Mathematica Pannonica
22/2 (2011), 259–277

EXACT TRAVELING WAVE SOLUTIONS OF (2+1)-DIMENSIONAL NONLINEAR EVOLUTION EQUATIONS BY USING THE GENERALIZED TANH-METHOD

Elsayed M. E. **Zayed**

Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

Hanan M. Abdel **Rahman**

Department of Basic Sciences, Higher Technological Institute, Tenth Of Ramadan City, Egypt

Received: April 2011

MSC 2010: 60 K 35

Keywords: The generalized tanh-method, Riccati equation, the $(2+1)$-dimensional Nizhnik–Novikov–Veselov equations, the $(2+1)$-dimensional Burgers equations, the $(2+1)$-dimensional Wu–Zhang equations.

Abstract: The Riccati equation involving parameters and symbolic computation are used to uniformly construct different forms of traveling wave solutions for nonlinear evolution equations. It is shown that the sign of the parameters can be applied in judging the existence of various forms of traveling wave solutions. In this paper, the generalized tanh-method is demonstrated on some nonlinear equations which include the $(2+1)$-dimensional Nizhnik–Novikov–Veselov equations(NNV), the $(2+1)$-dimensional Burgers equations and the $(2+1)$dimensional Wu–Zhang (WZ) equations.

E-mail addresses: e.m.e.zayed@hotmail.com, hananmetwali@hotmail.com

1. Introduction

It is well known [1–38] that many important phenomena and dynamics processes can be described by special nonlinear partial differential equations (NPDEs). When a nonlinear PDE is used to characterize physical properties such as propagation or aggregation, it is of fundamental physical interest to solve the nonlinear PDE in a closed form. In the past several decades both mathematician and physicists have made many attempts in this direction. Various methods (see [1–38]) which are used to solve the nonlinear PDEs have been developed. Among them are the inverse scattering method [1, 16], the Bäcklund transformation method [8, 16], the tanhsech method [10, 13, 14, 24], the extended tanh-method [7, 12], the sine-cosine method [21, 28–30], the homogeneous balance method [7], the Jacobi elliptic function method [18] and so on. In recent years, the direct method for exact solutions of NPDEs becomes more and more attractive partly due to the availability of computer symbolic systems which allows us to perform the complex and tedious algebraic calculation on computer. It helps us to find new exact solutions of nonlinear partial differential equations. One of the most effective direct method to construct the exact solutions of NPDEs is that the tanh-method. The tanh-method and its extended are widely used by the authors [22, 23, 25–27, 31, 32] and by the references given therein. A generalized tanh-function method [33] has been presented to find the new exact solutions of nonlinear partial differential equations. The goal of this work is to extend the generalized tanh-function method to solve the $(2+1)$-dimensional Nizhnik–Novikov–Veselov (NNV), the $(2+1)$-dimensional Burgers equations and the $(2+1)$-dimensional Wu–Zhang (WZ) equations. In this method, the Riccati equation involving parameters and symbolic computation are used to uniformly construct different forms of traveling wave solutions for nonlinear evolution equations. It is shown that the sign of these parameters can be applied in judging the existence of various forms of traveling wave solutions. The objective of this article is to use the generalized tanh-method for solving the $(2+1)$dimensional Nizhnik–Novikov–Veselov equations [17, 19, 34]:

$$
\begin{aligned}
u_t + ku_{xxx} + ru_{yyy} + su_x + qu_y &= 3k(uv)_x + 3r(uw)_y,\\
u_x &= v_y,\\
u_y &= w_x,
\end{aligned}
\tag{1.1}
$$

and the (2+1)-dimensional Painlevé integrable Burgers equations [3, 17, 34, 35, 37]:

$$\begin{aligned} -u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha\beta u_{xx} &= 0, \\ u_x - v_y &= 0, \end{aligned} \tag{1.2}$$

as well as the $(2+1)$-dimensional Wu–Zhang equations [12, 36]

$$\begin{aligned} u_t + uu_x + vu_y + w_x &= 0, \\ v_t + uv_x + vv_y + w_y &= 0, \\ w_t + (uw)_x + (vw)_y + \frac{1}{3}(u_{xxx} + u_{xyy} + v_{xxy} + v_{yyy}) &= 0, \end{aligned} \tag{1.3}$$

where $r, k, s, q, \alpha, \beta$ are real parameters. In the past years, many people studied the Nizhnik–Novikov–Veselov equations. For instance, Boiti et al. [2] solved NNV equations via the inverse scattering transformation. Ren [17] and Xia [34] also obtained the solutions of the NNV equations. Lou [11] analyzed the coherent structures of the NNV equation by separation of variables approach. Recently, Zayed [37] found the exact solutions of system (1.1) by using the $\frac{G'}{G}$-expansion method. Zayed et al. [35] discussed the $(2+1)$-dimensional Burgers system (1.2) which is Painlevé integrable and then used the generalized multiple Riccati equations rational expansion method to get some of its solutions. Wang et al. [20] have obtained some traveling wave solutions of the system (1.2) by using the Riccati equation rational expansion method. Cai et al. [3] used the F-expansion method to generate some new exact solutions of the system (1.2). Zayed [37] also discussed the system (1.2) using the $\frac{G'}{G}$-expansion method and found some exact solutions of it. The Wu–Zhang system of equations (1.3) describes the nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of the uniform depth. In system (1.3) $w - 1$ is the elevation of the water wave, u is the surface velocity of water along the x direction and v is the surface velocity of water along the y direction. The explicit solutions of the system (1.3) are very helpful for costal and civil engineers to apply the nonlinear water wave model in harbor and costal design. Therefore, the explicit solutions and the numerical results of the system (1.3) are fundamental interest in fluid dynamics. Recently, Zayed et al. [36] have solved the system (1.3) using the modified variational iteration method. The outline of the generalized tanh-method can be described as follows.

