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THE b-ADIC DIAPHONY OF DIGITAL (0, 1)-SEQUENCES: A CENTRAL LIMIT THEOREM AND A RELATION TO THE CLASSICAL DIAPHONY

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Abstract: The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence in the unit interval. In this paper we show that the b-adic diaphony of digital $(0,1)$-sequences satisfies a central limit theorem. Further we show a relation between the functions $\chi_b^{\delta_j}$ and ψ_b, which appear in the formulas for the classical diaphony and the b-adic diaphony of digital $(0,1)$-NUT-sequences respectively. This relation implies that the expected value of the squared classical diaphony over all digital $(0,1)$-NUT-sequences is given up to a constant by the squared b-adic diaphony, which has the same value for any digital $(0,1)$-sequence over $\mathbb{Z}_b$.

1. Introduction

The classical diaphony F_N (see [18] or [6, Def. 1.29] or [13, Ex. 5.27, p. 162]) of the first N elements of the sequences $\omega = (x_n)_{n\geq 0}$ in $[0,1)$ is given by

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$$F_N(\omega) := \left(2 \sum_{k=1}^{\infty} \frac{1}{k^2} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathtt{i} k x_n} \right|^2 \right)^{1/2}.$$

It is well known that the classical diaphony is a quantitative measure for the irregularity of distribution of the first N points of a sequence: A sequence ω is uniformly distributed modulo one if and only if $\lim\limits_{N\to\infty} F_N(\omega)=0$.

In [12] Hellekalek and Leeb introduced the notion of dyadic diaphony which was later generalized by Grozdanov and Stoilova [10] for general integers $b \geq 2$. This b-adic diaphony is similar to the classical diaphony but with the trigonometric functions replaced by Walsh functions in base b. Before we give the exact definition of the b-adic diaphony we recall the definition of Walsh functions.

Let b be an integer. For a non-negative integer k with base b representation $k = \kappa_{a-1} b^{a-1} + \cdots + \kappa_1 b + \kappa_0$ and a real $x \in [0,1)$ with base b representation $x = \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the x_i must be different from $b-1$) the k-th Walsh function in base b is defined as

$${}_b\mathrm{wal}_k(x) := \mathrm{e}^{2\pi \mathtt{i}(x_1\kappa_0 + \cdots + x_a \kappa_{a-1})/b}.$$

Now we give the definition of the b-adic diaphony (see [10] or [12]).

Definition 1. Let $b \geq 2$ be an integer. The b-adic diaphony of the first N elements of a sequence $\omega = (x_n)_{n\geq 0}$ in $[0,1)$ is defined by

$$F_{b,N}(\omega) := \left(\frac{1}{b} \sum_{k=1}^{\infty} \frac{1}{b^{2a(k)}} \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_k(x_n) \right|^2 \right)^{1/2},$$

where $a(k)$ is the unique determined integer a such that $b^a \leq k < b^{a+1}$. If $b = 2$ we also speak of dyadic diaphony.

The b-adic diaphony is a quantitative measure for the irregularity of distribution of a sequence: a sequence ω is uniformly distributed modulo one if and only if $\lim_{N\to\infty} F_{b,N}(\omega) = 0$. This was shown in [12] for the case $b = 2$ and in [10] for the general case. Further it is shown in [2] that the b-adic diaphony is – up to a factor depending on b and the dimension s – the worst case error for quasi-Monte Carlo integration of functions from a certain Hilbert space $H_{\mathrm{wal},s,\gamma}$, which has been introduced in [4].

Throughout this paper let b be a prime and let $\mathbb{Z}_b$ be the finite field of prime order b. We consider the b-adic diaphony of a special class of sequences in $[0,1)$, namely of so-called digital $(0,1)$-sequences over $\mathbb{Z}_b$. Here 1 is the dimension of the sequence and 0 is the (best possible) quality parameter. This is a special case of digital (t,s)-sequences over $\mathbb{Z}_b$ which were introduced by Niederreiter [14, 15] in a more general setting and provide at the moment the most efficient method to generate sequences with excellent distribution properties.

