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CLASSES OF SEMIGROUPS WITH COMPATIBLE NATURAL PARTIAL ORDER II

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Dedicated to the memory of Professor Gyula I. Maurer (1927–2012)

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Abstract: In continuation of the survey, "Classes of semigroups with compatible natural partial order I" (Math. Pann. 22 (2011), 165–198), the problem of (right) compatibility of the natural partial order of a semigroup is dealt with for the classes of E-inversive, of eventually regular, and of regular semigroups, respectively. Again, as far as possible the structure of the resulting semigroups is described and methods to construct them are given. Furthermore, general observations concerning the natural partial order of strong semilattices of semigroups, (iterated) inflations of semigroups, and generalized Rees matrix semigroups are included.

Introduction

This is the continuation of the survey, "Classes of semigroups with compatible natural partial order I" (Math. Pann. 22 (2011), 165–198). Recall that the natural partial order on a semigroup S is defined by

$$a \leq_S b \text{ if and only if } a = xb = by,\ xa = a (= ay) \text{ for some } x, y \in S^1 ([18]).$$

Also, $\leq_S$ is right (left, two-sided) compatible if $a \leq_S b$ implies $ac \leq_S bc$ ($ca \leq_S cb$, both) for any $c \in S$.

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In part I necessary and/or sufficient conditions for a semigroup S were given in order that the natural partial order $\leq_S$ of S be right (two-sided) compatible (Sec. 1). Furthermore, semigroups S for which $\leq_S$ is trivial or total were considered (Sec. 2). Also several well-known classes of semigroups with right (two-sided) compatible natural partial orders were specified (Sec. 3). In particular, (E-)medial semigroups were dealt with (Sec. 4). Throughout, sufficient conditions for right compatibility of $\leq_S$ with multiplication by particular elements were given.

In this second part, we give answers to the compatibility problem for the classes of E-inversive (Sec. 5), of eventually regular (Sec. 6), and of regular semigroups (Sec. 7). In an appendix, (1) a criterion for the right (two-sided) compatibility of $\leq_S$ for strong semilattices of semigroups, in which every element has a right and a left identity, is proved, (2) iterated inflations of a trivially-ordered semigroup are introduced, thus providing a method for the construction of semigroups with an arbitrary number of elements and a non-trivial two-sided compatible natural partial order (the same can be achieved by strong semilattices of trivially-ordered semigroups, in which every element has a right and a left identity), (3) semigroups, in which every element has a right and a left identity, are dealt with, and (4) generalized Rees matrix semigroups are considered in more detail.

A list of semigroups S for which $\leq_S$ is right (two-sided) compatible and which are dealt with in parts I and II, is given in the Introduction of part I. The numbering of the sections in part II is continued from part I.

5. E-inversive semigroups

A semigroup S is *E-inversive* if for every $a \in S$ there exists $x \in S$ such that $ax \in E(S)$ ([30]; see [3], EX.3.2(8)). Defining for any $a \in S$:

$$W_r(a) = \{x \in S | ax \in E(S)\}, W(a) = \{x \in S | x = xax\},$$

a semigroup S is E-inversive if and only if for every $a \in S : W_r(a) \neq \phi$, resp. $W(a) \neq \phi$. The elements of $W(a)$ are called weak inverses of $a \in S$. For basic properties of E-inversive semigroups see [21].

Examples: Semigroups with zero; eventually regular semigroups, in particular, regular or groupbound semigroups, hence periodic or finite semigroups; simple semigroups S with $E(S) \neq \phi$ (if $a \in S, e \in E(S)$, then according to P.M. Higgins, $e = xay$ for some $x, y \in S$ and $a \cdot yex \in E(S)$);

in particular, any Bruck-semigroup over a monoid; a generalized Rees matrix semigroup $S = \mathcal{M}(I, T, \Lambda; P)$ is E-inversive if and only if T is E-inversive (sufficiency: if $(i, a, \lambda) \in S$ then for any $j \in I$, $\mu \in \Lambda$ there is $x \in T$ with $x = x \cdot p_{\mu i} a p_{\lambda j} \cdot x$, hence $(i, a, \lambda)(j, x, \mu) \in E(S)$; necessity: if $a \in T$ then $a \cdot p_{\lambda j} b p_{\mu i} \in E(T)$ for $j \in I, b \in T, \mu \in \Lambda$, given by $(i, a, \lambda)(j, b, \mu) \in E(S)$); inflations of an E-inversive semigroup; strong semilattices $S = \langle Y, S_\alpha; \varphi_{\alpha,\beta} \rangle$ of semigroups such that $(Y, \leq_Y)$ has a least element $\mu \in Y$ and S_μ is E-inversive.

First we indicate the E-inversive semigroups S whose natural partial order is the identity relation which is evidently two-sided compatible. Note that if S has a zero and $\leq_S$ is trivial, then S consists of one element only (see Sec. 2).

Result 5.1 ([19])**.** *Let S be an E-inversive semigroup. Then $\leq_S$ is the identity relation if and only if S is completely simple.*

Corollary ([20])**.** *Let S be a finite semigroup. Then $\leq_S$ is the identity relation if and only if S is simple.*

Result 5.2. *Let S be an E-inversive semigroup. Then $\leq_S$ is the identity relation and $E(S)a \subseteq aS^1$ for any $a \in S$ if and only if S is a right group.*

Proof. Sufficiency is clear. Necessity: By Res. 5.1, S is completely simple, i.e., $S = \mathcal{M}(I, G, \Lambda; P)$ and $E(S) \neq \phi$. Since $E(S)a \subseteq aS^1$ for any $a \in S$, it follows that $|I| = 1$, thus S is right simple. Therefore, S is a right group, by [3], Th. 1.27. $\Diamond$

For monoids we have

Result 5.3. *Let S be an E-inversive monoid. Then $\leq_S$ is the identity relation if and only if S is a group.*

Proof. Necessity: Let $e \in E(S)$; then $e \leq_S 1_S$ implies that $e = 1_S$ and $E(S) = \{1_S\}$. Thus for every $a \in S$ there exists $a' \in S$ such that $aa' = 1_S$. Therefore S is a group. $\Diamond$

By Res. 3.2, for a semigroup S satisfying $Sa \subseteq aS^1$ for every $a \in S$, $\leq_S$ is right compatible. If S is E-inversive we have more precisely:

Result 5.4. *Let S be an E-inversive semigroup such that $Sa \subseteq aS^1$ for every $a \in S$. Then $\leq_S$ is right compatible and non-trivial if and only if S is not a right group.*

Proof. Necessity: For any right group S, $\leq_S$ is the identity relation – see Sec. 2. Sufficiency: If $\leq_S$ was the identity relation then by Res. 5.2, S is a right group. $\Diamond$

Remark. In particular, for any finite commutative semigroup S which is not an abelian group, $\leq_S$ is non-trivial and two-sided compatible (since a commutative right group is an abelian group).

Result 5.5. *Let S be an E-inversive monoid such that $a \leq_S b$ implies $a = bf$ for some $f \in E(S)$. Then $\leq_S$ is right compatible and non-trivial if and only if $E_S a \subseteq aS$ for every $a \in S$ and S is not a group.*

Proof. Necessity holds by Res. 1.8(ii) and Res. 5.3. Sufficiency follows from Res. 1.11 and Res. 5.3. $\lozenge$

Remark. In particular, for any finite monoid S with central idempotents which is not a group, $\leq_S$ is non-trivial and two-sided compatible.

Next we consider E-inversive semigroups S, which are *E-unitary*: $ea, e \in E(S)$ implies $a \in E(S)$. Note that for an E-unitary semigroup S, also $ae, e \in E(S)$ implies $a \in E(S)$: since $eae \in E(S)$ and $eae \cdot a \in E(S)$, it follows that $a \in E(S)$. Furthermore, if S is E-inversive and E-unitary then $E(S)$ is a subsemigroup of S: let $e, f \in E(S)$; then $ef \cdot x \in E(S)$ for some $x \in S$; thus $e \cdot fx \in E(S)$ implies $fx \in E(S)$, and therefore $x \in E(S)$; consequently, $ef \cdot x \in E(S)$ implies $ef \in E(S)$.

For certain semigroups we have a characterization of E-unitariness, which is known for inverse semigroups: following M. Petrich we call a semigroup S *pure* if $e \leq_S a$, $e \in E(S)$, $a \in S$, implies $a \in E(S)$. Note that for inverse semigroups the general conditions in (i) and (ii) of Res. 5.6 below are always satisfied.

Result 5.6. (i) *Let S be a semigroup such that $E(S)a \subseteq aS^1$ for any $a \in S$. Then S is E-unitary if and only if S is pure.*

(ii) *Let S be an E-inversive semigroup. Then S is E-unitary if and only if S is pure and $E(S)$ is a subsemigroup of S.*

Proof. (i) Necessity: Let $e \leq_S a$, $e \in E(S)$. Then by [20], $e = fa$ for some $f \in E(S)$. Since S is E-unitary, it follows that $a \in E(S)$.

Sufficiency: Let $e, ea \in E(S)$. Then the element $f := ea = ax$ $(x \in S^1)$ satisfies $f \leq_S a$ and $f \in E(S)$; therefore $a \in E(S)$.

(ii) Necessity: The first part is proved as in (i); the second holds by the observation above. Sufficiency: Let $e, ea \in E(S), a \in S$. Then we have for $a' \in W_r(a)$ that $aa' \in E(S)$ and $f := aa' \cdot ea \in E(S)$. Since $f = (aa'e)a = a(a'ea)$ and $aa' \cdot e \in E(S)$, we get $f \leq_S a$ and thus $a \in E(S)$. $\lozenge$

Remarks. (i) By the proof of necessity in Res. 5.6 (i) *any* E-unitary semigroup is pure. (ii) Let S be a trivially-ordered semigroup such that

$E(S)a \subseteq aS^1$ for any $a \in S$; then S is E-unitary. In particular, any trivially-ordered monoid is E-unitary (since $E(S) = \{1_S\}$); more generally, any unipotent monoid is so. Furthermore, any inflation S of a unipotent monoid T such that $1_T \in T$ is *not* inflated is E-unitary (since $E(S) = \{1_T\}$).

Result 5.7. *Let S be an E-inversive E-unitary monoid. Then $\leq_S$ is right compatible if and only if $E(S)a \subseteq aS$ for every $a \in S$. In particular, $\leq_S$ is two-sided compatible if and only if $aE(S) = E(S)a$ for any $a \in S$.*

Proof. We show that $a \leq_S b$ implies $a = eb = bf$ $(e, f \in E(S))$. Let $a \leq_S b$; then $a = xb$, $xa = a$ $(x \in S)$. Thus, for (any) $a' \in W_r(a)$, $x \cdot aa' = aa' \in E(S)$, so that $x \in E(S)$. Similarly, $a = bf$ for some $f \in E(S)$. The first statement now follows from Res. 1.12, the second from Res. 1.13. (See also Res. 5.5.) $\Diamond$

Remarks. (i) For any E-inversive E-unitary *semigroup* S, the natural partial order has the form: $a \leq_S b$ if and only if $a = eb = bf$ for some $e, f \in E(S^1)$.

(ii) By [8], Th. 3.14, the E-inversive E-unitary monoids S are precisely the *generalized F-monoids* (i.e., there is a group congruence ρ on S such that the identity ρ-class of S contains a greatest element with respect to $\leq_S$). Examples are given in [8].

By Remark (i) above we obtain the following

Corollary. *Let S be an E-inversive E-unitary semigroup with central idempotents. Then $\leq_S$ is two-sided compatible.*

Remark. All semilattices and groups satisfy the conditions of the Corollary; more generally, any Clifford semigroup with injective linking homomorphisms (see [13], V. Ex. 10). Also, any inflation S of a group G such that the identity of G is *not* inflated has these properties (if $S \neq G$ then S is not regular); concerning $\leq_S$ see Sec. 8(C).

In case that an E-inversive semigroup is rectangular (see Remark (iii) following Res. 4.7) we have

Result 5.8. *Let S be an E-inversive rectangular semigroup. Then $\leq_S$ is two-sided compatible; more precisely, $a \leq_S b$ implies $ac = bc$ and $ca = cb$ for every $c \in S$.*

Proof. By [3], Ex. 3.2(11), S is an inflation of a rectangular group T. Since $\leq_T$ is the identity relation on T (see Sec. 2), the claim follows (see Sec. 8(C)). $\Diamond$

Remarks. (i) By [33], a semigroup S is E-inversive and rectangular if and only if S is an inflation of a rectangular group.

(ii) By [30], see also [3], Ex. 3.2 (9), an E-inversive semigroup S is rectangular if and only if S is stationary on the right (and on the left) – see Remark (iv) following Res. 4.7. Therefore, Res. 5.8 also follows from Res. 4.8 and its dual.