2. The description of the generalized tanh-function method

In this section, we describe the generalized tanh-function method as follows:

Consider the general nonlinear PDE

$$u_t = P(u, u_x, u_{xx}, \dots), \tag{2.1}$$

where the independent variables x and t are combined into a new variable, $\xi = k(x - \omega t)$, where k and ω represent the wave number and velocity of the traveling wave, respectively. Therefore, $u(x,t)$ replaced by $u(\xi)$ which defines the traveling wave solution of Eq. (2.1). Thus Eq. (2.1) is then transformed into the following ODE:

$$-k\omega u' = P(u, ku', k^2u'', \dots). \tag{2.2}$$

Hence, under the transformation $\xi = k(x - \omega t)$, the PDE (2.1) has been reduced to an ordinary differential equation (ODE) given by (2.2). The resulting ODE is then solved by a finite series of tanh functions of the form

$$u(\xi) = \sum_{j=0}^{n} a_j \tanh^j \xi, \tag{2.3}$$

where n is a positive integer which can be determined by balancing between the highest derivatives with the nonlinear term in (2.2) where $a_0, a_1, \dots, a_n$ are parameters to be determined. The main idea of the generalized tanh-function method [21–33] is to replace $\tanh \xi$ in (2.3) by the solutions $Y(\xi)$ of the Riccati equation which are listed in Table 1 below. The Riccati equation is given by

$$Y' = A + BY + CY^2, \tag{2.4}$$

where $Y' = \frac{dY}{d\xi}$ while A, B and C are constants. It is formulated that the Riccati equation has several kinds of solutions in different cases are listed in Table 1.

Table 1. The relation between the values of (A, B, C) and the corresponding $Y(\xi)$ in the Riccati equation is as follows:

A	B	C	$Y(\xi)$
0	1	-1	$\frac{1}{2}+\frac{1}{2}\tanh\frac{\xi}{2}$
0	-1	1	$\frac{1}{2}-\frac{1}{2}\coth\frac{\xi}{2}$
$\frac{1}{2}$	0	$-\frac{1}{2}$	$\coth\xi \pm \not{c}\mathrm{sch}\,\xi,\ \tanh\xi \pm \not{+} i\,\mathrm{sech}\,\xi$
1	0	-1	$\tanh\xi, \coth\xi$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec\xi+\tan\xi, \csc\xi - \not{c}\mathrm{ot}\,\xi$
$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\sec\xi - \not{t}\mathrm{an}\,\xi, \csc\xi+\cot\xi$
$1(-1)$	0	$1(-1)$	$\tan\xi, \cot\xi$
0	0	$\neq 0$	$\frac{-1}{C\xi+m}$, m is a constant
arbitrary constant	0	0	$A\xi$
arbitrary constant	$\neq 0$	0	$\frac{\exp(B\xi)-A}{B}$

Therefore, the solution of Eq. (2.1) can be written in the form

$$u(x,t) = u(\xi) = \sum_{j=0}^{n} a_j Y^j. \tag{2.5}$$

By balancing the highest order derivatives with the nonlinear term in Eq. (2.2), we can determine n. Substituting (2.5) along with (2.4) into Eq. (2.2) and collect the coefficients of Y^j $(j = 0, 1, 2, ..., n)$, then set each coefficient to zero produce algebraic equations in terms of $a_0, a_1, ..., a_n$, A, B and C. Solving these algebraic equations, selecting A, B, C, $Y(\xi)$ from Table 1 and substituting them along with $a_0, a_1, \ldots, a_n$ into (2.5) we obtain the exact solutions of Eq. (2.1).

3. On solving the (2+1)-dimensional Nizhnik–Novikov–Veselov equations

In order to apply the generalized tanh-method to Eqs. (1.1), we use the transformations $u(x,y,t) = U(\xi)$, $v(x,y,t) = V(\xi)$, $w(x,y,t) = Z(\xi)$, where $\xi = \eta x + \lambda y - \rho t$ and

$$u(x,y,t) = U(\xi) = \sum_{i=0}^{m} a_i Y^i, \tag{3.1}$$

$$v(x,y,t)=V(\xi)=\sum_{i=0}^{n} b_i Y^i,$$

$$w(x,y,t)=Z(\xi)=\sum_{i=0}^{l} c_i Y^i.$$

Then Eqs. (1.1) are transformed into the following form:

$$\begin{aligned}(3.2)\qquad &-\rho U'+k\eta^3U'''+r\lambda^3U'''+s\eta U'+q\lambda U'-\\ &-3k\eta(UV'+U'V)-3r\lambda(UZ'+U'Z)=0,\\ &\eta U'-\lambda V'=0,\\ &\lambda U'-\eta Z'=0.\end{aligned}$$

Balancing U''' term with the UV', U''' term with the UZ' term in the first equation and U' term with V' or U' term with Z' in the third equation in (3.2) gives

$$\begin{aligned}(3.3)\qquad &m+3=m+n+1,\\ &m+3=m+l+1,\\ &m+1=n+1, m+1=l+1.\end{aligned}$$

Consequently, we have $m=n=l=2$. Thus, the solutions have the forms

$$\begin{aligned}(3.4)\qquad u(x,y,t)&=U(\xi)=a_0+a_1Y+a_2Y^2,\\ v(x,y,t)&=V(\xi)=b_0+b_1Y+b_2Y^2,\\ w(x,y,t)&=Z(\xi)=c_0+c_1Y+c_2Y^2.\end{aligned}$$

Substituting (3.4) along with (2.4) into (3.2) and setting the coefficients of the powers of $Y(\xi)$ to zero, then we obtain the following system of algebraic equations:

(3.5)

$$\begin{aligned}&-\rho Aa_1-3Aa_1b_0\eta k-3Aa_0b_1k+6A^2a_2B\eta^3k+Aa_1B^2\eta^3k+\\ &+2A^2a_1\eta^3Ck+Aa_1\lambda q-3Aa_1c_0\lambda r-3Aa_0c_1\lambda r+\\ &+6A^2a_2B\lambda^3r+Aa_1B^2\lambda^3r+2A^2a_1C\lambda^3r+Aa_1\eta s=0,\\ &-2\rho Aa_2-\rho a_1B-6Aa_2b_0\eta k-3a_1Bb_0\eta k-6Aa_1b_1\eta k-3a_0Bb_1\eta k-\\ &-6Aa_0b_2\eta k+14Aa_2B^2\eta^3k+a_1B^3\eta^3k+16A^2a_2\eta^3Ck+8Aa_1B\eta^3Ck,\\ &2Aa_2\lambda q+a_1B\lambda q-6Aa_2c_0\lambda r-3a_1Bc_0\lambda r-6Aa_1c_1\lambda r-3a_0Bc_1\lambda r-\end{aligned}$$

$$
\begin{aligned}
&-6Aa_0c_2\lambda r+14Aa_2B^2\lambda^3r+a_1B^3\lambda^3r+16A^2a_2C\lambda^3r+8Aa_1BC\lambda^3r+\\
&+2Aa_2cs+a_1B\eta s=0,\\
&-2\rho a_2B-\rho a_1C-6a_2Bb_0\eta k-9Aa_2b_1\eta k-6a_1Bb_1\eta k-9Aa_1b_2\eta k-\\
&-6a_0Bb_2\eta k+8a_2B^3\eta^3k-3a_1b_0\eta Ck-3a_0b_1\eta Ck+52Aa_2B\eta^3Ck+\\
&+7a_1B^2\eta^3Ck+8Aa_1\eta^3C^2k+2a_2B\lambda q+a_1C\lambda q-6a_2Bc_0\lambda r-\\
&-3a_1Cc_0\lambda r-9Aa_2c_1\lambda r-6a_1Bc_1\lambda r-3a_0Cc_1\lambda r-9Aa_1c_2\lambda r-\\
&-6a_0Bc_2\lambda r+8a_2B^3\lambda^3r+52Aa_2BC\lambda^3r+\\
&+7a_1B^2C\lambda^3r+8Aa_1C^2\lambda^3r+2a_2B\eta s+a_1\eta Cs=0,\\
&-2\rho a_2C-9a_2Bb_1\eta k-12Aa_2b_2\eta k-9a_1Bb_2\eta k-6a_2b_0\eta Ck-6a_1b_1\eta Ck-\\
&-6a_0b_2\eta Ck+38a_2B^2\eta^3Ck+40Aa_2\eta^3C^2k+12a_1B\eta^3C^2k+2a_2C\lambda q-\\
&-6a_2Cc_0\lambda r-9a_2Bc_1\lambda r-6a_1Cc_1\lambda r-12Aa_2c_2\lambda r-9a_1Bc_2\lambda r-\\
&-6a_0Cc_2\lambda r+38a_2B^2C\lambda^3r+40Aa_2C^2\lambda^3r+12a_1BC^2\lambda^3r+2a_2\eta Cs=0,\\
&-12a_2Bb_2\eta k-9a_2b_1\eta Ck-9a_1b_2\eta Ck+54a_2B\eta^3C^2k+6a_1\eta^3C^3k-\\
&-9a_2Cc_1\lambda r-12a_2Bc_2\lambda r-9a_1Cc_2\lambda r+54a_2BC^2\lambda^3r+\\
&+6a_1C^3\lambda^3r=0,\\
&-12a_2b_2\eta Ck+24a_2\eta^3C^3k-12a_2Cc_2\lambda r+24a_2C^3\lambda^3r=0,\\
&Aa_1\eta-Ab_1\lambda=0,\qquad 2Aa_2\eta+a_1B\eta-Bb_1\lambda-2Ab_2\lambda=0,\\
&2a_2B\eta+a_1\eta C-2Bb_2\lambda-b_1C\lambda=0,\qquad 2a_2\eta C-2b_2C\lambda=0,\\
&-A\eta c_1+Aa_1\lambda=0,\qquad -B\eta c_1-2A\eta c_2+2Aa_2\lambda+a_1B\lambda=0,\\
&-\eta Cc_1-2B\eta c_2+2a_2B\lambda+a_1C\lambda=0,\qquad -2\eta Cc_2+2a_2C\lambda=0.
\end{aligned}
$$

The algebraic equations (3.5) can be solved by Mathematica and give the following solutions:

$$
\begin{aligned}
\rho=\frac{1}{\eta\lambda}\Big[&-3a_0\eta^3k-3b_0\eta^2\lambda k+B^2\eta^4\lambda k+8A\eta^4C\lambda k+\lambda^2q\eta-\\
&-3c_0\lambda^2-3a_0\lambda^3\eta r+B^2\lambda^4r\eta+8A\eta C\lambda^4r+\eta^2\lambda s\Big],\\
a_1=2B\lambda\eta C,&\qquad b_1=2B\eta^2C,\qquad c_1=2BC\lambda^2,\\
a_2=2\eta\lambda C^2,&\qquad b_2=2\eta^2C^2,\qquad c_2=2C^2\lambda^2.
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) into (3.4) we have

$$
\begin{aligned}
u(x,y,t)&=a_0+2B\lambda\eta CY(\xi)+2\eta\lambda C^2Y^2(\xi),\\
v(x,y,t)&=b_0+2B\eta^2CY(\xi)+2\eta^2C^2Y^2(\xi),\\
w(x,y,t)&=c_0+2BC\lambda^2Y(\xi)+2C^2\lambda^2Y^2(\xi).
\end{aligned}
\tag{3.7}
$$

From Table 1, choosing $A = 0$, $B = 1$, $C = -1$, $Y(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)$ and inserting them into (3.7) we obtain the exact solutions