Before we give the definition of digital $(0,1)$-sequences over $\mathbb{Z}_b$ we introduce some notation: For a vector $\mathbf{c} = (c_1, c_2, \ldots) \in \mathbb{Z}_b^\infty$ and $m \in \mathbb{N}$ we denote the vector in $\mathbb{Z}_b^m$ consisting of the first m components of $\mathbf{c}$ by $\mathbf{c}(m)$, i.e. $\mathbf{c}(m) = (c_1, \ldots, c_m)$. Further let $\mathbf{c}_i$ denote the ith row vector of the matrix C.

Definition 2. Let b be a prime and C a $\mathbb{N} \times \mathbb{N}$ matrix over $\mathbb{Z}_b$ with the property that for every $m \in \mathbb{N}$ the vectors

$$\mathbf{c}_1(m), \ldots, \mathbf{c}_m(m)$$

are linearly independent.

For $n \geq 0$ let $n = n_0 + n_1 b + n_2 b^2 + \cdots$ be the base b representation of n. Multiply the vector $\mathbf{n} = (n_0, n_1, \ldots)^\top \in \mathbb{Z}_b^\infty$ by the matrix C,

$$C \cdot \mathbf{n} =: (x_n(1), x_n(2), \ldots)^\top \in \mathbb{Z}_b^\infty,$$

and set

$$x_n := \frac{x_n(1)}{b} + \frac{x_n(2)}{b^2} + \cdots .$$

Every sequence $(x_n)_{n \geq 0}$ constructed this way is called digital $(0,1)$-sequence over $\mathbb{Z}_b$. The matrix $C = (c_{i,j})_{i,j \geq 1}$ is called generator matrix of the sequence. If the generator matrix C is a non-singular upper triangular matrix the sequence is called a digital $(0,1)$-NUT-sequence.

For example if we choose as generator matrix the $\mathbb{N} \times \mathbb{N}$ identity matrix, then the resulting digital $(0,1)$-sequence over $\mathbb{Z}_b$ is the well known van der Corput sequence in base b. Hence the concept of digital $(0,1)$-sequences over $\mathbb{Z}_b$ is a generalization of the construction principle of the van der Corput sequence.

To guarantee that the points x_n belong to $[0,1)$ (and not just to $[0,1]$) we need the condition that for each $n \geq 0$ infinitely many of the $x_n(i)$ are different from $b-1$. This condition is always satisfied if we assume that for each $r \geq 1$ we have $c_{i,r} = 0$ for all sufficiently large i. Throughout this article we assume that the generator matrix fulfills this condition (see [15, p. 72] where this condition is called (S6)). More information about (t,s)-sequences can be found in the books [15] and [3].

In this article we pursue two main goals:

1. We want to extend the central limit theorem for the dyadic diaphony of digital $(0,1)$-sequences over $\mathbb{Z}_2$ (see [16, Cor. 2.4]) to the b-adic diaphony for primes $b \geq 2$. This will be done in Th. 4 in Sec. 2.

2. We want to show a relation between the functions $\chi_b^{\delta_j}$ and ψ_b, which appear in the formulas for the classical diaphony and the b-adic diaphony of digital $(0,1)$-NUT-sequences in base b, see Prop. 5 in Sec. 2. This relation will imply that the expected value of the squared classical diaphony over all digital $(0,1)$-NUT-sequences is given up to a constant by the squared b-adic diaphony, which has the same value for any digital $(0,1)$-sequence over $\mathbb{Z}_b$, see Th. 6 in Sec. 2.

For the classical diaphony it was proven by Faure [7, Th. 4] that

$$(NF_N(\omega))^2 = \frac{4\pi^2}{b^2} \sum_{j=1}^{\infty} \chi_b^{\delta_j} \left(\frac{N}{b^j} \right), \tag{1}$$

where ω is a digital $(0,1)$-NUT-sequence over $\mathbb{Z}_b$. Here $\chi_b^{\delta_j}$ are certain functions depending on the generator matrix C, which are defined in the following way (see [7]):

Let b be a prime and σ be a permutation of $\mathbb{Z}_b$. Set

$$Z_b^\sigma := \left(\frac{\sigma(0)}{b}, \dots, \frac{\sigma(b-1)}{b} \right).$$