Finally, we consider E-inversive semigroups S such that whenever $ax = a$ $(a, x \in S)$ then $ax' = a$ for some $x' \in W_r(x)$. Examples:

(1) Infinite monogenic semigroups with a zero adjoined, i.e., $S = \langle c\rangle^0$; S is E-inversive since $0 \in S$. Let $ax = a$; if $a \neq 0$ then also $x \neq 0$ and thus $a = c^m$, $x = c^n$ for some $m, n > 0$; therefore $c^{n+m} = c^m$, a contradiction. Hence $a = 0$ and $ax' = a$ for (every) $x' \in W_r(x)$.

(2) Nilsemigroups: S is E-inversive since $0 \in S$. If $ax = a$, then $ax^k = a$ for every $k > 0$, hence $a = a0 = 0$ if $x^n = 0$. Thus $ax' = a$ for (every) $x' \in W_r(x)$.

(3) Completely regular semigroups: S is E-inversive since S is regular. If $ax = a$ and $x = xx'x$ for $x' \in S$ such that $xx' = x'x$, then $x' \in W_r(x)$ and $ax' = ax \cdot x' = a \cdot xx' = a \cdot x'x = ax \cdot x'x = ax = a$.

(4) Groupbound semigroups: S is E-inversive since for any $a \in S$ there is $k > 0$ such that $a^k \in G_e$ (=subgroup of S with identity $e \in E(S)$). If $ax = a$ then $ax^n = a$ for every $n > 0$, hence for some $k > 0 : ax^k = a$ where $x^k \in G_e$. Thus $ae = ax^k(x^k)^{-1} = a(x^k)^{-1}$ and so $a = ax^k = a \cdot ex^k = a(x^k)^{-1} \cdot x^k = ae$. Therefore $x' := x^{k-1}(x^k)^{-1} \in S$ satisfies: $xx' = e \in E(S)$, i.e., $x' \in W_r(x)$, and $ax' = ax \cdot x^{k-1}(x^k)^{-1} = ae = a$ (for $k = 1$ take $x' = x^{-1}$).

Result 5.9. *Let S be an E-inversive semigroup such that $ax = a$ $(a, x \in S)$ implies $ax' = a$ for some $x' \in W_r(x)$. If $E(S)a \subseteq aS^1$ for every $a \in S$, then $\leq_S$ is right compatible.*

Proof. We show that $a \leq_S b$ $(a, b \in S)$ implies $a = bf$ for some $f \in E(S^1)$. Let $a <_S b$; then $a = by$, $ay = a$, for some $y \in S$. Thus also $ay' = a$ for some $y' \in W_r(y)$, and $a = ay' = byy' = bf$ with $f = yy' \in E(S)$. Hence the statement follows from Res. 1.11. $\Diamond$

Remarks. (i) For any E-inversive semigroup S satisfying the first condition in Res. 5.9, $a \leq_S b$ if and only if $a = xb = bf$ for some $x \in S^1$, $f \in E(S^1)$.

(ii) The condition "$E(S)a \subseteq aS^1$ for every $a \in S$" is not necessary for $\leq_S$ to be right compatible: let $S = \mathcal{M}(I, G, \Lambda, P)$ be a completely simple semigroup with $|I| > 1$. If $(i, g, \lambda)(j, x, \mu) = (i, g, \lambda)$ then $\mu = \lambda$

and $x = p_{\lambda j}^{-1}$; hence $(j, x, \mu) \in E(S)$ and $(j, x, \mu) \in W_r(j, x, \mu)$. Furthermore, $\leq_S$ is the identity relation (see Sec. 2), hence is (right) compatible. But for $e = (i, p_{\lambda i}^{-1}, \lambda) \in E(S)$, $a = (j, g, \lambda) \in S$ with $i \neq j$, we have $ea \notin aS^1$. However, for monoids we get from Res. 1.8 (ii) and Res. 5.9:

Result 5.10. *Let S be an E-inversive monoid such that $ax = a$ $(a, x \in S)$ implies $ax' = a$ for some $x' \in W_r(x)$. Then $\leq_S$ is right compatible if and only if $E(S)a \subseteq aS$ for every $a \in S$.*

With respect to compatibility of $\leq_S$ with multiplication by particular elements we have

Result 5.11. *Let S be an E-inversive semigroup such that $\leq_S$ is right (left) compatible with multiplication by idempotents. Then $\leq_S$ is right (left) compatible with multiplication by all weak inverses of any element of S.*

Proof. Since for any $c \in S$, every $c' \in W(c)$ is a regular element of S, the claim follows by Res. 1.14. $\diamondsuit$

Remark. Let S be an E-inversive semigroup with commuting idempotents such that $ax = a$ $(a, x \in S)$ implies $ax' = a$ for some $x' \in W_r(a)$; then by Remark (i) following Res. 5.9, $\leq_S$ is right compatible with multiplication by idempotents.

6. Eventually regular semigroups

A semigroup S is *eventually regular* (called: π-regular in [17]) if for every $a \in S$ there is $n > 0$ such that $a^n \in S$ is a regular element. Examples: regular semigroups and groupbound semigroups, in particular, periodic or finite or nil semigroups. Also, every inflation S of any of these semigroups T is eventually regular (for every $a \in S, a^2 \in T$), but not regular if $S \neq T$ $(axa \in T$ for any $a, x \in S)$.

With respect to the form of the natural partial order of an eventually regular semigroup, we observe the following:

(1) Let $\mathrm{Reg}(S)$ be the set of all regular elements of an eventually regular semigroup S. Then we have:

$a \leq_S b$ if and only if $a = rb = bs,\ ra = a = as,$ for some $r, s \in \mathrm{Reg}(S^1)$.

Indeed, let $a <_S b$, i.e., $a = xb = by$, $xa = a = ay$, for some $x, y \in S$. Then $a = x^k b = x^k a$ for every $k > 0$. Since $x^n \in \mathrm{Reg}(S)$ for some $n > 0$, we get $a = rb$, $ra = a$, for some $r \in \mathrm{Reg}(S)$. Similarly, $a = bs$, $as = a$, for some $s \in \mathrm{Reg}(S)$. This gives the following

Result 6.1. *Let S be an eventually regular semigroup. If* $\mathrm{Reg}(S)a \subseteq aS^1$ *for every $a \in S$ then $\leq_S$ is right compatible.*

(2) Let $CR(S)$ be the set of all completely regular elements of an eventually regular semigroup S. If $\mathrm{Reg}(S) = CR(S)$ then we have:

$$a \leq_S b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S^1).$$

Indeed, let $a <_S b$; then by (1), $a = rb = bs$, $ra = a = as$ for some $r, s \in \mathrm{Reg}(S) = CR(S)$. Thus $r = rr'r$, $r'r = r'r$ for some $r' \in S$ and $a = ra = rr'r \cdot a = rr' \cdot ra = r'r \cdot a = r' \cdot ra = r' \cdot rb = eb$ with $e = r'r \in E(S)$.

Similarly, $a = bf$ for $f = ss' \in E(S)$. This characterization of $\leq_S$ yields (by Results 1.11, 1.12, and 1.13):

Result 6.2. *Let S be an eventually regular semigroup such that* $\mathrm{Reg}(S) = CR(S)$. *Then the following hold:*

(i) *If $E(S)a \subseteq aS^1$ for every $a \in S$ then $\leq_S$ is right compatible.*

(ii) *If S is a monoid then $\leq_S$ is right compatible if and only if $E(S)a \subseteq aS$ for every $a \in S$.*

(iii) *If S is a monoid then $\leq_S$ is two-sided compatible if and only if $aE(S) = E(S)a$ for every $a \in S$.*

Remark. By [17] (Theorem of Shevrin–Veronesi), an eventually regular semigroup S satisfies $\mathrm{Reg}(S) = CR(S)$ if and only if S is a semilattice of archimedean semigroups each of which contains a primitive idempotent. For further characterizations of such semigroups see [17], Th. 1.5.7 and Th. 3.5.4.

(3) Finally, we consider eventually regular semigroups S with central idempotents. Their natural partial order has a very particular form:

$$a \leq_S b \text{ if and only if } a = eb \text{ for some } e \in E(S^1).$$

Indeed, such a semigroup S is groupbound: let $a \in S$; then $a^n = a^n x a^n$ for some $n > 0$, $x \in S$, and thus $a^n = x(a^n)^2 = (a^n)^2 x$; it follows from [25], IV.1.2 that a^n is contained in a subgroup of S. Therefore, $a \leq_S b$ if and only if $a = eb = bf$ for some $e, f \in E(S^1)$ (see the Introduction). Since the idempotents of S are central, the statement follows. This characterization of $\leq_S$ gives

Result 6.3. *Let S be an eventually regular semigroup with central idempotents. Then $\leq_S$ is two-sided compatible.*

Examples of eventually regular semigroups with central idempotents are given by:

(i) Nilsemigroups S (since $E(S) = \{0\}$); note that S is not regular.

(ii) Inflations S of a Clifford semigroup $T, S = \bigcup_{\alpha \in T} T_\alpha$. If $a \in S$, $a \in T_\alpha$ say, and $e \in E(S) = E(T)$, then e is central in T, hence $ae = \alpha e = e\alpha = ea$. Note that S is not regular and that $\leq_S$ is not trivial if $S \neq T$ (see Sec. 8(C)).

Two further classes of eventually regular semigroups should be mentioned in this context:

Result 6.4. *Let S be an eventually regular semigroup such that $E(S)$ forms a right zero semigroup. Then $\leq_S$ is right compatible.*

Proof. Let $a, b \in S$; then $a^n = a^n x a^n$, $b^m = b^m y\, b^m$ for some $m, n > 0$, $x, y \in S$. Since $a^n x, b^m y \in E(S)$, we have $a^n = (b^m y \cdot a^n x)a^n \in bS$. Therefore, S is a right archimedian semigroup. It follows by Res. 3.11 that $\leq_S$ is right compatible. $\Diamond$

Remark. By [17], Th. 1.4.6, a semigroup S satisfies the conditions in Res. 6.4 if and only if S is a nil-extension of a right group. For example: any inflation S of a right group T (note that $\leq_S$ is not trivial if $S \neq T$; see Sec. 8(C)).

Together with its dual, Res. 6.4 gives

Result 6.5. *Let S be an eventually regular semigroup containing exactly one idempotent. Then $\leq_S$ is two-sided compatible.*

Remark. By [17], Th. 1.4.7, a semigroup S satisfies the condition in Res. 6.5 if and only if S is a nil-extension of a group G. If G is finite then S is powerjoined (and $\leq_S$ is two-sided compatible, by Res. 3.16): let $a, b \in S \backslash G$ and $k = |G|$; then $a^m, b^n \in G$ for some $m, n > 0$ and $a^{mk} = 1_G = b^{nk}$. For example: any inflation of a (finite) group.

With respect to compatibility of $\leq_S$ with multiplication by particular elements we have by Res. 1.14:

Result 6.6. *Let S be an eventually regular semigroup such that $\leq_S$ is right (left) compatible with multiplication by idempotents. Then $\leq_S$ is right (left) compatible with multiplication by some power of any element of S.*

Remark. Let S be an eventually regular semigroup such that $\mathrm{Reg}(S)e \subseteq eS$ for any $e \in E(S)$; then $\leq_S$ is right compatible with multiplication by idempotents (see observation (1) at the beginning of this section).

An important subclass of the class of all eventually regular semigroups is that of *groupbound* semigroups. Examples: nilsemigroups, periodic (in particular, finite) semigroups, completely regular semigroups, eventually regular semigroups with central idempotents (see the paragraph preceding Res. 6.3), and every inflation S of any of these (since $a^2 \in T$ for any $a \in S$).

Since the natural partial order of a groupbound semigroup S has the particular form: $a \leq_S b$ if and only if $a = eb = bf$ for some $e, f \in E(S)$ (see the Introduction), the Corollary of Res. 1.1 gives

Result 6.7. *Let S be a groupbound semigroup. Then $\leq_S$ is right compatible if and only if $aeb \in abS^1$ for all $a, b \in S, e \in E(S)$, such that $ae \in S^1 a$.*

Furthermore we obtain (see Res. 4.7 and its dual)

Result 6.8. *Let S be a groupbound semigroup with $aeb = ab$ for all $a, b \in S$, $e \in E(S)$. Then $\leq_S$ is two-sided compatible; more precisely, $a \leq_S b$ implies $ac = bc$ and $ca = cb$ for every $c \in S$.*

Remark. Let S be an inflation of a group $G : S = \bigcup_{g \in G} T_g$. Then S is groupbound ($a^2 \in G$ for every $a \in S$) and satisfies $aeb = ab$ for all $a, b \in S, e \in E(S)$, since $E(S) = E(G) = \{1_G\}$. Note that any inflation S of a completely regular semigroup T (that is, of a union of groups) is also groupbound, but in general S does not satisfy $aeb = ab$ for all $a, b \in S$, $e \in E(S)$: by [27], II.1.4, T is a semilattice Y of completely simple semigroups T_α ($\alpha \in Y$); if $|Y| > 1$, $\alpha >_Y \beta$, $a, b \in T_\alpha$, and $e \in E(T_\beta)$, then $ab \in T_\alpha$, but $aeb \in T_\beta$. However, if $\leq_T$ is right compatible then so is $\leq_S$ (see Sec. 8(C)).