(3.8)

$$u_1(x,y,t) = a_0 - 2\lambda\eta\left\{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\} + 2\eta\lambda\left\{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\}^2,$$

$$v_1(x,y,t) = b_0 - 2\eta^2\left\{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\} + 2\eta^2\left\{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\}^2,$$

$$w_1(x,y,t) = c_0 - 2\lambda^2\left\{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\} + 2\lambda^2\left\{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\}^2,$$

where $\xi = \eta x + \lambda y - \rho t$ and $\rho = \frac{1}{\eta\lambda}\big[-3a_0\eta^3 k - 3b_0\eta^2\lambda k + \eta^4\lambda k + \lambda^2 q\eta - -3c_0\lambda^2 - 3a_0\lambda^3\eta r + \lambda^4 r\eta + \eta^2\lambda s\big]$. From Table 1, choosing $A=0$, $B=-1$, $C = 1$, $Y(\xi) = \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)$ and inserting them into (3.7) we obtain the exact solutions

(3.9)

$$u_2(x,y,t) = a_0 - 2\lambda\eta\left\{\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\} + 2\eta\lambda\left\{\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\}^2,$$

$$v_2(x,y,t) = b_0 - 2\eta^2\left\{\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\} + 2\eta^2\left\{\frac{1}{2} + \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\}^2,$$

$$w_2(x,y,t) = c_0 - 2\lambda^2\left\{\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\} + 2\lambda^2\left\{\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\}^2,$$

where $\xi = \eta x + \lambda y - \rho t$ and $\rho = \frac{1}{\eta\lambda}\big[-3a_0\eta^3 k - 3b_0\eta^2\lambda k + \eta^4\lambda k + \lambda^2 q\eta - -3c_0\lambda^2\eta r - 3a_0\lambda^3\eta r - \lambda^4 r\eta + \eta^2\lambda s\big]$.

From Table 1, choosing $A = \frac{1}{2}$, $B = 0$, $C = -\frac{1}{2}$, $Y(\xi) = \coth\xi \pm \pm \operatorname{csch}\xi$ or $Y(\xi) = \tanh\xi \pm i\operatorname{sech}\xi$ and inserting them into (3.7) we obtain the exact solutions

$$(3.10)\qquad \begin{aligned} u_3(x,y,t) &= a_0 + \frac{\eta\lambda}{2}\{\coth\xi \pm \operatorname{csch}\xi\}^2, \\ v_3(x,y,t) &= b_0 + \frac{\eta^2}{2}\{\coth\xi \pm \operatorname{csch}\xi\}^2, \\ w_3(x,y,t) &= c_0 + \frac{\lambda^2}{2}\{\coth\xi \pm \operatorname{csch}\xi\}^2, \end{aligned}$$

$$(3.11)\qquad \begin{aligned} u_4(x,y,t) &= a_0 + \frac{\eta\lambda}{2}\{\tanh\xi \pm i\operatorname{sech}\xi\}^2,\\ v_4(x,y,t) &= b_0 + \frac{\eta^2}{2}\{\tanh\xi \pm i\operatorname{sech}\xi\}^2,\\ w_4(x,y,t) &= c_0 + \frac{\lambda^2}{2}\{\tanh\xi \pm i\operatorname{sech}\xi\}^2, \end{aligned}$$

where $\xi = \eta x + \lambda y - \rho t$ and $\rho = \frac{1}{\eta\lambda}\big[-3a_0\eta^3 k - 3b_0\eta^2\lambda k - 2\eta^4\lambda k + \lambda^2 q\eta - -3c_0\lambda^2 - 3a_0\lambda^3\eta r - 2\eta\lambda^4 r + \eta^2\lambda s\big]$.

From Table 1, choosing $A = 1$, $B = 0$, $C = -1$, $Y(\xi) = \tanh\xi$ or $Y(\xi) = \coth\xi$ and inserting them into (3.7) we obtain the exact solutions

$$(3.12)\qquad \begin{aligned} u_5(x,y,t) &= a_0 + 2\eta\lambda^2\tanh^2\xi,\\ v_5(x,y,t) &= b_0 + 2\eta^2\tanh^2\xi,\\ w_5(x,y,t) &= c_0 + 2\lambda^2\tanh^2\xi, \end{aligned}$$

and

$$(3.13)\qquad \begin{aligned} u_6(x,y,t) &= a_0 + 2\eta\lambda^2\coth^2\xi,\\ v_6(x,y,t) &= b_0 + 2\eta^2\coth^2\xi,\\ w_6(x,y,t) &= c_0 + 2\lambda^2\coth^2\xi, \end{aligned}$$

where $\xi = \eta x + \lambda y - \rho t$ and $\rho = \frac{1}{\eta\lambda}\big[-3a_0\eta^3 k - 3b_0\eta^2\lambda k - 8\eta^4\lambda k + \lambda^2 q\eta - -3c_0\lambda^2\eta r - 3a_0\lambda^3\eta r - 8\eta\lambda^4 r + \eta^2\lambda s\big]$.

From Table 1, choosing $A = \frac{1}{2}$, $B = 0$, $C = \frac{1}{2}$, $Y(\xi) = \sec\xi + \tan\xi$ or $Y(\xi) = \csc\xi - \cot\xi$ and inserting them into (3.7) we obtain the exact solutions

$$(3.14)\qquad \begin{aligned} u_7(x,y,t) &= a_0 + \frac{\eta\lambda}{2}\{\sec\xi + \tan\xi\}^2,\\ v_7(x,y,t) &= b_0 + \frac{\eta^2}{2}\{\sec\xi + \tan\xi\}^2,\\ w_7(x,y,t) &= c_0 + \frac{\lambda^2}{2}\{\sec\xi + \tan\xi\}^2, \end{aligned}$$

$$(3.15)\qquad \begin{aligned} u_8(x,y,t) &= a_0 + \frac{\eta\lambda}{2}\{\csc\xi - \cot\xi\}^2,\\ v_8(x,y,t) &= b_0 + \frac{\eta^2}{2}\{\csc\xi - \cot\xi\}^2,\\ w_8(x,y,t) &= c_0 + \frac{\lambda^2}{2}\{\csc\xi - \cot\xi\}^2, \end{aligned}$$

where $\xi = \eta x + \lambda y - \rho t$ and $\rho = \frac{1}{\eta\lambda}\left[-3a_0\eta^3 k - 3b_0\eta^2\lambda k + 2\eta^4\lambda k + \lambda^2 q\eta - \right.$ $\left. -3c_0\lambda^2\eta r - 3a_0\lambda^3\eta r + 2\eta\lambda^4 r + \eta^2\lambda s\right]$ while a_0, b_0 and c_0 are arbitrary constants.