For any integer h, $0 \leq h \leq b-1$, the function $\varphi_{b,h}^\sigma : \mathbb{R} \to \mathbb{R}$ is defined as follows. Let k be an integer with $1 \leq k \leq b$. Then for $x \in \left[\frac{k-1}{b}, \frac{k}{b}\right)$ we set

$$\varphi_{b,h}^\sigma(x) := \begin{cases} A\left(\left[0, \frac{h}{b}\right), k, Z_b^\sigma\right) - hx & \text{if } 0 \leq h \leq \sigma(k-1) \\ (b-h)x - A\left(\left[\frac{h}{b}, 1\right), k, Z_b^\sigma\right) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

where $A([\alpha, \beta), k, Z_b^\sigma)$ is the number of indices n such that $1 \leq n \leq k$ and $Z_b^\sigma(n) \in [\alpha, \beta) \subseteq [0,1)$; then the function $\varphi_{b,h}^\sigma$ is extended to the reals by periodicity. Note that $\varphi_{b,0}^\sigma = 0$.

Thus we have associated b functions with the pair (b, σ):

$$\varphi_{b,0}^\sigma = 0, \varphi_{b,1}^\sigma, \dots, \varphi_{b,b-1}^\sigma.$$

For a given pair (b, σ) we set now

$$\chi_b^\sigma := b \sum_{h=0}^{b-1} (\varphi_{b,h}^\sigma)^2 - \left(\sum_{h=0}^{b-1} \varphi_{b,h}^\sigma \right)^2.$$

For $x \in \mathbb{Z}$ let $[x]_b$ denote the remainder of x, when divided by b, i.e. $[x]_b \in \{0, 1, \dots, b-1\}$. For simplicity we omit from now on the subscript b, i.e. $[x] = [x]_b$. For $j \geq 1$ the permutation δ_j of $\mathbb{Z}_b$ is defined as

$$\delta_j := \begin{pmatrix} 0 & 1 & 2 & \cdots & b-1 \\ 0 & c_j^j & [2c_j^j] & \cdots & [(b-1)c_j^j] \end{pmatrix},$$

where c_j^j are the diagonal entries of the generator matrix C.

For the b-adic diaphony the author showed in [8, Th. 6] that

$$(NF_{b,N}(\omega))^2 = \frac{12}{b^2}\sum_{u=1}^{\infty}\psi_b\left(\frac{N}{b^u}\right) \tag{2}$$

for any digital $(0,1)$-sequence over $\mathbb{Z}_b$. Here the function ψ_b does not depend on the generator matrix C, so the b-adic diaphony is invariant for all digital $(0,1)$-sequences over $\mathbb{Z}_b$. The function ψ_b is defined as follows:

Definition 3. Let β be an integer in $\{1,\dots,b-1\}$. For $x \in \left[\frac{j}{b}, \frac{j+1}{b}\right)$, $j \in \{0,\dots,b-1\}$ we set

$$\psi_b^\beta(x) := \frac{b^2(b^2-1)}{12}\left|\frac{1}{b}\frac{z_\beta^j - 1}{z_\beta - 1} + z_\beta^j\left(x - \frac{j}{b}\right)\right|^2,$$

where $z_\beta = \mathrm{e}^{\frac{2\pi i}{b}\beta} = {}_b\mathrm{wal}_1\left(\frac{\beta}{b}\right)$; then the function is extended to the reals by periodicity. The function ψ_b is now defined as the mean of the functions ψ_b^β:

$$\psi_b(x) := \frac{1}{b-1}\sum_{\beta=1}^{b-1}\psi_b^\beta(x).$$

The functions $\chi_b^{\delta_j}$ appearing in (1) and ψ_b appearing in (2) have a similar structure (see [1, Propr. 3.3, Propr. 3.5 (ii)] and [8, Lemma 11]):

1. $\chi_b^{\delta_j}(x) = \psi_b(x) = \frac{b^2(b^2-1)}{12}x^2$ for $x \in \left[0, \frac{1}{b}\right)$,
2. $\chi_b^{\delta_j}$ and ψ_b are continuous,
3. on intervals of the form $\left[\frac{k}{b}, \frac{k+1}{b}\right)$ the functions $\chi_b^{\delta_j}$ and ψ_b are a translation of the parabola $\frac{b^2(b^2-1)}{12}x^2$.