Again we have a sufficient condition, which for monoids is also necessary (see Results 1.12 and 1.13):

Result 6.9. *Let S be a groupbound semigroup. If $E(S)a \subseteq aS^1$ for every $a \in S$ then $\leq_S$ is right compatible.*

Result 6.10. *Let S be a groupbound monoid. Then $\leq_S$ is right compatible if and only if $E(S)a \subseteq aS^1$ (equivalently, $E(S)a \subseteq aE(S)$) for every $a \in S$. In particular, $\leq_S$ is two-sided compatible if and only if $aE(S) = E(S)a$ for every $a \in S$.*

7. Regular semigroups

Recall that for a regular semigroup S, $a \leq_S b$ if and only if $a = eb = bf$ for some $e, f \in E(S)$. First, we indicate the regular semigroups S for which the natural partial order is trivial. Since S is E-inversive, Res. 5.1 gives

Result 7.1. *Let S be a regular semigroup. Then $\leq_S$ is the identity relation if and only if S is completely simple.*

Corollary ([4])**.** *Let B be a band. Then $\leq_B$ is the identity relation if and only if B is a rectangular band (i.e., $efe = e$ for any $e, f \in B$).*

With respect to the general problem of compatibility we first have, by Res. 1.14:

Result 7.2. *For a regular semigroup S, $\leq_S$ is right (left, two-sided) compatible if and only if $\leq_S$ is right (left, two-sided) compatible with multiplication by idempotents.*

Remark. Let S be a regular semigroup such that $aef \in afS$ for any $a \in S$, $e, f \in E(S)$; then obviously $\leq_S$ is right compatible with multiplication by idempotents. For example, let $S = Y \times G$, where Y is a semilattice and G is a group; thus the idempotents of S commute and the condition is trivially satisfied (note that S is inverse – see Res. 7.8). Also, every right group S has this property, since $E(S)$ forms a right zero band (see Res. 7.28, Corollary). Note that a regular semigroup S has this property if and only if $aeb \in abS$ for all $a, b \in S$, $e \in E(S)$ (necessity: if $b = bb'b \in S$ then $aeb = (a \cdot e \cdot bb')b \in a \cdot bb'Sb \subseteq abS$) – see Res. 7.11 below.

Besides of this result there are three characterizations of regular semigroups S, for which $\leq_S$ is right compatible.

Result 7.3. *([1]) Let S be a regular semigroup. Then $\leq_S$ is right compatible if and only if S is locally $\mathcal{L}$-unipotent (i.e., for every $e \in E(S)$, each $\mathcal{L}$-class of eSe contains exactly one idempotent).*

For monoids this yields a simple characterization of right compatibility.

Result 7.4. *Let S be a regular monoid. Then $\leq_S$ is right compatible if and only if $E(S)$ forms a right regular band (i.e., $efe = fe$ for all $e, f \in E(S)$).*

Proof. Necessity holds by Res. 1.8(i). Sufficiency: For any $c = cc'c \in S$, $ac = e \cdot bc = bfc = b \cdot fcc' \cdot c = b \cdot cc'fcc' \cdot c = bc \cdot z$ $(z \in S)$; hence $ac \leq_S bc$. $\Diamond$

Remark. By [32] (see also [27], II. 4.10), for a regular semigroup S, $E(S)$ is a right regular band if and only if S is $\mathcal{L}$-unipotent. In [32] such a semigroup is called *right inverse.* Since sufficiency in Res. 7.4 holds for any regular semigroup we obtain the following

Corollary. *Let S be a right inverse semigroup. Then $\leq_S$ is right compatible.*

Remark. Let S be a right inverse semigroup; then $a \leq_S b$ if and only if $a = eb$ for some $e \in E(S)$: necessity holds since S is regular; conversely, $a = eb = (e \cdot bb')b = (bb' \cdot e \cdot bb')b = bx$ $(x \in S)$ implies $a \leq_S b$ (note that $x \in E(S)$ choosing $b' \in S$ such that $b = bb'b$, $b' = b'bb'$).

In the case of a completely regular semigroup S, note first that every local submonoid $S_e = eSe$ $(e \in E(S))$ of S is again completely regular: for $a \in S_e$ let $a^* = ea'e \in S_e$ with $a' \in S$ such that $aa'a = a$, $aa' = a'a$; then (since $ae = ea = a$) $aa^*a = a$, $aa^* = aea'e = aa'e = = a'ae = a'a = aa' = eaa' = ea'a = ea'ea = a^*a$. By Res. 7.3 and the Remark above, $\leq_S$ is right compatible if and only if for any $e \in E(S)$, $E(S_e)$ is a right regular band. By [27], V.3.1 (dual), $E(S_e)$ is a right regular band if and only if S_e is a semilattice of right groups. Thus we obtain

Result 7.5. *Let S be a completely regular semigroup. Then $\leq_S$ is right compatible if and only if S is locally a semilattice of right groups.*

Remark. For completely regular *monoids* S we have by Res. 7.4 that $\leq_S$ is right compatible if and only if S is a right regular orthogroup. A construction of the latter was given in [35] (see also [27], V.2.5).

With respect to two-sided compatibility, Res. 7.3 gives

Result 7.6. *([23]) Let S be a regular semigroup. Then $\leq_S$ is two-sided compatible if and only if S is locally inverse.*

Again, for monoids we obtain a simple characterization:

Result 7.7. *Let S be a regular monoid. Then $\leq_S$ is two-sided compatible if and only if S is inverse.*

Proof. Necessity: By Res. 7.4 and its dual, $E(S)$ is a commutative band. Hence S is an inverse semigroup. Sufficiency holds by Res. 7.8 below. $\Diamond$

By [27], IV.2.3, this result implies the following

Corollary. *Let S be a completely regular monoid. Then $\leq_S$ is two-sided compatible if and only if S is a Clifford semigroup.*

Result 7.8 ([31])**.** *Let S be an inverse semigroup. Then $\leq_S$ is two-sided compatible.*

The second characterization of right compatibility is given in

Result 7.9 ([27], II.4.11)**.** *Let S be a regular semigroup. Then $\leq_S$ is right* *compatible if and only if S satisfies $\mathcal{L}$-majorization (i.e., if $a, b, c \in S$ are such that $a \geq_S b$, $a \geq_S c$ and $b \,\mathcal{L}\, c$, then $b = c$; equivalently, if $e, f, g \in E(S)$ are such that $e \geq_S f$, $e \geq_S g$, then $fgf = gf$).*

By [27], II.4.14, for completely regular semigroups the conjunction of $\mathcal{L}$- and $\mathcal{R}$-majorization is equivalent to $\mathcal{D}$-majorization. Thus we obtain by Res. 7.9 and its dual the following

Corollary ([27], IV.1.6)**.** *Let S be a completely regular semigroup. Then $\leq_S$ is two-sided compatible if and only if S satisfies $\mathcal{D}$-majorization.*

The third characterization of right compatibility follows from the proof of the Corollary of Res. 1.1:

Result 7.10. *Let S be a regular semigroup. Then $\leq_S$ is right compatible if and only if $aeb \in abS$ for all $a, b \in S$, $e \in E(S)$, such that $ae \in E(S)a$.*

Using the last remark following Res. 7.2 we obtain the following

Corollary. *Let S be a regular semigroup. Then $\leq_S$ is right compatible if and only if $aef \in afS$ for any $a \in S$, $e, f \in E(S)$, such that $ae \in E(S)a$.*

This result again allows to specify particular classes of regular semigroups, for which $\leq_S$ is right or even two-sided compatible.

Result 7.11. *Let S be a regular semigroup such that $aeb \in abS$ for all $a, b \in S$, $e \in E(S)$ (equivalently, $aef \in afS$ for any $a \in S$, $e, f \in E(S)$). Then $\leq_S$ is right compatible.*

Remarks. (i) A regular semigroup S satisfies the condition in Res. 7.11 if and only if $efg \in egS$ for any $e, f, g \in E(S)$ – sufficiency: let $a = aa'a$, $b = bb'b \in S$, $e \in E(S)$; then for some $x \in S$, $aeb = a(a'a \cdot e \cdot bb')b = = a(a'a \cdot bb' \cdot x)b \in abS$.

(ii) Every right inverse semigroup (see the Corollary of Res. 7.4) satisfies the condition in Res. 7.11, by Remark (i): $fg = gfg$. Furthermore, any rectangular group S (in particular, right or left group) satisfies this condition – and its dual: $aeb \in Sab$ (note that in this case, $\leq_S$ is the identity relation). Also, any regular semigroup with commuting idempotents, that is, every inverse semigroup, satisfies the condition in Res. 7.11 and its dual – see Res. 7.8. More generally, every regular E-medial semigroup, i.e., any generalized inverse semigroup, satisfies both conditions – see Remarks (i) and (iv) following Res. 7.18 below. Finally, every completely simple semigroup $S = \mathcal{M}(I, G, \Lambda; P)$ satisfies both conditions; note that in this case, $\leq_S$ is the identity relation (see Res. 7.1) and that in general, $E(S)$ is not a subsemigroup of S.

(iii) Let S be a regular semigroup; then S satisfies $aeb \in abS$ for any $a, b \in S$, $e \in E(S)$ *and* $E(S)$ is a subsemigroup of S if and only if $E(S)$ is right seminormal band (i.e., $egefg = efg$ for all $e, f, g \in E(S)$): concerning sufficiency, let $a = aa'a, b = bb'b \in S, e \in E(S)$; then $aeb = = a(a'a \cdot e \cdot bb')b = a(a'a \cdot bb' \cdot a'a \cdot e \cdot bb') \in abS$; with respect to necessity, let $e, f, g \in E(S)$; then for some $x \in S, efg = egx$ and $eg \cdot efg = eg \cdot egx = = egx = efg$. It follows that the class of regular semigroups, for which $\leq_S$ is right compatible, properly contains the class of regular semigroups S such that $E(S)$ forms a right seminormal band (see the last example in (ii) above and Res. 7.27 below).

As a particular case we obtain (see Res. 4.7):

Result 7.12. *Let S be a regular semigroup such that $aeb = ab$ for all $a, b \in S$, $e \in E(S)$. Then $\leq_S$ is the identity relation; in fact, S is a rectangular group.*

Proof. We have for all $e, f \in E(S) : efe = e^2 = e$, thus $E(S)$ is a rectangular band. It follows by [25], IV.3.5 that S is a rectangular group. Therefore, $\leq_S$ is the identity relation – see Sec. 2. $\Diamond$

Res. 1.20 implies for regular semigroups:

Result 7.13. *Let S be a regular semigroup such that $aeb = eab$ ($aeb = = abe$) for all $a, b \in S$, $e \in E(S)$ (equivalently, $aef = eaf$ ($efa = eaf$) for any $a \in S$, $e, f \in E(S)$). Then $\leq_S$ is two-sided compatible.*

Remark. Let S be a regular semigroup satisfying $aeb = eab$ for any $a, b \in S$, $e \in E(S)$. Then S is completely regular: for any $a \in S$, $a = a \cdot a'a = a(a' \cdot aa' \cdot a) = a(aa' \cdot a' \cdot a) \in a^2S$; it follows by [3], Th. 4.3, that S is completely regular. Thus, S satisfies the identity $ax^\circ b = x^\circ ab$, where $x^\circ \in S$ denotes the identity of the subgroup of S to which $x \in S$ belongs. Hence we obtain by [27], IV.2.12 (dual), that a regular semigroup S satisfies $aeb = eab$ for every $a, b \in S$, $e \in E(S)$, if and only if S is a strong semilattice Y of right groups S_α ($\alpha \in Y$). Note that $\leq_S$ is not trivial if $|Y| > 1$, since for $\alpha >_Y \beta$ and $e \in E(S_\alpha)$, $e\varphi_{\alpha,\beta} <_S e$ ($e\varphi_{\alpha,\beta} \in E(S_\beta)$). Observe also that S is E-medial – see Res. 7.18 below.

Result 7.14. *Let S be a regular semigroup such that $aba = ba$ for all $a, b \in S$. Then $\leq_S$ is right compatible; in fact, S is a right regular band.*

Proof. For every $a \in S$, $a = aa'a = a'a \in E(S)$, and for all $e, f \in E(S) = S$, $efe = fe$. The statement now follows from Res. 1.9, Corollary. $\Diamond$

As a particular case of Res. 7.13 we get

Result 7.15. *Let S be a regular semigroup such that $axy = ayx$ for* *all $a, x, y \in S$ (i.e., S is right commutative). Then $\leq_S$ is two-sided compatible.*

Remarks. (i) In Res. 7.15, "right" can be replaced by "left".