4. On solving the (2+1)-dimensional Painlevé integrable Burgers equations

In this section, we will use the generalized tanh-function method to solve Eqs. (1.2). To this end, we use the transformations $u(x, y, t) = U(\xi)$, $v(x, y, t) = V(\xi)$, $\xi = \eta(x + \lambda y - \rho t)$ where

$$u(x,y,t) = U(\xi) = \sum_{i=0}^{m} a_i Y^i, \qquad v(x,y,t) = V(\xi) = \sum_{i=0}^{n} b_i Y^i. \tag{4.1}$$

Then Eqs. (1.2) become

$$\rho U' + \lambda U'U + \alpha V U' + \eta\lambda^2\beta U'' + \eta\alpha\beta U'' = 0, \qquad U' - \lambda V' = 0. \tag{4.2}$$

Balancing the highest derivatives term with highest nonlinear terms in Eqs. (4.2) gives $m = n = 1$. Thus, the solutions have the forms:

$$U(\xi) = a_0 + a_1 Y(\xi), \qquad V(\xi) = b_0 + b_1 Y(\xi). \tag{4.3}$$

Substituting (4.3) along with (2.4) into (4.2) and equating the coefficients of the powers of $Y(\xi)$ to zero, then we obtain the following system of algebraic equations:

(4.4)

$$\alpha A a_1 b_0 + \alpha A a_1 \beta B \eta + A a_0 a_1 \lambda + A a_1 \beta B \eta \lambda^2 + A a_1 \rho = 0,$$
$$\alpha a_1 B b_0 + \alpha A a_1 b_1 + \alpha a_1 \beta B^2 \eta + 2\alpha A a_1 \beta C \eta + A a_1^2 \lambda + a_0 a_1 B \lambda + $$
$$+ a_1 \beta B^2 \eta \lambda^2 + 2 A a_1 \beta C \eta \lambda^2 + a_1 B \rho = 0,$$
$$\alpha a_1 B b_1 + \alpha a_1 b_0 C + 3\alpha a_1 \beta B C \eta + a_1^2 B \lambda + a_0 a_1 C \lambda + 3 a_1 \beta B C \eta \lambda^2 + a_1 C \rho = 0,$$
$$\alpha a_1 b_1 C + 2\beta a_1 \beta C^2 \eta + a_1^2 C \lambda + 2 a_1 \beta C^2 \eta \lambda^2 = 0,$$
$$A a_1 - A b_1 \lambda = 0, \qquad a_1 B - B b_1 \lambda = 0, \qquad a_1 C - b_1 C \lambda = 0.$$

The algebraic equations (4.4) can be solved by Mathematica and give the following set of solutions:

$$\rho = (-\alpha b_0 - \alpha\beta B\eta - a_0\lambda - \beta B\eta\lambda^2), \quad a_1 = -2\beta C\eta\lambda, \quad b_1 = -2\beta\eta C. \tag{4.5}$$

Substituting (4.5) into (4.3) we have

$$\begin{aligned} u(x,y,t) &= a_0 - 2\beta\eta\lambda C Y(\xi), \\ v(x,y,t) &= b_0 - 2\beta\eta C Y(\xi). \end{aligned} \tag{4.6}$$

From Table 1, choosing $A = 0$, $B = 1$, $C = -1$, $Y(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)$ and inserting them into (4.6) we obtain the exact solutions

$$\begin{aligned} u_1(x,y,t) &= a_0 + 2\beta\eta\lambda \left\{ \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right) \right\}, \\ v_1(x,y,t) &= b_0 + 2\beta\eta \left\{ \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right) \right\}, \end{aligned} \tag{4.7}$$

where $\xi = \eta(x + \lambda y - \rho t)$ and $\rho = (-\alpha b_0 - \alpha\beta\eta - a_0\lambda - \beta\eta\lambda^2)$.

From Table 1, choosing $A = 0$, $B = -1$, $C = 1$, $Y(\xi) = \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)$ and inserting them into (4.6) we obtain the exact solutions

$$\begin{aligned} u_2(x,y,t) &= a_0 - 2\beta\eta\lambda \left\{ \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right) \right\}, \\ v_2(x,y,t) &= b_0 - 2\beta\eta \left\{ \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right) \right\}, \end{aligned} \tag{4.8}$$

where $\xi = \eta(x + \lambda y - \rho t)$ and $\rho = (-\alpha b_0 + \alpha\beta\eta - a_0\lambda + \beta\eta\lambda^2)$.

From Table 1, choosing $A = \frac{1}{2}$, $B = 0$, $C = -\frac{1}{2}$, $Y(\xi) = \coth\xi \pm \operatorname{csch}\xi$ or $Y(\xi) = \tanh\xi \pm i\operatorname{sech}\xi$ and inserting them into (4.6) we obtain the exact solutions

$$\begin{aligned} u_3(x,y,t) &= a_0 + \beta\eta\lambda\{\coth\xi \pm \operatorname{csch}\xi\}, \\ v_3(x,y,t) &= b_0 + \beta\eta\{\coth\xi \pm \operatorname{csch}\xi\}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} u_4(x,y,t) &= a_0 + \beta\eta\lambda\{\tanh\xi \pm i\operatorname{sech}\xi\}, \\ v_4(x,y,t) &= b_0 + \beta\eta\{\tanh\xi \pm i\operatorname{sech}\xi\}, \end{aligned} \tag{4.10}$$

where $\xi = \eta(x + \lambda y - \rho t)$ and $\rho = (-\alpha b_0 - a_0\lambda)$.