We use the following notation: For $c \in \{1,\dots,b-1\}$ let χ_b^c denote the function χ_b^σ with the permutation σ of $\mathbb{Z}_b$ given by.

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & \cdots & b-1 \\ 0 & c & [2c] & \cdots & [(b-1)c] \end{pmatrix}.$$

Note that $\chi_b^{\delta_j} \in \{\chi_b^1,\dots,\chi_b^{b-1}\}$ for all $j \in \mathbb{N}$. For $b=2$ and $b=3$ the functions ψ_b and χ_b^c are the same. This can be easily calculated. Hence we have for any digital $(0,1)$-NUT-sequences ω over $\mathbb{Z}_2$ and $\mathbb{Z}_3$

$$F_{2,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega) \quad \text{and} \quad F_{3,N}(\omega) = \frac{\sqrt{3}}{\pi} F_N(\omega), \tag{3}$$

respectively. I.e., the classical diaphony and the b-adic diaphony of any digital $(0,1)$-NUT-sequences ω over $\mathbb{Z}_b$ are for $b = 2, 3$ up to the constant $\frac{\sqrt{3}}{\pi}$ the same. For greater values of b this is no longer true. The aim of this paper is now to prove a relation between the functions χ_b^c and ψ_b, see Prop. 5. This will imply that the squared (invariant) b-adic diaphony of a digital $(0,1)$-sequence over $\mathbb{Z}_b$ is the expected value over all $(0,1)$-NUT-sequences of the squared classical diaphony, see Th. 6.

2. Results

We will show a central limit theorem for the b-adic diaphony of digital $(0,1)$-sequences over $\mathbb{Z}_b$.

Theorem 4. *Let b be a prime and ω be a digital $(0,1)$-sequence over $\mathbb{Z}_b$. Then for any real y we have*

$$\frac{1}{b^m}\left|\left\{N < b^m \colon (NF_{b,N}(\omega))^2 \leq \frac{b^2-1}{6b}\log_b N + y\frac{12}{b^2}\sqrt{\frac{\log_b N(b^4-1)b^2}{25920}}\right\}\right| =$$
$$= \Phi(y) + o(1) \qquad (\text{as } m \to \infty),$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \mathrm{e}^{-\frac{t^2}{2}} \mathrm{d}t$$

denotes the normal distribution function and $\log_b$ denotes the logarithm to the base b. I.e., the squared b-adic diaphony of a digital $(0,1)$-sequence over $\mathbb{Z}_b$ satisfies a central limit theorem.

The proof of this theorem will be given in Sec. 3.

For $b = 2$ and $b = 3$ the functions ψ_b and χ_b^c are the same. For greater values of b this is no longer true, i.e. the functions χ_b^c depend on the parameter c, but we have the following nice relation between ψ_b and χ_b^c:

Proposition 5. *Let b be a prime and ψ_b and χ_b^c as above, then we have*

$$\psi_b(x) = \frac{1}{b-1} \sum_{c=1}^{b-1} \chi_b^c(x). \tag{4}$$

The proof of this theorem will be given in Sec. 4.

For $b=2$ and $b=3$ the classical and the b-adic diaphony are up to a constant the same, see (3). For $b \geq 5$ such a simple relation is no longer true, but we still have a relation between the classical diaphony and the b-adic diaphony of digital $(0,1)$-NUT-sequences over $\mathbb{Z}_b$.

Theorem 6. *The expected value of the squared classical diaphony over all* $(0,1)$*-NUT-sequences over* $\mathbb{Z}_b$ *is given by*

$$\mathbb{E}((NF_N(\omega))^2) = \frac{\pi^2}{3}(NF_{b,N})^2,$$

where $(NF_{b,N})^2$ *is the squared b-adic diaphony of any digital* $(0,1)$*-sequence over* $\mathbb{Z}_b$*. (Note that the b-adic diaphony is invariant for all digital* $(0,1)$*-sequences over* $\mathbb{Z}_b$*.)*