(ii) By [26], IV.4.5(3) ([22], Cor. 10.1), a semigroup S satisfies the conditions in Res. 7.15, if and only if S is a strong semilattice of left abelian groups. Note that S is medial – see the following Res. 7.16.

Since every regular element of a semigroup has a right and a left identity, we obtain from Res. 4.1 and its dual the following generalization of Res. 7.15:

Result 7.16. *Let S be a regular medial semigroup (i.e., $axyb = ayxb$ for all $a, x, y, b \in S$). Then $\leq_S$ is two-sided compatible.*

Remarks. (i) By [26], IV. 3.10(8) (see also [22], Th. 9.10), a semigroup S is regular and medial if and only if S is a strong semilattice of rectangular abelian groups.

(ii) A regular semigroup S is medial if and only if $exyf = eyxf$ for any $e, f \in E(S)$, $x, y \in S$: concerning sufficiency we have for $a = aa'a, b = bb'b \in S$ that $a'a \cdot xy \cdot bb' = a'a \cdot yx \cdot bb'$ (see Res. 4.12 and Remark (iii) following it).

(iii) A regular medial semigroup S is completely regular (see also (i)): for any $a \in S$, $a = aa'a = a \cdot a' \cdot aa' \cdot a = a \cdot aa' \cdot a' \cdot a \in a^2S$, hence the claim follows from [3], Th. 4.3. Thus by [27], IV.2.16, (xii) (δ), a semigroup is regular and medial if and only if it is a normal orthogroup with abelian subgroups. In particular, $E(S)$ is a normal band, hence S is a generalized inverse semigroup (see Res. 7.28, Remark, below).

A particular class of medial semigroups is that of externally commutative semigroups (see [22], Lemma 11.1). Hence Res. 7.16 and Remark (iii) following Res. 4.2 give

Result 7.17. *Let S be a regular semigroup such that $axb = bxa$ for all $a, b, x \in S$. Then $\leq_S$ is two-sided compatible; in fact, S is commutative.*

Remark. By [22], Th. 11.4, an externally commutative semigroup S is regular if and only if S is a commutative Clifford semigroup. Note that such semigroups are inverse and have a two-sided compatible, in general non-trivial, natural partial order.

Res. 7.16 still holds for the larger class of E-medial semigroups: $aefb = afeb$ for any $a, b \in S$, $e, f \in E(S)$ – see Sec. 4. In fact, Res. 4.5 and its dual give

Result 7.18. *Let S be a regular E-medial semigroup. Then $\leq_S$ is two-*

sided compatible.

Remarks. (i) Every regular E-medial semigroup S satisfies $aeb \in abS$ and $aeb \in Sab$ for all $a, b \in S, e \in E(S)$. Hence Res. 7.18 also follows from Res. 7.11 and its dual.

(ii) In Res. 7.18, regularity of S is not necessary. Let $S = \bigcup_{\alpha \in T} T_\alpha$ be a proper inflation of a rectangular group T. Then S is non-regular and E-medial (since T is), but still $\leq_S$ is two-sided compatible since $\leq_T$ is the identity relation (see Sec. 8(C)).

(iii) Any inverse semigroup S is regular and E-medial and $\leq_S$ is two-sided compatible (see Res. 7.8). Also every rectangular (non)abelian group is regular and E-medial; in this case, $\leq_S$ is the identity relation.

(iv) A regular semigroup S is E-medial if and only if $E(S)$ is a normal band (i.e., $efgh = egfh$ for all $e, f, g, h \in E(S)$): with respect to sufficiency, for any $a = aa'a$, $b = bb'b \in S$, $a'a \cdot fg \cdot bb' = a'a \cdot gf \cdot bb'$ for all $f, g \in E(S)$. Equivalently, S is a generalized inverse semigroup (see Remark and Corollary following Res. 7.28, below).

As a particular case of Res. 7.18, we obtain (see Res. 4.10):

Result 7.19. *Let S be a regular semigroup such that $axeb = aexb$ for all $a, x, b \in S, e \in E(S)$. Then $\leq_S$ is two-sided compatible.*

Remarks. (i) The converse of Res. 7.19 holds for certain semigroups. In fact, we have: Let S be a regular semigroup such that $E(S)$ is an ideal of S and the semilattice Y of rectangular bands E_α $(\alpha \in Y)$; then $\leq_S$ is two-sided compatible and for any $a, b, x \in S$ there is $\alpha \in Y$ with $axeb$, $aexb \in E_\alpha$, if and only if $axeb = aexb$ for any $a, b, x \in S$, $e \in E(S)$ (see Res. 4.11).

(ii) Let S be a regular semigroup satisfying the condition in Res. 7.19. Then first, S is completely regular: for any $a \in S$, $a = aa' \cdot a = = a \cdot a' \cdot aa' \cdot a = a \cdot aa' \cdot a' \cdot a \in a^2S$; it follows by [3], Th. 4.3, that S is completely regular. Thus, S satisfies the identity $axy^o a = ay^o xa$, where $y^o \in S$ denotes the identity of the subgroup of S, to which $y \in S$ belongs. Hence we obtain by [27], IV.2.7, that a regular semigroup S satisfies $axeb = aexb$ for all $a, x, b \in S$, $e \in E(S)$, if and only if S is a strong semilattice of rectangular groups (equivalently, S is a normal orthogroup). In this case, S is a generalized inverse semigroup (see Res. 7.28, Corollary and Remark, below).

Corollary ([27], IV.2.7. and IV.1.6)**.** *Let S be a normal orthogroup (i.e., a completely regular semigroup such that $E(S)$ is a normal band). Then*

$\leq_S$ is two-sided compatible.

Result 7.20. *Let S be a regular semigroup such that $eaf = afe$ ($eaf = fea$) for any $a \in S, e, f \in E(S)$. Then $\leq_S$ is two-sided compatible.*

Proof. Every idempotent $e \in E(S)$ is central in S, since for any $a \in S$: $ea = e \cdot a \cdot a'a = a \cdot a'a \cdot e = ae$. Hence S is a Clifford semigroup; in particular, S is inverse and so $\leq_S$ is two-sided compatible (by Res. 7.8). $\Diamond$

Remark. By the proof of Res. 7.20, a regular semigroup S satisfies $eaf = afe$ for any $a \in S, e, f \in E(S)$, if and only if S is a Clifford semigroup.

As a particular case we obtain (see Res. 1.22):

Corollary. *Let S be a regular semigroup such that $abe = bea$ ($eab = bea$) for all $a, b \in S$, $e \in E(S)$. Then $\leq_S$ is two-sided compatible.*

Remark. By the proof of Res. 1.22, a regular semigroup S satisfies the condition in the Corollary if and only if S is commutative, that is, S is a strong semilattice of abelian groups. For the particular case that S is regular and satisfies $axy = xya$ for any $a, x, y \in S$ (i.e., S is (1,2)-commutative), the characterization of S as commutative Clifford semigroup is known (see [22], Cor. 15.9).

Concerning a generalization of regular medial semigroups, different from the E-medial case, we obtain from Res. 1.24 and its dual:

Result 7.21. *Let S be a regular semigroup such that $axyb \in aySxb$ for all $a, x, y, b \in S$. Then $\leq_S$ is two-sided compatible.*

Remarks. (i) Every regular medial semigroup S satisfies the condition in Res. 7.21 since any $y \in S$ has a right (and a left) identity in S. Also, any regular E-externally medial semigroup S has this property – but in this case S is already medial (see Res. 4.12 and Remark (iii) following it).

(ii) Let S be a regular semigroup satisfying the condition in Res. 7.21. Then again, S is completely regular: for any $a \in S$, $a = aa'a = a \cdot a' \cdot aa' \cdot a \in a \cdot aa' \cdot S \cdot a' \cdot a \subseteq a^2S$, hence a is completely regular by [3], Th. 4.3. Thus we obtain by [25], IV.4.9(1) that a regular semigroup S satisfies $axyb \in aySxb$ for all $a, x, y, b \in S$ if and only if S is a strong semilattice of completely simple semigroups. Note that the condition given in Res. 7.21 does not characterize the regular semigroups S for which $\leq_S$ is two-sided compatible. For example, any inverse semigroup S, which is not a Clifford semigroup, has a two-sided natural partial order; but S does not satisfy the indicated condition (if so S

would be an inverse union of groups by the above – see [3], Lemma 4.8).

Corollary ([27], IV.1.6)**.** *Let S be a completely regular semigroup. Then $\leq_S$ is two-sided compatible if and only if S is a strong semilattice of completely simple semigroups.*

These semigroups are also characterized in the Corollary of Res. 7.9 and the following

Result 7.22 ([27], IV.1.6)**.** *Let S be a completely regular semigroup. Then $\leq_S$ is two-sided compatible if and only if S is a normal cryptogroup (i.e., $\mathcal{H}$ is a congruence on S such that $S/\mathcal{H}$ is a normal band).*

For a particular class of semigroups considered in Res. 7.21 we have

Result 7.23. *Let S be a regular semigroup such that $axy \in aySx$ ($xya \in \in ySxa$) for all $a, x, y \in S$. Then $\leq_S$ is two-sided compatible.*

Remark. By [25], IV.4.9(2), a semigroup S satisfies the conditions in Res. 7.23 if and only if S is a strong semilattice of left (right) groups.

Another sufficient condition for right compatibility follows from Res. 3.2:

Result 7.24. *Let S be a regular semigroup such that $Sa \subseteq aS$ for every $a \in S$. Then $\leq_S$ is right compatible.*

Remarks. (i) By [25], IV. 3.10, a semigroup S satisfies the conditions in Res. 7.24 if and only if S is a semilattice of right groups (equivalently, S is completely regular such that $E(S)$ is a right regular band). A construction of these semigroups was given in [35] (see also [27], V.2.5). Note that $\leq_S$ is non-trivial if the semilattice Y has more than one element: if $\alpha <_Y \beta$, $e \in E(S_\alpha)$, $b \in S_\beta$, then $a := eb <_S b$ ($a \in S_\alpha$).

(ii) Examples of semigroups considered in Res. 7.24 are given in the following way: let H be a commutative semigroup with $E(H) \neq \phi$, I and Λ sets with $|I| = 1$, $|\Lambda| > 1$, $P = (p_{\lambda 1})$ with $p_{\lambda 1} = e$ for every $\lambda \in \Lambda$, $e \in E(H)$ fixed, and $T = \mathcal{M}(I, H, \Lambda; P)$ the Rees matrix semigroup over H. Then $S = \mathrm{Reg}(T) \neq \phi$, since $(1, e, \lambda) \in E(T)$ for every $\lambda \in \Lambda$. S is a regular semigroup by [10], since $fg \in S = \mathrm{Reg}(T)$ for all $f, g \in E(S) = = E(T)$ ($fg \cdot x \cdot fg = fg$ with $x = (1, e, \lambda) \in T$). Clearly, $Sa \subseteq aS$ for any $a \in S$.

(iii) Generalizing Res. 7.24, let S be a (regular) semigroup such that $E(S)e \subseteq eS$ for any $e \in E(S)$. Then $E(S)$ is a right regular band (if $e, f \in E(S)$ then $fe = ex$ for some $x \in S$, hence $efe = fe$). Since the converse is evident, we obtain that a regular semigroup S satisfies $E(S)e \subseteq eS$ for any $e \in E(S)$ if and only if S is a right inverse semigroup (see Remark and Corollary following Res. 7.4). In particular, if S is a regular semigroup satisfying $Se \subseteq eS$ for any $e \in E(S)$, then S is right inverse.

Result 7.25. *Let S be a regular semigroup such that $aS = Sa$ for every $a \in S$. Then $\leq_S$ is two-sided compatible.*

Remarks. (i) By [25], II.4.10, a semigroup S satisfies the conditions in Res. 7.25 if and only if S is a Clifford semigroup.

(ii) Generalizing Res. 7.25, let S be a (regular) semigroup which satisfies $eE(S) \subseteq Se$, $E(S)e \subseteq eS$ for any $e \in E(S)$. Then the idempotents of S commute (if $e, f \in E(S)$ then $ef = xe$, $fe = ey$ for some $x, y \in S$, hence $ef = efe = fe$). Since the converse is evident, we obtain that a regular semigroup S satisfies the indicated conditions if and only if S is an inverse semigroup.

In the following we consider in a general setting regular semigroups S, for which $E(S)$ forms a subsemigroup, that is, *orthodox* semigroups. For example, right (left) inverse or inverse semigroups, more generally $S = B \times T$ where B is a band and T an inverse semigroup, orthogroups and regular E-medial semigroups are orthodox. First, we have with respect to trivially-ordered semigroups of this kind:

Result 7.26. *Let S be an orthodox semigroup. Then $\leq_S$ is the identity relation if and only if $E(S)$ is a rectangular band (equivalently, S is a rectangular group).*

Proof. Necessity holds by Res. 7.1, Corollary.

