z) = \frac{1}{2}\int_0^\infty \left[(t+x)(t+y)(t+z)\right]^{-1/2} dt, \tag{2.2}$$

where at most one of the nonnegative variables x, y, z is 0 (see [3, (9.2-1)]). Third R-hypergeometric function R_C is the degenerate case of R_F:

$$R_C(x, y) = R_F(x, y, y). \tag{2.3}$$

Bounds for R_F expressed in terms of R_C are obtained in [4] and [5].

Four lemniscate functions arcsl, arcslh, arctl, and arctlh admit representations in terms of R_B. B. C. Carlson [2, (4.1)] has shown that

$$\text{arcsl}\, x = xR_B(1, 1 - x^4) \tag{2.4}$$

($|x| \le 1$) and

$$\text{arcslh}\, x = xR_B(1, 1 + x^4) \tag{2.5}$$

($x \in \mathbb{R}$). In [6] the author defined functions arctl and arctlh as follows

$$\text{arctl}\, x = xR_B(1 + x^4, 1) \tag{2.6}$$

($x \in \mathbb{R}$) and

$$\text{arctlh}\, x = xR_B(1 - x^4, 1) \tag{2.7}$$

($|x| \le 1$).

For later use let us recall that the inverse circular and inverse hyperbolic functions can be represented in terms of R_C [3, Ex. 6.9-16]:

$$\arcsin x = xR_C(1 - x^2, 1), \quad \text{arcsinh}\, x = xR_C(1 + x^2, 1), \tag{2.8}$$

$$\arctan x = xR_C(1, 1 + x^2), \quad \text{arctanh}\, x = xR_C(1, 1 - x^2). \tag{2.9}$$

These formulas will be utilized in Sec. 4.

3. Bivariate means used in this paper

In this section we recall definitions of several bivariate means of nonnegative numbers x and y of which at most one is 0. Tactically we will assume that $x \ne y$.

The letters G, A, and Q will stand for the geometric, arithmetic, and the root-mean-square means of x and y, i.e.,

$$G = \sqrt{xy}, \quad A = \frac{x+y}{2}, \quad Q = \sqrt{\frac{x^2+y^2}{2}}\,.$$

Other means employed in this paper include

$$\begin{aligned} L &= A\frac{z}{\text{arctanh}\, z}\,, \quad & P &= A\frac{z}{\arcsin z}\,, \\ T &= A\frac{z}{\arctan z}\,, \quad & M &= A\frac{z}{\text{arcsinh}\, z}\,, \end{aligned} \tag{3.1}$$

where

$$z = \frac{x-y}{x+y}\,. \tag{3.2}$$

Here L is the logarithmic mean of positive numbers x and y, P and T are the first and second Seiffert mean (see [10, 11]), and the mean M was introduced in [8].

Another mean which plays a crucial role in this paper is the lemniscate mean $LM(x, y) \equiv LM$ $(x > 0,\ y \geq 0)$. It is an iterative mean, i.e.,

$$LM = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n\,,$$

where

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = (x_{n+1}x_n)^{1/2}$$

$(n = 0, 1, \ldots)$ (see [2] and [1]). For more properties of this mean the interested reader is referred to [6]. The lemniscatic mean can be expressed in terms of four lemniscate functions as follows

$$LM(x,y) = \frac{\sqrt{x^2-y^2}}{(\mathrm{arcsl}\ \sqrt[4]{1-(y/x)^2})^2} = \frac{\sqrt{x^2-y^2}}{(\mathrm{arctl}\ \sqrt[4]{(x/y)^2-1})^2} \tag{3.3}$$

$(0 \leq y < x)$ and

$$LM(x,y) = \frac{\sqrt{y^2-x^2}}{(\mathrm{arcslh}\ \sqrt[4]{(y/x)^2-1})^2} = \frac{\sqrt{y^2-x^2}}{(\mathrm{arctlh}\ \sqrt[4]{1-(x/y)^2})^2} \tag{3.4}$$

$(x < y)$. See [6, (6.1)–(6.2)].

It is worth mentioning that

$$LM(x,y) = [R_B(x^2, y^2)]^{-2} \tag{3.5}$$

(see [2, (3.10) and (3.14)]).

Another quadruple of bivariate means was introduced in [6, (6.4)]. They are defined as follows

$$U = LM(G,A), \quad V = LM(A,G), \quad R = LM(A,Q), \quad S = LM(Q,A). \tag{3.6}$$

It has been demonstrated that [6, (6.6)–(6.7)]

$$\begin{aligned} U &= A\left(\frac{\sqrt{z}}{\mathrm{arctlh}\ \sqrt{z}}\right)^2, \quad V = A\left(\frac{\sqrt{z}}{\mathrm{arcsl}\ \sqrt{z}}\right)^2, \\ R &= A\left(\frac{\sqrt{z}}{\mathrm{arcslh}\ \sqrt{z}}\right)^2, \quad S = A\left(\frac{\sqrt{z}}{\mathrm{arctl}\ \sqrt{z}}\right)^2, \end{aligned} \tag{3.7}$$

where z is defined in (3.2).