Proof. For $(0,1)$-NUT-sequences the diagonal entries must be in $\{1,\dots,b-1\}$. If we consider the expectation of the diaphony over all $(0,1)$-NUT-sequences, each diagonal entry appears with the same probability $1/(b-1)$. The functions $\chi_b^{\delta_j}$ can therefore be replaced by $\frac{1}{b-1}\sum_{c=1}^{b-1}\chi_b^c(x) = \psi_b(x)$ and we have

$$\begin{aligned}\mathbb{E}((NF_N(\omega))^2) &= \frac{4\pi^2}{b^2}\sum_{j=1}^{\infty}\frac{1}{b-1}\sum_{c=1}^{b-1}\chi_b^c\left(\frac{N}{b^j}\right) = \\ &= \frac{4\pi^2}{b^2}\sum_{j=1}^{\infty}\psi_b\left(\frac{N}{b^j}\right) = \frac{\pi^2}{3}(NF_{b,N})^2. \qquad \diamondsuit\end{aligned}$$

3. Proof of Theorem 4

For the proof of this theorem we will need the following central limit theorem [17, Th. 2.3.1].

Theorem 7. *Suppose* S_m *is given by*

$$S_m = \sum_{j=1}^{m} f_j(X_j),$$

where $X_j = \{b^j U\}$*,* U *a random variable that is uniformly distributed modulo* 1*, and* f_j *are uniformly bounded and uniformly Lipschitz continuous. If the variance of* S_m *satisfies* $\mathbb{V}(S_m) \geq m^\alpha$ *for some* $\alpha > 2/3$*, then* S_m *satisfies a central limit theorem:*

$$\frac{S_m - \mathbb{E}(S_m)}{\sqrt{\mathbb{V}(S_m)}} \longrightarrow \mathcal{N}(0,1).$$

Remark 8. One can easily check, that the above theorem also holds with the slightly weaker condition $\mathbb{V}(S_m) \geq m^\alpha$ for all $m \geq m_0$, for some $\alpha > 2/3$ and $m_0 \in \mathbb{N}$.

Lemma 9. *The function ψ_b is Lipschitz-continuous.*

Proof. Let $x, y \in \left[\frac{j}{b}, \frac{j+1}{b}\right]$, $j \in \{0, \ldots, b-1\}$. On this interval ψ_b is a translation of the parabola $\frac{b^2(b^2-1)}{12}x^2$ and therefore Lipschitz-continuous with constant C_j.

Let $x \in \left[\frac{j}{b}, \frac{j+1}{b}\right), y \in \left[\frac{l}{b}, \frac{l+1}{b}\right)$, $l, j \in \{0, \ldots, b-1\}$, $l > j$, and $C := \max_{j \in \{0,\ldots,b-1\}} C_j$, then we have

$$\begin{aligned}
&|\psi_b(y) - \psi_b(x)| \leq \\
&\leq \left|\psi_b(y) - \psi_b\left(\frac{l}{b}\right)\right| + \left|\psi_b\left(\frac{l}{b}\right) - \psi_b\left(\frac{l-1}{b}\right)\right| + \\
&\quad + \cdots + \left|\psi_b\left(\frac{j+1}{b}\right) - \psi_b^\beta(x)\right| \leq \\
&\leq C\left|y - \frac{l}{b}\right| + C\left|\frac{l}{b} - \frac{l-1}{b}\right| + \cdots + C\left|\frac{j+1}{b} - x\right| = \\
&= C|y - x|.
\end{aligned}$$

So we have shown that ψ_b is Lipschitz-continuous. ♢

We will apply Th. 7 to

$$S_m = \sum_{j=1}^{m} \psi_b(Xb^j), \tag{5}$$

where X is uniformly distributed on $[0, 1)$. The function ψ_b is bounded (see [8, Lemma 12]) and Lipschitz continuous (see Lemma 9). Now we have to compute $\mathbb{E}(S_m)$ and $\mathbb{V}(S_m)$.