Sufficiency: Let $a \leq_S b$ $(a, b \in S)$; then $a = eb = bf$ $(e, f \in E(S))$ and thus for $b = bb'b$, $a = bf = bb' \cdot bf = bb' \cdot eb = bb' \cdot e \cdot bb' \cdot b = bb' \cdot b = b$.

The statement in the parenthesis holds by [25], IV.3.5. $\diamondsuit$

Remark. Let S be a regular semigroup such that $E(S)$ is a trivially-ordered subsemigroup; then S is trivially-ordered (by Res. 7.1, Corollary, and Res. 7.26).

Result 7.27 ([9])**.** *Let S be an orthodox semigroup. Then $\leq_S$ is right compatible if and only if $E(S)$ is a right seminormal band (i.e., $eg \cdot efg = $ $= efg$ for all $e, f, g \in E(S)$).*

Proof. Necessity holds by Res. 1.6(i).

Sufficiency: If $b = bb'b$ and $c = cc'c \in S$ then $ac = e \cdot bc = bfc =$ $= b(b'b \cdot f \cdot cc')c = b(b'b \cdot cc' \cdot b'b \cdot f \cdot cc')c = bc \cdot z$ $(z \in S)$; hence $ac \leq_S bc$. $\diamondsuit$

Corollary ([26], II.3.15.1, II.3.8)**.** *For a band B, $\leq_B$ is right compatible if and only if B is right seminormal.*

Concerning two-sided compatibility we have

Result 7.28 ([12], p. 48). *Let S be an orthodox semigroup. Then $\leq_S$ is two-sided compatible if and only if $E(S)$ is a normal band (i.e., $efgh = = egfh$ for all $e, f, g, h \in E(S)$).*

Proof. Necessity holds by Res. 1.7(i).

Sufficiency: If $b = bb'b$ and $c = cc'c \in S$, then $ac = e \cdot bc = bfc = = b(b'b \cdot f \cdot cc' \cdot cc')c = b(b'b \cdot cc' \cdot f \cdot cc')c = bc \cdot x$ ($x \in S$), $ca = cb \cdot f = ceb = = c(c'c \cdot c'c \cdot e \cdot bb')b = c(c'c \cdot e \cdot c'c \cdot bb')b = y \cdot cb$ ($y \in S$); hence $ac \leq_S bc$ and $ca \leq_S cb$. ◊

Corollary ([13], Ex. IV.12). *For a band B, $\leq_B$ is two-sided compatible if and only if B is normal.*

Remark. Regular semigroups S, for which $E(S)$ forms a normal band, are called *generalized inverse* semigroups in [34] (see also [12]). Hence we get the following

Corollary ([12]). *Let S be an orthodox semigroup. Then $\leq_S$ is two-sided compatible if and only if S is generalized inverse.*

Remark. In the Corollary, the condition that $E(S)$ is a subsemigroup can not be dropped – see the paragraph preceding Res. 1.6 (also, compare with Res. 7.6).

In the lattice of subvarieties of the variety of all bands, the class of right seminormal bands has several interesting subclasses (see [26], p. 29). In particular, we obtain from Res. 7.27 that for a regular semigroup S, $\leq_S$ is right compatible in the following cases: $E(S)$ is a right zero, a rectangular, a right normal, a right regular, or a right quasinormal band (i.e., $efgf = egf$ for all $e, f, g \in E(S)$). More precisely, we have by Res. 7.26:

Corollary. *Let S be a regular semigroup such that $E(S)$ forms a rectangular band (in particular, a right zero band). Then $\leq_S$ is the identity relation.*

Remark. By [25], IV. 3.9, for a regular semigroup S, $E(S)$ is a right zero band if and only if S is a right group. Note that in the Corollary "right" can be replaced by "left".

Result 7.29. *Let S be a regular semigroup such that $E(S)$ is a right normal band (i.e., $efg = feg$ for all $e, f, g \in E(S)$). Then $\leq_S$ is two-sided compatible.*

Proof. Since a right normal band is normal, the statement follows form

Res. 7.28. $\Diamond$

Remarks. (i) In Res. 7.29, "right" can be replaced by "left".

(ii) Let S be a regular semigroup with $E(S)$ a right normal band; then $a \leq_S b$ if and only $a = eb$ for some $e \in E(S)$ (a right normal band is right regular, hence S is right inverse – see the Remark following Res. 7.4, Corollary).

(iii) For a regular semigroup $S, E(S)$ is a right normal band if and only if S satisfies the identity $efa = fea$ for any $e, f \in E(S), a \in S$ (necessity: $ef \cdot aa' = fe \cdot aa'$ for $a = aa'a$).

(iv) A *completely regular* semigroup S satisfies $efa = fea$ for any $e, f \in E(S)$, $a \in S$, if and only if S is a strong semilattice of right groups, equivalently, S is a right normal orthogroup (by the duals of [27], IV.2.16(xiii) and IV.2.12).

(v) More generally, let S be a regular semigroup satisfying $efg \in \in fSg$ for any $e, f, g \in E(S)$. Then we still have: $a \leq_S b$ if and only if $a = eb$ for some $e \in E(S)$ (sufficiency: $a = eb = e \cdot bb' \cdot bb' \cdot b = bb' \cdot s \cdot b = by$ for some $s, y \in S$). Note that a (regular) semigroup S satisfies the indicated condition if and only if $E(S)$ is a right regular band (S is right inverse): concerning necessity, $ef = eff = fsf (s \in S)$ implies $fef = f^2sf = ef$ for any $e, f \in E(S)$; sufficiency: if $e, f, g \in E(S)$ then $efg = fef \cdot g \in fSg$ (see Res. 7.4, Corollary and Remark).

As a particular case of Res. 7.29 we have

Result 7.30. *Let S be a regular semigroup such that $eaf = aef$ ($efa = = eaf$) for all $e, f \in E(S), a \in S$. Then $\leq_S$ is two-sided compatible.*

Remark. Let S be a semigroup satisfying the conditions in Res. 7.30. Then S satisfies $eab = aeb$ for any $e \in E(S), a, b \in S$: if $b = bb'b \in S$ then $ea \cdot bb' = ae \cdot bb'$. Since the converse is evident, a characterization of such semigroups S as strong semilattices of right groups is given in the Remark following Res. 7.13. Note that S is completely regular (by [3], Th. 4.3): for any $a \in S, a = aa'a = aa' \cdot a \cdot a'a = a \cdot aa' \cdot a'a \in a^2S$. Hence this characterization of S also holds by Remark (iv) following Res. 7.29.

Summarizing we obtain by Res. 7.27 and its Corollary (resp., their duals) and Res. 7.28 and its Corollary, the following sharpening of Res. 7.2:

Corollary. *Let S be an orthodox semigroup. Then $\leq_S$ is right (left, two-sided) compatible if and only if the natural partial order on $E(S)$ is right (left, two-sided) compatible with multiplication by idempotents.*

Remark. If S is a monoid (not necessarily regular) such that the natural partial order on $E(S)$ is two-sided compatible with multiplication by idempotents, then $E(S)$ is a commutative subsemigroup of S, i.e., a semilattice: indeed, let $e, f \in E(S)$; since $e \leq_S 1_S$ we have $ef \leq_S 1_S f = f \leq_S 1_S$; hence by [20], Lemma 2.1, $ef \in E(S)$. It follows by Res. 7.28, Corollary, that $E(S)$ is a normal band; hence (because of $1_S \in S$) $ef = fe$ for any $e, f \in E(S)$. In particular, we obtain that a regular monoid is inverse if and only if the natural partial order on $E(S)$ is two-sided compatible with multiplication by idempotents (see Res. 7.8).

8. Appendix

(A) In Sec. 7, particular classes of regular semigroups for which the natural partial order is two-sided compatible were specified. Besides of these there are two easy methods constructing such semigroups. Both of them yield locally inverse semigroups (see Res. 7.6).

(1) Regular Rees matrix semigroups $S = \mathrm{Reg}(T)$, where $T = \mathcal{M}(I, U, \Lambda; P)$ is a Rees matrix semigroup over an inverse semigroup U (see [16]).

(2) Regular semidirect products $S = U \times_\varphi C$, where U is an inverse semigroup and C is a completely simple semigroup (see [14]).

(B) *Strong semilattices of semigroups*

At several points in the above considerations, strong semilattices of semigroups have appeared. With respect to their natural partial order, a general observation can be made. Denoting by $\leq_\alpha$ the natural partial order on the semigroup S_α $(\alpha \in Y)$ we have

Result 8.1. *Let $S = \langle Y, S_\alpha; \varphi_{\alpha,\beta} \rangle$ be such that for any $\alpha \in Y$, every $a \in S_\alpha$ has a right and a left identity in S_α. Then $\leq_S$ is right (two-sided) compatible if and only if $\leq_\alpha$ is right (two-sided) compatible on S_α for any $\alpha \in Y$.*

Proof. First we show:

$a \leq_S b$ $(a \in S_\alpha, b \in S_\beta)$ if and only if $\alpha \leq_Y \beta$ and $a \leq_\alpha b\varphi_{\beta,\alpha}$.

(For the particular case that each S_α, $\alpha \in Y$, is a monoid, see [20]).

Suppose that $a \leq_S b$, $a \in S_\alpha$, $b \in S_\beta$. If $a = b$ then $\alpha = \beta$ and $a = b\varphi_{\alpha,\alpha}$. If $a \neq b$ then $a = xb = by$, $xa = a$, for some $x, y \in S$. For $x \in S_\gamma$, $y \in S_\delta$ say, $a = xb$ implies that $\alpha = \gamma\beta \leq_Y \gamma, \beta$; furthermore, $a = xb = (x\varphi_{\gamma,\gamma\beta})\ (b\varphi_{\beta,\gamma\beta}) = (x\varphi_{\gamma,\alpha})\ (b\varphi_{\beta,\alpha})$, $(x\varphi_{\gamma,\alpha})a = (x\varphi_{\gamma,\alpha})\cdot(a\varphi_{\alpha,\alpha}) = (x\varphi_{\gamma,\alpha\gamma})(a\varphi_{\alpha,\alpha\gamma}) = xa = a$, and similarly, $a = (b\varphi_{\beta,\alpha})(y\varphi_{\delta,\alpha})$, i.e., $a \leq_\alpha b\varphi_{\beta,\alpha}$ (note that $x\varphi_{\gamma,\alpha}, y\varphi_{\delta,\alpha} \in S_\alpha$). Conversely, let $a \in S_\alpha$, $b \in S_\beta$, be such that $\alpha \leq_Y \beta$, $a \leq_\alpha b\varphi_{\beta,\alpha}$. If $a = b\varphi_{\beta,\alpha}$ then by hypothesis, $a = xa = ay$ for some $x, y \in S_\alpha$. Thus $a = xa = x(b\varphi_{\beta,\alpha}) = (x\varphi_{\alpha,\alpha})(b\varphi_{\beta,\alpha}) = (x\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}) = xb$, and similarly, $a = by$; therefore $a \leq_S b$. If $a <_\alpha b\varphi_{\beta,\alpha}$, then $a = x(b\varphi_{\beta,\alpha})=(b\varphi_{\beta,\alpha})y$, $xa = a$, for some $x, y \in S_\alpha$. Hence $a = (x\varphi_{\alpha,\alpha})\ (b\varphi_{\beta,\alpha})=(x\varphi_{\alpha,\alpha\beta})\ (b\varphi_{\beta\alpha,\beta})= xb$, and similarly, $a = by$; therefore $a \leq_S b$.

The proof of Res. 8.1 now follows.

Sufficiency: If $a \leq_S b$, $a \in S_\alpha$, $b \in S_\beta$, and $c \in S_\gamma$ say, then $\alpha \leq_Y \beta$ implies $\alpha\gamma \leq_Y \beta\gamma$, and $a \leq_\alpha b\varphi_{\beta,\alpha}$ implies $a\varphi_{\alpha,\alpha\gamma} \leq_{\alpha\gamma} (b\varphi_{\beta,\alpha})\varphi_{\alpha,\alpha\gamma} = b\varphi_{\beta,\alpha\gamma}$ (because any homomorphism is order-preserving). Since $\leq_{\alpha\gamma}$ is right compatible on $S_{\alpha\gamma}$ we get

$$\begin{aligned} ac = (a\varphi_{\alpha,\alpha\gamma})(c\varphi_{\gamma,\alpha\gamma}) &\leq_{\alpha\gamma} (b\varphi_{\beta,\alpha\gamma})(c\varphi_{\gamma,\alpha\gamma}) = \\ &= \big[(b\varphi_{\beta,\beta\gamma})\varphi_{\beta\gamma,\alpha\gamma}\big] \cdot \big[(c\varphi_{\gamma,\beta\gamma})\varphi_{\beta\gamma,\alpha\gamma}\big] = \\ &= \big[(b\varphi_{\beta,\beta\gamma})(c\varphi_{\gamma,\beta\gamma})\big]\varphi_{\beta\gamma,\alpha\gamma} = (bc)\varphi_{\beta\gamma,\alpha\gamma}, \end{aligned}$$

that is, $ac \leq_S bc$. Necessity is obvious. $\Diamond$

Remarks. (i) Note that for $S = \langle Y, S_\alpha; \varphi_{\alpha,\beta}\rangle$, $\leq_\alpha$ $(\alpha \in Y)$ is the restriction of $\leq_S$ on S to S_α: let $a <_S b$ with $a, b \in S_\alpha$; then $a = xb = by$, $xa = a = ay$ for some $x, y \in S, x \in S_\gamma$, $y \in S_\delta$ say; hence $\alpha = \gamma\alpha = \alpha\delta$ and $a = (x\varphi_{\gamma,\alpha})(b\varphi_{\alpha,\alpha}) = (b\varphi_{\alpha,\alpha})(y\varphi_{\delta,\alpha})$, $(x\varphi_{\gamma,\alpha})(a\varphi_{\alpha,\alpha}) = a = (a\varphi_{\alpha,\alpha})(y\varphi_{\delta,\alpha})$, i.e., $a = x'b = by'$, $x'a = a = ay'$ for $x' = x\varphi_{\gamma,\alpha}$, $y' = y\varphi_{\delta,\alpha} \in S_\alpha$.