From Table 1, choosing $A = 1$, $B = 0$, $C = -1$, $Y(\xi) = \tanh\xi$ or $Y(\xi) = \coth\xi$ and inserting them into (4.6) we obtain the exact solutions

$$\begin{aligned} u_5(x,y,t) &= a_0 + 2\beta\eta\lambda\tanh\xi, \\ v_5(x,y,t) &= b_0 + 2\beta\eta\tanh\xi, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} u_6(x,y,t) &= a_0 + 2\beta\eta\lambda\coth\xi, \\ v_6(x,y,t) &= b_0 + 2\beta\eta\coth\xi, \end{aligned} \tag{4.12}$$

where $\xi = \eta(x + \lambda y - \rho t)$ and $\rho = (-\alpha b_0 - a_0\lambda)$, a_0, b_0 are arbitrary constants.

From Table 1, choosing $A = \frac{1}{2}$, $B = 0$, $C = \frac{1}{2}$, $Y(\xi) = \sec\xi + \tan\xi$ or $Y(\xi) = \csc\xi - \cot\xi$, inserting them into (4.6) we obtain the exact solutions

$$\begin{aligned} u_7(x,y,t) &= a_0 - \beta\eta\lambda\{\sec\xi + \tan\xi\}, \\ v_7(x,y,t) &= b_0 - \beta\eta\{\sec\xi + \tan\xi\}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} u_8(x,y,t) &= a_0 - \beta\eta\lambda\{\csc\xi - \cot\xi\}, \\ v_8(x,y,t) &= b_0 - \beta\eta\{\csc\xi - \cot\xi\}, \end{aligned} \tag{4.14}$$

where $\xi = \eta(x + \lambda y - \rho t)$ and $\rho = (-\alpha b_0 - a_0\lambda)$ while a_0, b_0 are arbitrary constants.

5. On solving the (2+1)-dimensional Wu–Zhang equations

In order to apply the generalized tanh-function method to Eqs. (1.3), we use the transformation $u(x,y,t) = U(\xi)$, $v(x,y,t) = V(\xi)$, $w(x,y,t) = = Z(\xi)$ with $\xi = \eta x + \lambda y - \rho t$, where

$$\begin{aligned} u(x,y,t) &= U(\xi) = \sum_{i=0}^{m} a_i Y^i, \\ v(x,y,t) &= V(\xi) = \sum_{i=0}^{n} b_i Y^i, \\ w(x,y,t) &= Z(\xi) = \sum_{i=0}^{l} c_i Y^i. \end{aligned} \tag{5.1}$$

Then Eqs. (1.3) are transformed into the following forms:

(5.2)

$$\begin{aligned}
&-\rho U' + \eta UU' + \lambda VU' + \eta Z' = 0,\\
&-\rho V' + \eta UV' + \lambda VU' + \lambda Z' = 0,\\
&-\rho Z' + \eta(UZ)' + \lambda(VZ)' + \frac{1}{3}\{\eta^3 U''' + \eta\lambda^2 U''' + \eta^2\lambda V''' + \lambda^3 V'''\} = 0.
\end{aligned}$$

Balancing the highest derivatives term with highest nonlinear terms in Eqs. (5.2) gives so that $m = 1$, $n = 1$, $l = 2$. Thus, the solutions have the forms

$$\begin{aligned}
U(\xi) &= a_0 + a_1 Y,\\
V(\xi) &= b_0 + b_1 Y, \qquad (5.3)\\
Z(\xi) &= c_0 + c_1 Y + c_2 Y^2.
\end{aligned}$$

Substituting (5.3) along with (2.4) into Eqs. (5.2) and equating the coefficients of the powers of Y to zero, then we obtain the following system of algebraic equations:

(5.4)

$$\begin{aligned}
&-\rho A a_1 + A a_0 a_1 \eta + A c_1 \eta + A a_1 b_0 \lambda = 0,\\
&-\rho a_1 B + A a_1^2 \eta + a_0 a_1 B \eta + B c_1 \eta + 2 A c_2 \eta + a_1 B b_0 \lambda + A a_1 b_1 \lambda = 0,\\
&-\rho a_1 C + a_1^2 B \eta + a_0 a_1 C \eta + C c_1 \eta + 2 B c_2 \eta + a_1 B b_1 \lambda + a_1 b_0 C \lambda = 0,\\
&a_1^2 C \eta + 2 C c_2 \eta + a_1 b_1 C \lambda = 0,\\
&-\rho A b_1 + A a_0 b_1 \eta + A b_0 b_1 \lambda + A c_1 \lambda = 0,\\
&-\rho B b_1 + A a_1 b_1 \eta + a_0 B b_1 \eta + B b_0 b_1 \lambda + A b_1^2 \lambda + B c_1 \lambda + 2 A c_2 \lambda = 0,\\
&-\rho b_1 C + a_1 B b_1 \eta + a_0 b_1 C \eta + B b_1^2 \lambda + b_0 b_1 C \lambda + C c_1 \lambda + 2 B c_2 \lambda = 0,\\
&a_1 b_1 C \eta + b_1^2 C \lambda + 2 C c_2 \lambda = 0,\\
&-\rho A c_1 + A a_1 c_0 \eta + A a_0 c_1 \eta + \frac{1}{3} A a_1 B^2 \eta^3 + \frac{2}{3} A^2 a_1 C \eta^3 +\\
&+ A b_1 c_0 \lambda + A b_0 c_1 \lambda + \frac{1}{3} A B^2 b_1 \eta^2 \lambda + \frac{2}{3} A^2 b_1 C \eta^2 \lambda + \frac{1}{3} A a_1 B^2 \eta \lambda^2 +\\
&+ \frac{2}{3} A^2 a_1 C \eta \lambda^2 + \frac{1}{3} A B^2 b_1 \lambda^3 + \frac{2}{3} A^2 b_1 C \lambda^3 = 0,\\
&-\rho B c_1 - 2\rho A c_2 + a_1 B c_0 \eta + 2 A a_1 c_1 \eta + a_0 c_1 \eta + 2 A a_0 c_2 \eta +\\
&+ \frac{1}{3} a_1 B^3 \eta^3 + A a_1 B C \eta^3 + B b_1 c_0 \lambda + B b_0 c_1 \lambda + 2 A b_1 c_1 \lambda +\\
&+ 2 A b_0 c_2 \lambda + B^3 b_1 \eta^2 \lambda + \frac{8}{3} A B b_1 C \eta^2 \lambda + \frac{1}{3} a_1 B^3 \eta \lambda^2 +
\end{aligned}$$