4. Inequalities involving lemniscate functions

In this section we give several inequalities involving lemniscate functions defined in the first two sections of this paper.

Our first result reads as follows.

Theorem 4.1. *Let* $0 < |x| < 1$ *and let* $t = x^2$*. Then*

$$\begin{aligned}\frac{\arctan t}{t} &< \left(\frac{\operatorname{arctl} x}{x}\right)^2 < \left(\frac{\operatorname{arcslh} x}{x}\right)^2 < \frac{\operatorname{arcsinh} t}{t} < 1 < \\ &< \frac{\arcsin t}{t} < \left(\frac{\operatorname{arcsl} x}{x}\right)^2 < \left(\frac{\operatorname{arctlh} x}{x}\right)^2 < \frac{\operatorname{arctanh} t}{t}\,.\end{aligned} \tag{4.1}$$

Proof. In order to establish (4.1) we shall utilize a long inequality

$$L < U < V < P < A < M < R < S < T$$

(see [6, (6.10)]). Using (3.1) and (3.4) with $x := 1 + x^2$, $y := 1 - x^2$ we obtain the assertion because $z = x^2$ (see (3.2)). The proof is complete. $\Diamond$

For later use, let us record four formulas for the lemniscatic mean of some pairs of functions.

It follows from (3.3) and (3.4) that

$$LM(1, \sqrt{1 - x^4}) = \left(\frac{x}{\operatorname{arcsl} x}\right)^2 \tag{4.2}$$

($|x| \le 1$) and

$$LM(\sqrt{1 - x^4}, 1) = \left(\frac{x}{\operatorname{arctlh} x}\right)^2 \tag{4.3}$$

($|x| < 1$). Using again (3.4) and (3.5) we obtain

$$LM(\sqrt{1 + x^4}, 1) = \left(\frac{x}{\operatorname{arctl} x}\right)^2 \tag{4.4}$$

and

$$LM(1, \sqrt{1 + x^4}) = \left(\frac{x}{\operatorname{arcslh} x}\right)^2. \tag{4.5}$$

We are in a position to prove the following.

Theorem 4.2. *Let* $0 < |x| < 1$*. Then*

$$\left(\frac{\operatorname{arcsl} x}{x}\right)^4 < \frac{\operatorname{arctlh} x}{x\sqrt[4]{1 - x^4}} \tag{4.6}$$

and

$$\left(\frac{\operatorname{arctlh} x}{x}\right)^4 < \frac{\operatorname{arcsl} x}{x\sqrt{1-x^4}} \,. \tag{4.7}$$

If $x \neq 0$, *then*

$$\left(\frac{\operatorname{arcslh} x}{x}\right)^4 < \frac{\operatorname{arctl} x}{x\sqrt[4]{1+x^4}} \tag{4.8}$$

and

$$\left(\frac{\operatorname{arctl} x}{x}\right)^4 < \frac{\operatorname{arcslh} x}{x\sqrt{1+x^4}} \,. \tag{4.9}$$

Proof. Inequalities (4.6)–(4.9) are the special cases of the following one

$$x^2 y LM(y,x) < [LM(x,y)]^4 \tag{4.10}$$

($x > 0$, $y > 0$, $x \neq y$) which has been established in [6, (5.12)]. Inequality (4.6) is obtained using (4.10) with $x = 1$ and $y = \sqrt{1-x^4}$ followed by application of (4.2). The remaining inequalities (4.7)–(4.9) can be proven in an analogous manner. For instance, letting in (4.10) $x := \sqrt{1-x^4}$, $y = 1$ and using (4.3), we obtain (4.7). For the proof of (4.8)–(4.9) we let in (4.10) $x = 1$, $y = \sqrt{1+x^4}$ and $x := \sqrt{1+x^4}$, $y = 1$, respectively, followed by application of (4.4) and (4.5), respectively. This completes the proof. ♢

Inequalities (4.6)–(4.7) can be regarded as the lemniscate counterparts of the two-sided inequality (2.1) in [7]. Similarly, inequalities (4.8)–(4.9) are the lemniscate counterparts of (2.2) in [7].

The last theorem of this section reads as follows.

Theorem 4.3. *Let* $|x| < 1$. *Then*

$$\left(\frac{\operatorname{arcsl} x}{x}\right)^2 < \frac{\arctan x}{x}\,\frac{\operatorname{arctanh} x}{x} \,, \tag{4.11}$$

$$1 < \frac{\operatorname{arcsl} x}{x}\,\frac{\operatorname{arcslh} x}{x} \,, \tag{4.12}$$

$$1 < \frac{\operatorname{arctl} x}{x}\,\frac{\operatorname{arctlh} x}{x} \,, \tag{4.13}$$

$$\frac{\operatorname{arcsl} x}{x}\,\frac{\operatorname{arctlh} x}{x} < (1-x^4)^{-1/4} \tag{4.14}$$

$$\frac{\text{arctl } x}{x} \frac{\text{arcslh } x}{x} < (1+x^4)^{-1/4}. \tag{4.15}$$