(ii) In particular, it follows from Res. 8.1 that under the hypothesis of the existence of left and right identities, $\leq_S$ is two-sided compatible if each $\leq_\alpha$ $(\alpha \in Y)$ is the identity relation. Note that $\leq_S$ is not trivial if $|Y| > 1$: for $\alpha <_Y \beta$ and $b \in S_\beta$, $a := b\varphi_{\beta,\alpha} <_S b$ (see the proof of sufficiency in (iii) below).

(iii) As can be seen from the characterization of the natural partial order, candidates for comparable elements in *general* strong semilattices of semigroups $S = \langle Y,\ S_\alpha;\ \varphi_{\alpha,\beta}\rangle$ are elements and their images. For these we have: If $b \in S_\beta$ and $\alpha <_Y \beta$, then $a := b\varphi_{\beta,\alpha} <_S b$ if and only if $a \in S_\alpha$ has a left and right identity in S_α. Necessity: If $a <_S b$ then $a = xb = by$, $xa = a = ay$ for some $x, y \in S$, $x \in S_\gamma$, $y \in S_\delta$, say. Hence $\gamma\alpha = \alpha = \alpha\delta$ and $(x\varphi_{\gamma,\gamma\alpha})(a\varphi_{\alpha,\alpha\gamma}) = a = (a\varphi_{\alpha,\alpha\delta})(y\varphi_{\delta,\delta\alpha})$, i.e., $x'a = a = ay'$ for $x' = x\varphi_{\gamma,\alpha}$, $y' = y\varphi_{\delta,\alpha} \in S_\alpha$. Sufficiency: If $xa = a = ay$ for some $x, y \in S_\alpha$ then $xb = (x\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\beta\alpha}) = (x\varphi_{\alpha,\alpha})(b\varphi_{\beta,\alpha}) = xa = a$ and similarly, $by = a$, i.e., $a \leq_S b$; since $a \in S_\alpha$, $b \in S_\beta$ and $\alpha <_Y \beta$ we get $a <_S b$.

(C) *Inflations of semigroups*

Throughout the text particular inflations S of semigroups T have occurred. Again, general observations concerning their natural partial order can be made. Let $S = \bigcup_{\alpha \in T} T_\alpha$ be such that any $\alpha \in T$ has a right and a left identity in T. Then by [5] Lemma 4.1,

$$a \leq_S b \ (a \in T_\alpha, b \in T_\beta) \text{ if and only if } a = b \text{ or } a = \alpha \leq_T \beta.$$

Note that if $S \neq T$ then $\leq_S$ is not trivial: for $x \in T_\alpha$, $x \neq \alpha$, we have $\alpha <_S x$.

If T is *any* semigroup then $\leq_S$ is right (two-sided) compatible if and only if $\leq_T$ is so (on T): concerning sufficiency, let $a <_S b, a \in T_\alpha, b \in T_\beta$; then $a = \alpha \leq_T \beta$, hence for any $c \in S$, $c \in T_\gamma$ say, $ac = \alpha\gamma \leq_T \beta\gamma = bc$; thus also $ac \leq_S bc$. In particular, it follows that for *any* trivially-ordered semigroup T, $\leq_S$ is two-sided compatible on S; more precisely, $a \leq_S b$ implies $ac = bc$ and $ca = cb$ for every $c \in S$.

(D) *Iterated inflations of semigroups*

Recall that a semigroup $S = \bigcup_{\alpha \in T} T_\alpha$ is an inflation of the semigroup T over $U \subseteq T$ if $T_\alpha = \{\alpha\}$ for any $\alpha \in T \backslash U$ (that is, only elements in U may be inflated; see [25]). Iterated inflations of this kind provide a method to construct semigroups S with an arbitrary number of elements and a nontrivial two-sided compatible natural partial order, with a pre-assigned number of layers in its diagram and a given number of minimal elements:

Starting with a trivially-ordered semigroup S_0 let S_1 be any inflation of S_0; define S_2 to be an inflation of S_1 over $S_1 \backslash S_0$ (that is, only the new elements in S_1 may be inflated, those in S_0 are not); generally: define S_{n+1} to be an inflation of S_n over $S_n \backslash S_{n-1}$, and let $S = S_k$ ($k \in \mathbb{N}$). In this way, if $|S_0| = 1$ then $(S, \leq_S)$ is a directed rooted tree. Generally, for any $\alpha \in S_0$ let T_α be the set of all elements in S obtained from $\alpha \in S_0$ by the iterated inflations performed. Then T_α is a directed rooted tree and $T_\alpha \cap T_\beta = \phi$ for all $\alpha \neq \beta$ in S_0. Hence, if $a, b \in S$ then $a \in T_\alpha, b \in T_\beta$, say, and $ab = \alpha\beta$ (by definition). Furthermore $a <_S b$ $(a \in T_\alpha, b \in T_\beta)$ implies $\alpha = \beta$ (see (C) above and (F) below). Thus, if $a <_S b$ and $c \in S$, then $a, b \in T_\alpha$, $c \in T_\gamma$, for some $\alpha, \gamma \in S_0$, so that $ac = \alpha\gamma = bc$ and $ca = \gamma\alpha = cb$.

(E) An other method for the *construction* of semigroups having the properties mentioned in (D) is that of forming strong semilattices of semigroups:

Let $S = \langle Y, S_\alpha; \varphi_{\alpha,\beta}\rangle$ be such that $(Y, \leq_Y)$ is a chain (with least element) and each S_α $(\alpha \in Y)$ is a trivially-ordered semigroup such that any $a \in S_\alpha$ has a left and a right identity in S_α. Then by Remark (ii) in (B) above, $\leq_S$ is two-sided compatible. Note that completely simple semigroups, in particular groups, provide examples of semigroups S_α of this kind (and of arbitrary size).

(F) *Elements with right (left) identity*

In the study of the natural partial order, elements having a right (left) identity play an important role – see, in particular, Res. 2.2: a necessary condition for $\leq_S$ to be non-trivial is that there are elements in S having a left and a right identity. Examples: Any element of a monoid has a right (and left) identity. Also, every regular element of a semigroup has this property. Furthermore, let $S = \mathcal{M}(I, T, \Lambda; P)$ be a Rees matrix semigroup over a semigroup T; then every element of S has a right (left) identity if and only if for any $a \in T, \lambda \in \Lambda$ $(i \in I)$, there exist $j \in I$ $(\kappa \in \Lambda)$ and $t \in T$ such that $ap_{\lambda j}\ t = a$ $(tp_{\kappa i}a = a)$. For example, this occurs if for any $a \in T$, $\lambda \in \Lambda$ $(i \in I)$ there exists $j \in I (\kappa \in \Lambda)$ with $ap_{\lambda j} = a$ $(p_{\kappa i}a = a)$. In particular, this holds if T is a monoid and each row (column) of P contains $1_T \in T$. In this case, every element of S has an idempotent right (left) identity. Note that in a semigroup S, for which $a \leq_S b$ iff $a = eb = bf$ $(e, f \in E(S^1))$ and such that $(S, \leq_S)$ has no maximal elements, any element has both an idempotent left and right identity.

More generally, if S is a semigroup without maximal elements then every element of S has a right and a left identity (see the proof of Res. 2.2). If in a semigroup S, every element has a right and a left identity then S is weakly reductive (by [25], III.1.14(1)), i.e., $ax = bx$, $xa = xb$ for all $x \in S$ implies $a = b$. This can be proved using the natural partial order of S: if $a'a = a = aa''$ for some $a', a'' \in S$ then $a = a'a = a'b, a = aa'' = ba''$, hence $a \leq_S b$; similarly, $b \leq_S a$, so that $a = b$. It follows that any semigroup without maximal elements is weakly reductive. With respect to the construction of semigroups without maximal elements we mention infinitely iterated inflations $S = S_\infty$ (see (D) above). Also we have the following

Result 8.2. *Let $S = \langle Y, S_\alpha, \varphi_{\alpha,\beta}\rangle$ be a strong semilattice of semigroups*

S_α such that $(Y, \leq_Y)$ has no maximal elements and each linking homomorphism is surjective. Then $(S, \leq_S)$ has no maximal elements if and only if any $a \in S$, $a \in S_\alpha$ say, has a right and a left identity in S_α.

Proof. Necessity: Let $a \in S$, $a \in S_\alpha$ say. Then $a <_S b$ for some $b \in S$, i.e., $a = xb = by$, $xa = a = ay$, for some $x, y \in S$. If $x \in S_\gamma$ say, then $a = xa$ implies $\alpha = \gamma\alpha \leq_Y \gamma$; therefore $a = xa = (x\varphi_{\gamma,\alpha\gamma})\cdot \cdot(a\varphi_{\alpha,\alpha\gamma}) = (x\varphi_{\gamma,\alpha})\ (a\varphi_{\alpha,\alpha}) = a'a$ for $a' = x\varphi_{\gamma,\alpha} \in S_\alpha$. Similarly if $y \in S_\delta$, $a = aa''$ for $a'' = y\varphi_{\delta,\alpha} \in S_\alpha$.

Sufficiency: Let $a \in S$, $a \in S_\alpha$ say; then $a'a = a = aa''$ for some $a', a'' \in S_\alpha$. Since there exists $\beta >_Y \alpha$, and there are $b, x, y \in S_\beta$ with $a = b\varphi_{\beta,\alpha}$, $a' = x\varphi_{\beta,\alpha}$, $a'' = y\varphi_{\beta,\alpha}$, we have: $a = a'a = (x\varphi_{\beta,\alpha})(b\varphi_{\beta,\alpha}) = = xb$, $a = aa'' = (b\varphi_{\beta,\alpha})\ (y\varphi_{\beta,\alpha}) = by$, $xa = (x\varphi_{\beta,\alpha\beta})(a\varphi_{\alpha,\alpha\beta}) = (x\varphi_{\beta,\alpha})\cdot \cdot(a\varphi_{\alpha,\alpha}) = a'a = a$; hence $a \leq_S b$; since $\alpha \neq \beta$, we get $a <_S b$. $\Diamond$

Remarks. (i) Note that the two general conditions on S are used only in the proof of sufficiency. Hence in *any* strong semilattice of semigroups without maximal elements *each* component is weakly reductive (by the observation made prior to 8.2).

(ii) Concerning inflations S of a semigroup T, no element $a \in S\backslash T$ has a right or a left identity (since $ax, xa \in T$ for every $x \in S$). Therefore any two elements in $S\backslash T$ are incomparable, by Res. 2.2.

(G) *Generalized Rees matrix semigroups*

At several occasions in the text Rees matrix semigroups $S = = \mathcal{M}(I, T, \Lambda; P)$ over semigroups T with $(\Lambda \times I)$ – sandwich matrix $P = = (p_{\lambda i})$, $p_{\lambda i} \in T$, have occurred. Recall that $S = I \times T \times \Lambda$ endowed with the operation: $(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu)$. Some general observations concerning their structure and their natural partial order will be made.

(i) $S = \mathcal{M}(I, T, \Lambda; P)$ is *commutative* if and only if $|I| = |\Lambda| = 1$ and $P = (p_{11})$ satisfies $ap_{11}b = bp_{11}a$ for any $a, b \in T$; in this case, $\leq_S$ is two-sided compatible. In particular, $S = \mathcal{M}(I, T, \Lambda; P)$ is commutative for *any* P if and only if $|I| = |\Lambda| = 1$ and T is externally commutative (see Res. 4.2). (This is one of the rare cases that S has a stronger property than T.)