$$+\frac{8}{3}Aa_1BC\eta\lambda^2+\frac{1}{3}B^3b_1\lambda^3+\frac{8}{3}ABb_1C\lambda^3=0,$$
$$-\rho Cc_1-2\rho Bc_2+a_1Cc_0\eta+2a_1Bc_1\eta+a_0Cc_1\eta+3Aa_1c_2\eta+2a_0Bc_2\eta+$$
$$+\frac{7}{3}a_1B^2C\eta^3+\frac{8}{3}Aa_1C^2\eta^3+b_1Cc_0\lambda+2Bb_1c_1\lambda+b_0Cc_1\lambda+2Bb_0c_2\lambda+$$
$$+3Ab_1c_2\lambda+\frac{7}{3}B^2b_1C\eta^2\lambda+\frac{8}{3}Ab_1C^2\eta^2\lambda+\frac{7}{3}a_1B^2C\eta\lambda^2+\frac{8}{3}Aa_1C^2\eta\lambda^2+$$
$$+\frac{7}{3}B^2b_1C\lambda^3+\frac{8}{3}Ab_1C^2\lambda^3=0,$$
$$-2\rho Cc_2+2a_1Cc_1\eta+3a_1Bc_2\eta+2a_0Cc_2\eta+4a_1BC^2\eta^3+2b_1Cc_1\lambda+$$
$$+3Bb_1c_2\lambda+2b_0Cc_2\lambda+4Bb_1C^2\eta^2\lambda+4a_1BC^2\eta\lambda^2+4Bb_1C^2\lambda^3=0,$$
$$3a_1Cc_2\eta+2a_1C^3\eta^3+3b_1Cc_2\lambda+2b_1C^3\eta^2\lambda+2a_1C^3\eta\lambda^2+2b_1C^3\lambda^3=0.$$

The algebraic equations (5.4) can be solved by Mathematica and give the following solutions:

$$c_0=-\frac{2AC}{3}(\eta^2+\lambda^2),\qquad c_1=-\frac{2BC}{3}(\eta^2+\lambda^2),$$
$$c_2=-\frac{2C^2}{3}(\eta^2+\lambda^2),\qquad b_1=-\frac{2c^2\lambda^2}{\sqrt{3}},\qquad a_1=\frac{-2C^2\eta\lambda}{\sqrt{3}},$$

(5.5)

$$a_0=\frac{1}{4C\eta^3+4C\eta\lambda^2}\Bigg\{4\rho C\eta^2-4b_0C\eta^2\lambda+4\rho C\lambda^2-4b_0C\lambda^3-\frac{4B\eta^2C^2\lambda^3}{\sqrt{3}}-$$
$$-\frac{4B\lambda^5C^2}{\sqrt{3}}-\frac{4B\eta^6C^2\lambda^2}{\sqrt{3}(\eta^2\lambda+\lambda^3)}-\frac{8B\eta^4\lambda^4C^2}{\sqrt{3}(\eta^2\lambda+\lambda^3)}-\frac{4B\eta^2\lambda^6C^2}{\sqrt{3}(\eta^2\lambda+\lambda^3)}\Bigg\}.$$

Substituting the solution (5.5) into (5.3) we have

(5.6)

$$u(x,y,t)=a_0-\frac{2C^2\eta\lambda}{\sqrt{3}}Y(\xi),$$
$$v(x,y,t)=b_0-\frac{2C^2\lambda^2}{\sqrt{3}}Y(\xi),$$
$$w(x,y,t)=-\frac{2BC}{3}(\eta^2+\lambda^2)-\frac{2BC}{3}(\eta^2+\lambda^2)Y(\xi)-\frac{2C^2}{3}(\eta^2+\lambda^2)Y^2(\xi).$$

From Table 1, choosing $A=0,\ B=1,\ C=-1,\ Y(\xi)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\xi}{2}\right)$ and inserting them into (5.6) we obtain the exact solutions

(5.7)

$$u_1(x,y,t) = a_0 + \frac{-2\eta\lambda}{\sqrt{3}}\left\{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\},$$
$$v_1(x,y,t) = b_0 - \frac{2\lambda^2}{\sqrt{3}}\left\{\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\},$$
$$w_1(x,y,t) = \frac{2}{3}(\eta^2+\lambda^2)\left[1+\left\{\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\}-\left\{\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\xi}{2}\right)\right\}^2\right],$$

where $a_0 = \frac{1}{-4\eta^3-4\eta\lambda^2}\Big\{-4\rho\eta^2 + 4b_0\eta^2\lambda - 4\rho\lambda^2 + 4b_0\lambda^3 + \frac{-4\eta^2\lambda^3-4\lambda^5}{\sqrt{3}} + + \frac{-4\eta^6-8\eta^4\lambda^4\lambda^2-4\eta^2\lambda^6}{\sqrt{3}(\eta^2\lambda+\lambda^3)}\Big\}$.