Proof. In order to establish the inequality (4.11) we use the following formula

$$\text{arcsl } x = x R_F(1-x^2, 1+x^2, 1)$$

(see [3, Ex. 8.3-7]) together with the inequality

$$R_F(x,y,z) < \big[R_C(z,x) R_C(z,y)\big]^{1/2}$$

(see [4, Thm. 3.3]) to obtain

$$\left(\frac{\text{arcsl } x}{x}\right)^2 < R_C(1, 1-x^2) R_C(1, 1+x^2).$$

Application of (2.9) to the right side of the last inequality gives the desired result (4.11). We shall establish now inequalities (4.12) and (4.13) using strict logarithmic convexity of the R_B function in its variables (see [9, Prop. 2.1]). In particular, one has

$$R_B(p,u) R_B(p,v) > \left[R_B\left(p, \frac{u+v}{2}\right)\right]^2$$

($p > 0$, $u \ge 0$, $v \ge 0$, $u \ne v$). Letting above $p = 1$, $u = 1 - x^4$, $v = 1 + x^4$ and next using (2.4) and (2.5) we obtain

$$\frac{\text{arcsl } x}{x} \frac{\text{arcslh } x}{x} = R_B(1, 1-x^4) R_B(1, 1+x^4) > \big[R_B(1,1)\big]^2 = 1.$$

Inequality (4.13) can be established in an analogous manner using strict logarithmic convexity of R_B in its first variable

$$R_B(u,p) R_B(v,p) > \left[R_B\left(\frac{u+v}{2}, p\right)\right]^2.$$

Letting above $u = 1+x^4$, $v = 1-x^4$, $p = 1$ and next using formulas (2.6) and (2.7) we obtain the desired result. Inequalities (4.14) and (4.15) can be obtained using, respectively, the following ones

$$AG < UV \tag{4.16}$$

and

$$AQ < RS. \tag{4.17}$$

We shall establish the last two inequalities at the end of the proof of this theorem. In order to demonstrate that (4.16) implies (4.14) we let

$x := 1 + x^2$ and $y := 1 - x^2$ $(0 < x^2 < 1)$. Then $G = (1 - x^4)^{1/2}$, $A = 1$, $Q = (1 + x^4)^{1/2}$ and $z = x^2$. Making use of (4.16) and (3.7) we obtain

$$(1 - x^4)^{1/2} < \Big(\frac{x}{\operatorname{arctlh} x}\,\frac{x}{\operatorname{arcsl} x}\Big)^2.$$

Hence (4.14) follows. Similarly, using (4.17) and (3.7) we obtain the assertion (4.15). In the proofs of the inequalities (4.16) and (4.17) we shall utilize the left inequality in

$$(x^3y^2)^{1/3} < LM(x, y) < \frac{3x + 2y}{5} \tag{4.18}$$

(see [6, Lemma 4.1]). Using the first formula in (3.6) we have

$$(G^3A^2)^{1/5} < U.$$

The second formula in (3.6) together with (4.18) gives

$$(A^3G^2)^{1/5} < V.$$

Multiplying the corresponding sides of the last two inequalities we obtain (4.16). Inequality (4.17) can be established in a similar manner. We omit further details. The proof is complete. $\Diamond$

5. Computable bounds for lemniscate functions

Simple lower and upper bounds for the lemniscate functions are obtained in the following.

Theorem 5.1. *Let* $0 < |x| < 1$. *Then*

$$\Big(\frac{5}{3 + 2(1 - x^4)^{1/2}}\Big)^{1/2} < \frac{\operatorname{arcsl} x}{x} < (1 - x^4)^{-1/10}, \tag{5.1}$$

$$\Big(\frac{5}{3 + 2(1 + x^4)^{1/2}}\Big)^{1/2} < \frac{\operatorname{arcslh} x}{x} < (1 + x^4)^{-1/10}, \tag{5.2}$$

$$\Big(\frac{5}{3(1 - x^4)^{1/2} + 2}\Big)^{1/2} < \frac{\operatorname{arctlh} x}{x} < (1 - x^4)^{-3/20}, \tag{5.3}$$

$$\Big(\frac{5}{3(1 + x^4)^{1/2} + 2}\Big)^{1/2} < \frac{\operatorname{arctl} x}{x} < (1 + x^4)^{-3/20}. \tag{5.4}$$

Proof. In order to obtain the two-sided inequality (5.1) we utilize (4.18) with $x = 1$ and $y = (1-x^4)^{1/2}$ followed by use of (4.2). The remaining inequalities (5.2)–(5.4) can be established in a similar way. This completes the proof. $\Diamond$

We close this section with the remark that the tighter and more complicated bounds for the lemniscate functions can be obtained using the two-sided inequality [6, (4.4)]

$$(xA^4)^{1/5} < LM(x,y) < \left(\frac{3A^2 + 2xA}{5}\right)^{1/2},$$

where A is the arithmetic mean of x and y.

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