(ii) S is *right (left) cancellative* if and only if $ap_{\lambda k}c = bp_{\mu k}c$ ($cp_{\kappa i}a = = cp_{\kappa j}b$) implies $\lambda = \mu$ ($i = j$) and $a = b$; in this case, $\leq_S$ is the identity relation (see Sec. 2). For instance, this occurs if T is right (left) cancellative and $|\Lambda| = 1$ ($|I| = 1$). Conversely, if S is right (left) cancellative and T is a monoid such that $1_T \in T$ is an entry of P, then T is right (left) cancellative. On the other hand, if P contains a right (left) zero of T then S is *not* right (left) cancellative.

(iii) S is *right (left) simple* – for any P – if and only if $|I| = 1$ ($|\Lambda| = 1$) and T is right (left) simple; in this case, $\leq_S$ is the identity relation (as is $\leq_T$) – see Sec. 2. In particular, S is a right (left) group if and only if $|I| = 1$ ($|\Lambda| = 1$) and T is a right (left) group (sufficiency: since T is right simple and left cancellative, so is S – see above; necessity: T is right simple since S is so, and $E(T) \neq \phi$, because $E(S) \neq \phi$ implies that $a = ap_{\lambda i}a$ for some $a \in T, i \in I, \lambda \in \Lambda$, whence $ap_{\lambda i} \in E(T)$ – see [3], Th. 1.27).

(iv) S is *simple* if and only if T is simple.

(v) S admits a primitive idempotent if T has a primitive idempotent e that is an entry of P: if $e = p_{\kappa k}$, say, then $(k, e, \kappa) \in E(S)$; let $(i, a, \lambda) \leq_S (k, e, \kappa)$, where $(i, a, \lambda) \in E(S)$, i.e., $a = ap_{\lambda i}a$; then $(i, a, \lambda) = (i, a, \lambda)(k, e, \kappa) = (k, e, \kappa)(i, a, \lambda)$, hence $i = k$, $\lambda = \kappa$, and $a = ap_{\lambda i} \cdot e = e \cdot p_{\lambda i}a$; since $ap_{\lambda i} \in E(T)$ it follows that $a \leq_T e$; therefore $a \in E(T)$, by [20], and $a = e$; thus $(k, e, \kappa) \in S$ is a primitive idempotent. Since every idempotent of a completely simple semigroup is primitive (see [25], IV.2.4) we obtain by (iv):

Result 8.3. *Let $S = \mathcal{M}(I, T, \Lambda, P)$ be such that T is completely simple and P contains an idempotent of T. Then S is completely simple (and $\leq_S$ is the identity relation).*

Remarks. (i) If T is a group then P can be normalized (see [3], §2.7); hence the (unique) idempotent $1_T \in T$ can be chosen to be an entry of P.

(ii) By the Rees theorem it follows from Res. 8.3 that S is isomorphic with a Rees matrix semigroup *over a group*.

(iii) Let I and Λ be finite sets and T be a finite simple semigroup (i.e., T is completely simple); then (for *any* P) S is completely simple, too (since S is simple and also finite). It follows that the converse of Res. 8.3 does not hold with respect to the property of P.

(iv) If T is any semigroup and if there is $e \in E(T)$ which is an entry of $P, e = p_{\kappa k}$ say, then $(k, f, \kappa) \in E(S)$ for any $f \in E(T)$ with $f \leq_T e$, and $(k, f, \kappa) \leq_S (k, e, \kappa)$.

Recall that a Rees matrix semigroup S over a group G is isomorphic with a rectangular group $S' = B \times G$ if and only if $E(S)$ forms a subsemigroup of S (see [25], IV. 3.3). Analyzing the proof we obtain the following generalization:

Result 8.4. *Let $S = \mathcal{M}(I, T, \Lambda; P)$ be a Rees matrix semigroup over a* *unipotent monoid T such that P contains a row and a column, each of which consists of invertible elements only. Then S is isomorphic with $S' = B \times T$, where B is a rectangular band, if and only if $E(S)$ forms a subsemigroup of S.*

Proof. Sufficiency: First, $E(S) = \{(i,a,\lambda) \in S \mid p_{\lambda i} \in T$ is invertible, $a = p_{\lambda i}^{-1}\}$ since $ap_{\lambda i}a = a$ implies $ap_{\lambda i}, p_{\lambda i}a \in E(T) = \{1_T\}$. By hypothesis, there are $k \in I$, $\kappa \in \Lambda$, such that $p_{\lambda k}$, $p_{\kappa i} \in T$ are invertible for any $\lambda \in \Lambda$, $i \in I$ (hence $E(S) \neq \phi$). We have $p_{\lambda k}p_{\kappa k}^{-1}p_{\kappa j} = p_{\lambda j}$ for any $\lambda \in \Lambda$, $j \in I$, since $(k, p_{\lambda k}^{-1}, \lambda)$, $(j, p_{\kappa j}^{-1}, \kappa) \in E(S)$, hence by hypothesis, $(k, p_{\lambda k}^{-1}p_{\lambda j}p_{\kappa j}^{-1}, \kappa) \in E(S)$ and $p_{\lambda k}^{-1}p_{\lambda j}p_{\kappa j}^{-1} = p_{\kappa k}^{-1}$. Giving I resp. Λ the multiplication of a left resp. right zero semigroup the direct product $B = I \times \Lambda$ is a rectangular band. It follows that $\varphi : S \to S' = B \times T$, $\varphi(i,a,\lambda) = ((i,\lambda), p_{\kappa k}^{-1}p_{\kappa i} \cdot a \cdot p_{\lambda k})$, is an isomorphism: if $(\alpha, a) \in S'$ with $\alpha = (i,\lambda)$, then $\varphi(i, p_{\kappa i}^{-1}p_{\kappa k}ap_{\lambda k}^{-1}, \lambda) = (\alpha, a)$; φ is injective since $p_{\lambda k}, p_{\kappa i} \in T$ are cancellable for any $i \in I, \lambda \in \Lambda$; φ is a homomorphism because of the above equation of any $p_{\lambda j} \in P$.

Necessity: Since $E(S') = \{(\alpha, 1_T) \in S' | \alpha \in B\}$, $E(S')$ is a subsemigroup of S'. It follows that $E(S)$ is a subsemigroup of S, too. $\diamondsuit$

Remarks. (i) If $S = \mathcal{M}(I,T,\Lambda;P)$ is of the above kind then $\leq_S$ is determined by the natural partial order of T (that on B is the identity relation, by Res. 7.1, Corollary): $(\alpha,a) \leq_S (\beta,b) \Leftrightarrow \alpha = \beta$, $a \leq_T b$. In fact, if $S = A \times B$ is the direct product of two semigroups A and B, then $(a_1,b_1) \leq_S (a_2,b_2)$ if and only if $a_1 \leq_A a_2$ and $b_1 \leq_B b_2$. In particular, $\leq_S$ is the identity relation if and only if $\leq_A$ and $\leq_B$ both are so.

(ii) In Res. 8.4, the general conditions are satisfied for example if T is a unipotent monoid and every element of some row and some column of P is equal to $1_T \in T$ (i.e., P is "normalized"). In this case, $E(S)$ is a subsemigroup of S if and only if *each* entry of P is $1_T \in T$: necessity holds by the proof of Res. 8.4; sufficiency is evident since $E(S) = \{(i, 1_T, \lambda) \in S \mid i \in I, \lambda \in \Lambda\}$. In this way we obtain two reminiscents of the completely simple case:

(1) If T is a unipotent monoid and if each entry of P is equal to $1_T \in T$, then S is isomorphic with $S' = B \times T$, B a rectangular band. If T is a group then S is a rectangular group.

(2) If T and P are as in (1) and if $|\Lambda| = 1$ then S is isomorphic with $S' = I \times T$ giving I the multiplication of a left zero semigroup. If T is a group then S is a left group. Generalizing we have

Result 8.5. *Let $S = \mathcal{M}(I, T, \Lambda; P)$ be such that T is a semigroup and P satisfies: $ap_{\lambda i}b = ab$ for any $a, b \in T$, $i \in I$, $\lambda \in \Lambda$. Then S is isomorphic with $S' = B \times T$, where B is the rectangular band $I \times \Lambda$.*

Proof. Giving I resp. Λ the multiplication of a left resp. right zero semigroup, $B = I \times \Lambda$ is a rectangular band. The mapping $\varphi : S \to S'$, $\varphi(i, a, \lambda) = ((i, \lambda), a)$, is an isomorphism. $\Diamond$

The condition in Res. 8.5 says that each entry of P is a "middle unit" for T; this is satisfied for example if T is a monoid and if each entry of P is $1_T \in T$. It is also fulfilled by any semigroup T such that $axb = ab$ for any $a, b, x \in T$ (by Res. 4.6, Remark, T is an inflation of a rectangular band). In this case, Res. 8.5 holds for *any* matrix P. Evidently, any semigroup T with zero multiplication has this property, too. Note that for any Rees matrix semigroup S considered in Res. 8.5, $\leq_S$ is determined by $\leq_T$; in particular, $\leq_S$ is the identity relation resp. is right (left) compatible if and only if $\leq_T$ is so.

For the natural partial order on generalized Rees matrix semigroups $S = \mathcal{M}(I, T, \Lambda; P)$ we have by [5]:

$$\begin{aligned}
&(i, a, \lambda) \leq_S (j, b, \mu) \Leftrightarrow i = j, \lambda = \mu, \quad \text{and} \\
&a = b \text{ or } a = xp_{\nu i}b = bp_{\lambda l}y, xp_{\nu i}a = a \ (= ap_{\lambda l}y), \\
&\text{for some } \ x, y \in T, l \in I, \nu \in \Lambda.
\end{aligned}$$

In particular, if T is trivially-ordered then so is S. The converse holds in case that T is a monoid and $1_T \in T$ is an entry of P: in fact, if a $\leq_T$ b and $1_T = p_{\kappa k}$, say, then $(k, a, \kappa) \leq_S (k, b, \kappa)$ and thus a = b. Furthermore, if T is an E-inversive monoid without zero and $1_T \in T$ is an entry of P then S is trivially-ordered if and only if T is a group, i.e., S is completely simple (see [5]). Finally, if $(i, a, \lambda) <_S (i, b, \lambda)$ for some $i \in I, \lambda \in \Lambda$, then $a <_T b$. The converse holds, for example, if P contains a middle unit of T; in particular, if T is a monoid and $1_T \in T$ is an entry of P. With respect to right (left) compatibility we have the following characterization:

Result 8.6. *Let $S = \mathcal{M}(I, T, \Lambda; P)$. Then $\leq_S$ is right (resp. left) compatible if and only if for any $a, b, c \in T, k \in I, \kappa \in \Lambda$, either $ap_{\lambda k}c = bp_{\lambda k}c$ (resp. $cp_{\kappa i}a = cp_{\kappa i}b$) or there are $m \in I$ (resp. $\sigma \in \Lambda$) and $t \in T$ such that $ap_{\lambda k}c = bp_{\lambda k}cp_{\kappa m}t$ (resp. $cp_{\kappa i}a = tp_{\sigma k}cp_{\kappa i}b$) whenever $a = xp_{\nu i}b = $ $= bp_{\lambda l}y$, $xp_{\nu i}a = a$ (resp. $ap_{\lambda l}y = a$), for some $x, y \in T, i, l \in I, \nu, \lambda \in \Lambda$.*

Proof. (Right compatibility). Necessity: Let $a, b, c \in T, k \in I, \kappa \in \Lambda$, and $a = xp_{\nu i}b = bp_{\lambda l}y, xp_{\nu i}a = a$ for some $x, y \in T$, $i, l \in I$, $\nu, \lambda \in \Lambda$. Then $(i, a, \lambda) \leq_S (i, b, \lambda)$, and thus $(i, a, \lambda)(k, c, \kappa) \leq_S (i, b, \lambda)(k, c, \kappa)$, i.e., $(i, ap_{\lambda k}c, \kappa) \leq_S (i, bp_{\lambda k}c, \kappa)$. Therefore either $ap_{\lambda k}c = bp_{\lambda k}c$ or there are $m \in I, t \in T$, such that $ap_{\lambda k}c = bp_{\lambda k}c.p_{\kappa m}.t$, since $(i, ap_{\lambda k}c, \kappa) = = (i, bp_{\lambda k}c, \kappa)(m, t, \kappa)$ for some $(m, t, \kappa) \in S$.