From Table 1, choosing $A = 0$, $B = -1$, $C = 1$, $Y(\xi) = \frac{1}{2} - -\frac{1}{2}\coth\left(\frac{\xi}{2}\right)$ and inserting them into (5.6) we obtain the exact solutions

(5.8)

$$u_2(x,y,t) = a_0 - \frac{2\eta\lambda}{\sqrt{3}}\left\{\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\},$$
$$v_2(x,y,t) = b_0 - \frac{2\lambda^2}{\sqrt{3}}\left\{\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\},$$
$$w_2(x,y,t) = \frac{2}{3}(\eta^2+\lambda^2)\left[1+\left\{\frac{1}{2}-\frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\}-\left\{\frac{1}{2}-\frac{1}{2}\coth\left(\frac{\xi}{2}\right)\right\}^2\right],$$

where $a_0 = \frac{1}{-4\eta^3-4\eta\lambda^2}\Big\{4\rho\eta^2 - 4b_0\eta^2\lambda + 4\rho\lambda^2 - 4b_0\lambda^3 + \frac{4\eta^2\lambda^3+4\lambda^5}{\sqrt{3}} + + \frac{4\eta^6+8\eta^4\lambda^4+4\eta^2\lambda^6}{\sqrt{3}(\eta^2\lambda+\lambda^3)}\Big\}$.

From Table 1, choosing $A = \frac{1}{2}$, $B = 0$, $C = -\frac{1}{2}$, $Y(\xi) = \coth\xi \pm \pm \operatorname{csch}\xi$ or $Y(\xi) = \tanh\xi \pm i\operatorname{sech}\xi$, inserting them into (5.6) we obtain the exact solutions

$$u_3(x,y,t) = a_0 - \frac{\eta\lambda}{2\sqrt{3}}\{\coth\xi \pm \operatorname{csch}\xi\},$$
$$v_3(x,y,t) = b_0 - \frac{\lambda^2}{2\sqrt{3}}\{\coth\xi \pm \operatorname{csch}\xi\}, \quad (5.9)$$
$$w_3(x,y,t) = -\frac{1}{6}(\eta^2+\lambda^2)\{\coth\xi \pm \operatorname{csch}\xi\}^2,$$

$$
\begin{aligned}
u_4(x,y,t) &= a_0 - \frac{\eta\lambda}{2\sqrt{3}}\{\tanh\xi \pm i\operatorname{sech}\xi\},\\
v_4(x,y,t) &= b_0 - \frac{\lambda^2}{2\sqrt{3}}\{\tanh\xi \pm i\operatorname{sech}\xi\},\\
w_4(x,y,t) &= -\frac{1}{6}(\eta^2+\lambda^2)\{\tanh\xi \pm i\operatorname{sech}\xi\}^2,
\end{aligned}
\tag{5.10}
$$

where $a_0 = \frac{1}{-2\eta^3-2\eta\lambda^2}\{-2\rho\eta^2 + 2b_0\eta^2\lambda - 2\rho\lambda^2 + 2b_0\lambda^3\}$.

From Table 1, choosing $A = 1$, $B = 0$, $C = -1$, $Y(\xi) = \tanh\xi$ or $Y(\xi) = \coth\xi$, inserting them into (5.6) we obtain the exact solutions

$$
\begin{aligned}
u_5(x,y,t) &= a_0 - \frac{2\eta\lambda}{\sqrt{3}}\tanh\xi,\\
v_5(x,y,t) &= b_0 - \frac{2\lambda^2}{\sqrt{3}}\tanh\xi,\\
w_5(x,y,t) &= -\frac{2}{3}(\eta^2+\lambda^2)\tanh^2\xi,
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
u_6(x,y,t) &= a_0 - \frac{2\eta\lambda}{\sqrt{3}}\coth\xi,\\
v_6(x,y,t) &= b_0 - \frac{2\lambda^2}{\sqrt{3}}\coth\xi,\\
w_6(x,y,t) &= -\frac{2}{3}(\eta^2+\lambda^2)\coth^2\xi,
\end{aligned}
\tag{5.12}
$$

where $a_0 = \frac{1}{-4\eta^3-4\eta\lambda^2}\{-4\rho\eta^2 + 4b_0\eta^2\lambda - 4\rho\lambda^2 + 4b_0\lambda^3\}$.

From Table 1, choosing $A = \frac{1}{2}$, $B = 0$, $C = \frac{1}{2}$, $Y(\xi) = \sec\xi + \tan\xi$ or $Y(\xi) = \csc\xi - \cot\xi$, inserting them into (5.6) we obtain the exact solutions

$$
\begin{aligned}
u_7(x,y,t) &= a_0 - \frac{\eta\lambda}{2\sqrt{3}}\{\sec\xi + \tan\xi\},\\
v_7(x,y,t) &= b_0 - \frac{\lambda^2}{2\sqrt{3}}\{\sec\xi + \tan\xi\},\\
w_7(x,y,t) &= -\frac{1}{6}(\eta^2+\lambda^2)\{\sec\xi + \tan\xi\}^2,
\end{aligned}
\tag{5.13}
$$

$$
(5.14)\qquad
\begin{aligned}
u_8(x,y,t) &= a_0 - \frac{\eta\lambda}{2\sqrt{3}}\{\csc\xi - \cot\xi\},\\
v_8(x,y,t) &= b_0 - \frac{\lambda^2}{2\sqrt{3}}\{\csc\xi - \cot\xi\},\\
w_8(x,y,t) &= -\frac{1}{6}(\eta^2+\lambda^2)\{\csc\xi - \cot\xi\}^2,
\end{aligned}
$$

where $a_0 = \frac{1}{2\eta^3+2\eta\lambda^2}\{2\rho\eta^2 - 2b_0\eta^2\lambda + 2\rho\lambda^2 - 2b_0\lambda^3\}$ and $\xi = \eta x + \lambda y - \rho t$.

6. Conclusion

In this article, the generalized tanh-function method is applied to find the traveling wave solutions of the coupled $(2+1)$-dimensional Nizhnik–Novikov–Veselov, the $(2+1)$-dimensional Painlevé integrable Burgers equations and the $(2+1)$-dimensional Wu–Zhang equations. The generalized tanh-function method is successfully used to establish these solutions. So this method provides a powerful mathematical tool to obtain more general exact solutions of many other nonlinear partial differential equations in mathematical physics.

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