Sufficiency: Let $(i, a, \lambda) <_S (j, b, \mu)$; then $j = i, \mu = \lambda, (a \neq b)$ and $a = xp_{\nu i}b = bp_{\lambda l}y, xp_{\nu i}a = a$, for some $x, y \in T, l \in I, \nu \in \Lambda$. Let $(k, c, \kappa) \in S$; then either $ap_{\lambda k}c = bp_{\lambda k}c$ or there are $m \in I, t \in T$, such that $ap_{\lambda k}c = bp_{\lambda k}cp_{\kappa m}t$. It follows also in the second case that $(i, a, \lambda)(k, c, \kappa) = (i, ap_{\lambda k}c, \kappa) \leq_S (i, bp_{\lambda k}c, \kappa) = (i, b, \lambda)(k, c, \kappa)$, since $a.p_{\lambda k}c = xp_{\nu i}b.p_{\lambda k}c$, $xp_{\nu i}a.p_{\lambda k}c = a.p_{\lambda k}c$, whence $(i, x, \nu)(i, bp_{\lambda k}c, \kappa) = = (i, ap_{\lambda k}c, \kappa) = (i, bp_{\lambda k}c, \kappa)(m, t, \kappa)$, $(i, x, \nu)(i, ap_{\lambda k}c, \kappa) = (i, ap_{\lambda k}c, \kappa)$. ◊

By this result we obtain examples of generalized Rees matrix semigroups $S = \mathcal{M}(I, T, \Lambda; P)$ with *right* compatible natural partial order:

(i) T is a right simple semigroup – note that $\leq_T$ is the identity relation (see Sec. 2), hence $\leq_S$ is so, too (see [5]). More generally,

(ii) T is a semigroup such that $aT = bT$ for any $a, b \in T$; for instance, T is a semigroup with zero multiplication (in this case, $\leq_S$ is even two-sided compatible).

(iii) T is a semigroup containing one or more right zeros and each entry of P is a right zero in T; in this case $ap_{\lambda k}c = bp_{\lambda k}c$ and $A \leq_S B$ implies $AC = BC$ for any $C \in S$.If in addition in each column of P all entries are equal (i.e, for any $k \in I$, $p_{\lambda k} = p_{\kappa k}$ for any λ, $\kappa \in \Lambda$) then S satisfies $AXB = AB$ for any $A, B, X \in S$; thus by Res. 4.6 and the Remark following it, S is an inflation of the rectangular band $E(S)$ and $\leq_S$ is even two-sided compatible (more precisely, we also have that $A \leq_S B$ implies $CA = CB$ for any $C \in S$). Note that $\leq_S$ is not the identity relation if there are $b \in T, p_{\lambda i} \in P$, such that $p_{\lambda i}b \neq b$: in fact, $(i, a, \lambda) <_S (i, b, \lambda)$ for $a := p_{\lambda i}b$ since $(i, a, \lambda) = (i, p_{\lambda i}, \lambda)\cdot \cdot(i, b, \lambda) = (i, b, \lambda)(i, b, \lambda)$, $(i, p_{\lambda i}, \lambda)(i, a, \lambda) = (i, a, \lambda)$ (observe that $(i, b, \lambda) \notin E(S)$ since $bp_{\lambda i}b = p_{\lambda i}b \neq b$, hence $S \neq E(S)$: see (C) above).

A somewhat stronger (and more illuminating) sufficient condition is given by

Result 8.7. *Let $S = \mathcal{M}(I, T, \Lambda; P)$ be such that for any $a, b, c \in T$, $k, l \in I$, $\lambda, \kappa \in \Lambda$, either $ap_{\lambda k}c = bp_{\lambda k}c$ ($cp_{\kappa i}a = cp_{\kappa i}b$) or there are $m \in I$ ($\sigma \in \Lambda$) and $t \in T$ satisfying: $a.p_{\lambda l}b.p_{\lambda k}c = a.p_{\lambda k}c.p_{\kappa m}t$ ($cp_{\lambda l}.bp_{\kappa l}.a = = tp_{\sigma k}.cp_{\lambda l}.a$). Then $\leq_S$ is right (left) compatible.*

Proof. (Right compatibility). Following the proof of sufficiency in Res. 8.6, we have in the case that $ap_{\lambda k}c \neq bp_{\lambda k}c$:

$$a.p_{\lambda k}c = xp_{\nu i}b.p_{\lambda k}c,\ xp_{\nu i}a.p_{\lambda k}c = ap_{\lambda k}c,$$

and by hypothesis,

$$a.p_{\lambda k}c = bp_{\lambda l}y.p_{\lambda k}c = bp_{\lambda k}cp_{\kappa m}t, \text{ for some } m \in I, t \in I;$$

and this implies right compatibility of $\leq_S$. $\Diamond$

From this result we can deduce further examples of generalized Rees matrix semigroups $S = \mathcal{M}(I, T, \Lambda; P)$ for which $\leq_S$ is *right* compatible:

(i) Evidently, the condition is satisfied if P is the zero-matrix or if T is a semigroup with zero-multiplication. In both cases, S is isomorphic with the direct product $S' = B \times T'$, where $B = I \times \Lambda$ is a rectangular band (giving I resp. Λ the multiplication of a left resp. right zero semigroup) and $T' = T$ (endowed with the zero-multiplication if $P = 0$): the mapping $\varphi : S \to S', \varphi(i, a, \lambda) = ((i, \lambda), a)$, is an isomorphism. Therefore, $\leq_S$ is even two-sided compatible since $(\alpha, a) \leq_S (\beta, b)$ if and only if $\alpha = \beta, a = 0; \leq_S$ is non-trivial if $|T| \neq 1$.

(ii) T is a left zero semigroup; in this case, $\leq_T$ is the identity relation, whence so is $\leq_S$ (and $\leq_S$ is even two-sided compatible).

(iii) T is a commutative semigroup and any two rows of P consist (up to permutations) of the same elements (i.e., for any $\lambda, \kappa \in \Lambda, l \in I$, there exists $m \in I$ such that $p_{\lambda l} = p_{\kappa m}$); in particular, in each column of P all entries are equal; more specifically, all entries of P are equal – note that in the last case, $\leq_S$ is two-sided compatible. More generally:

(iv) T is a right commutative semigroup and any two rows of P consist (up to permutations) of the same elements – note that $\leq_T$ is right compatible by Res. 3.6.

(v) T is an externally commutative semigroup and any two rows of P consist (up to permutations) of the same elements ($a.p_{\lambda l}bp_{\lambda k}.c = = a.p_{\lambda k}bp_{\lambda l}.c = ap_{\lambda k}.cp_{\lambda l}b = ap_{\lambda k}c.p_{\kappa m}t$ with $t = b$) – note that $\leq_T$ is two-sided compatible by Res. 4.2.

(vi) T is a semigroup satisfying $axb = ab$ for any $a, b, x \in T$ – in this case, $\leq_S$ is even two-sided compatible. Note that by the Remark following Res. 4.6, such semigroups T are precisely the inflations of a rectangular band, and $\leq_T$ is two-sided compatible. Furthermore by Res. 8.5, S is isomorphic with $S' = B \times T$ where B is a rectangular band.

(vii) T is a semigroup, each entry of P is a left zero of T, and in each row of P all entries are equal (i.e., for any $\lambda \in \Lambda, p_{\lambda l} = p_{\lambda k}$ for all $l, k \in I$). In this case, we have again $AXB = AB$ for any $A, B, X \in S$. Therefore, as in example (iii) following Res. 8.6, S is an inflation of the rectangular band $E(S)$, and $A \leq_S B$ implies $AC = BC$, $CA = CB$ for any $C \in S$ (see (C) above).

(viii) T is a monoid satisfying $Tc \subseteq cT$ for any $c \in T$ and each entry of P is $1_T \in T$ – note that $\leq_T$ is right compatible by Res. 3.2. Furthermore, S is isomorphic with $S' = B \times T$ where B is a rectangular band (by Res. 8.5).

In several examples above the natural partial order of the underlying semigroup T was right compatible and also that of the resulting semigroup $S = \mathcal{M}(I, T, \Lambda; P)$. There is no relation between these properties, in general:

(i) If $\leq_S$ is right compatible, then $\leq_T$ is not necessarily so: let $T = L^1$, where L is a left zero semigroup with at least two elements, and consider T^0; then $\leq_{T^0}$ is not right compatible (for $e, f \in L : e <_T 1$ and $ef = e \not\leq_{T^0} f = 1f$); but for $S = \mathcal{M}(I, T^0, \Lambda; P)$, where each entry of P is 0, $\leq_S$ is right (and left) compatible. But note that in case that T has a (right, left) identity $e \in T$, which is an entry of P, right compatibility of $\leq_S$ implies that of $\leq_T$: if $p_{\kappa k} = e$ say, then $a <_T b$ implies $a = xp_{\kappa k}b = = bp_{\kappa k}y, xp_{\kappa k}a = a$, for some $x, y \in T$; hence $(k, a, \kappa) <_S (k, b, \kappa)$, so that $(k, ac, \kappa) = (k, a, \kappa)(k, c, \kappa) \leq_S (k, b, \kappa)(k, c, \kappa) = (k, bc, \kappa)$ for any $c \in T$; thus $ac = bc$ or $ac = wp_{\nu k}.bc = bc.p_{\kappa l}z, wp_{\nu k}.ac = ac$, for some $w, z \in T, \nu \in \Lambda, l \in I$; therefore $ac \leq_T bc$ for any $c \in T$.

(ii) Conversely, right compatibility of $\leq_T$ does *not* imply that of $\leq_S$: for example, if T is the semilattice $0 <_T e <_T f$, $I = \Lambda = \{1, 2\}$, and P is given by $p_{11} = e, p_{12} = f, p_{21} = p_{22} = 0$, then $\leq_T$ is even two-sided compatible, but $\leq_S$ is not right compatible (see the first Remark in Sec. 4). A sufficient condition for the implication to hold is given by

Result 8.8. *Let $S = \mathcal{M}(I, T, \Lambda; P)$ be such that $\leq_T$ is right (resp. left) compatible. If for any $a, b \in T, k \in I, \kappa, \lambda \in \Lambda$ (resp. $i, k \in I, \kappa \in \Lambda$) there exists $m \in I$ with $ap_{\lambda k}bT \subseteq ap_{\lambda k}bp_{\kappa m}T$ (resp. $\mu \in \Lambda$ with $Tbp_{\kappa i}a \subseteq \subseteq Tp_{\mu k}bp_{\kappa i}a$), then $\leq_S$ is right (resp. left) compatible.*

Proof. (Right compatibility). Let $(i, a, \lambda) <_S (j, b, \mu)$. Then $i = j$, $\lambda = \mu$, $a \neq b$, and there are $x, y \in T$, $l \in I$, $\nu \in \Lambda$, such that $a = = xp_{\nu i}.b = b.p_{\lambda l}y, xp_{\nu i}.a = a$; hence $a <_T b$. Let $(k, x, \kappa) \in S$; then we have: $a.p_{\lambda k}c = xp_{\nu i}b.p_{\lambda k}c$ and $xp_{\nu i}a.p_{\lambda k}c = a.p_{\lambda k}c$. Furthermore, $a <_T b$ implies that $a.p_{\lambda k}c \leq_T b.p_{\lambda k}c$. If $ap_{\lambda k}c <_T bp_{\lambda k}c$ then $ap_{\lambda k}c = bp_{\lambda k}c.z$ for some $z \in T$. If follows by hypothesis that there are $m \in I, t \in T$, such that $bp_{\lambda k}cz = bp_{\lambda k}cp_{\kappa m}t$; thus $ap_{\lambda k}c = bp_{\lambda k}cp_{\kappa m}t$. Consequently,

$$(i, a, \lambda)(k, c, \kappa) = (i, ap_{\lambda k}c, \kappa) \leq_S (i, bp_{\lambda k}c, \kappa) = (i, b, \lambda)(k, c, \kappa),$$

since $(i, x, \nu)(i, bp_{\lambda k}c, \kappa) = (i, ap_{\lambda k}c, \kappa) = (i, bp_{\lambda k}c, \kappa)(m, t, \kappa)$, and

$$(i, x, \nu)(i, ap_{\lambda k}c, \kappa) = (i, ap_{\lambda k}c, \kappa).$$

If $ap_{\lambda k}c = bp_{\lambda k}c$, then clearly $(i, ap_{\lambda k}c, \kappa) = (i, bp_{\lambda k}c, \kappa)$. $\Diamond$

This result yields some classes of semigroups T such that right compatibility of $\leq_T$ implies that of $\leq_S$. The first example is independent of Res. 8.8.

(i) T is a semigroup for which $\leq_T$ is the identity relation (see Sec. 2); then $\leq_S$ is the identity relation, by [5].

(ii) T is a semigroup and each entry of P is a left zero of T – in this case, $\leq_S$ is also left compatible: in fact, $A \leq_S B$ implies $CA = CB$ for any $C \in S$. In particular, for any left (right) zero semigroup T and *any* matrix P, $S = \mathcal{M}(I, T, \Lambda; P)$ has a trivial, hence two-sided compatible natural partial order, since $\leq_T$ has this property. In fact, by Res. 8.5, S is a rectangular band: $S \cong S' = (I \times T) \times \Lambda$, where $I \times T$ is a left, and Λ a right zero semigroup. Note that S is not a left (right) zero semigroup if $|\Lambda| \neq 1$ ($|I| \neq 1$).

(iii) T is a monoid and each row of P contains $1_T \in T$. In this case, $\leq_S$ is the identity relation if and only if $\leq_T$ is so (by [5]). More generally,

(iv) T is a monoid and each row of P contains an invertible element of T.

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