Lemma 10. *With S_m as in* (5)*, we have*

$$\begin{aligned}
\mathbb{E}(S_m) &= \frac{mb(b^2-1)}{72}, \\
\mathbb{V}(S_m) &= \frac{mb^2(b^4-1)}{25920} + \frac{b^3(1+b+b^2)(1-b^{-2m})}{12960}.
\end{aligned}$$

Proof. In the following we will use the two formulas

$$\sum_{k=1}^{b-1} \frac{\mathrm{e}^{\frac{2\pi i}{b}kl}}{\sin^2\left(\frac{k\pi}{b}\right)} = \frac{b^2-1}{3} + 2|l|(|l| - b), \quad \text{for } l \in \{-b, \ldots, b\}, \tag{6}$$

(see [3, Cor. A.23]) and

$$\sum_{k=1}^{b-1} \frac{e^{\frac{2\pi i}{b}kl}-1}{e^{\frac{2\pi i}{b}k}-1} = b-l, \quad \text{for } l \in \{1,\dots,b\}, \tag{7}$$

which can be easily deduced from (6) in the following way.

$$\begin{aligned}\sum_{k=1}^{b-1} \frac{e^{\frac{2\pi i}{b}kl}-1}{e^{\frac{2\pi i}{b}k}-1} &= \sum_{k=1}^{b-1} \frac{(e^{\frac{2\pi i}{b}kl}-1)(e^{-\frac{2\pi i}{b}k}-1)}{(e^{\frac{2\pi i}{b}k}-1)(e^{-\frac{2\pi i}{b}k}-1)} = \\ &= \sum_{k=1}^{b-1} \frac{e^{\frac{2\pi i}{b}k(l-1)} - e^{\frac{2\pi i}{b}kl} - e^{\frac{2\pi i}{b}k}+1}{4\sin^2\left(\frac{k\pi}{b}\right)} \overset{(6)}{=} b-l.\end{aligned}$$

Let $\beta \in \{1,\dots,b-1\}$ and $j \in \{0,\dots,b-1\}$, with the abbreviations

$$A_{\beta,j} := \mathrm{Re}\left(\frac{1}{b}\frac{e^{\frac{2\pi i}{b}\beta j}-1}{e^{\frac{2\pi i}{b}\beta j}(e^{\frac{2\pi i}{b}\beta}-1)}\right) \quad \text{and} \quad B_{\beta,j} := \mathrm{Im}\left(\frac{1}{b}\frac{e^{\frac{2\pi i}{b}\beta j}-1}{e^{\frac{2\pi i}{b}\beta j}(e^{\frac{2\pi i}{b}\beta}-1)}\right)$$

the function ψ_b^β on the interval $\left[\frac{j}{b}, \frac{j+1}{b}\right)$ equals

$$\psi_b^\beta(x) = \frac{b^2(b^2-1)}{12}\left|A_{\beta,j} + iB_{\beta,j} + x - \frac{j}{b}\right|^2.$$

We have

$$\sum_{\beta=1}^{b-1} A_{\beta,j} = \sum_{\beta=1}^{b-1} \mathrm{Re}\left(\frac{1}{b}\frac{e^{\frac{2\pi i}{b}\beta j}-1}{e^{\frac{2\pi i}{b}\beta j}(e^{\frac{2\pi i}{b}\beta}-1)}\right) = -\frac{1}{b}\,\mathrm{Re}\left(\sum_{\beta=1}^{b-1}\frac{e^{\frac{2\pi i}{b}\beta(b-j)}-1}{e^{\frac{2\pi i}{b}\beta}-1}\right) \overset{(7)}{=} -\frac{j}{b} \tag{8}$$

and

$$\begin{aligned}\sum_{\beta=1}^{b-1}(A_{\beta,j}^2 + B_{\beta,j}^2) &= \sum_{\beta=1}^{b-1}\left|\frac{1}{b}\frac{e^{\frac{2\pi i}{b}\beta j}-1}{e^{\frac{2\pi i}{b}\beta j}(e^{\frac{2\pi i}{b}\beta}-1)}\right|^2 \\ &= \frac{1}{b^2}\sum_{\beta=1}^{b-1}\frac{\left(e^{\frac{2\pi i}{b}\beta j}-1\right)\left(e^{-\frac{2\pi i}{b}\beta j}-1\right)}{\left(e^{\frac{2\pi i}{b}\beta}-1\right)\left(e^{-\frac{2\pi i}{b}\beta}-1\right)} \\ &= \frac{1}{2b^2}\,\mathrm{Re}\left(\sum_{\beta=1}^{b-1}\frac{1-e^{\frac{2\pi i}{b}\beta j}}{\sin^2\left(\frac{\beta\pi}{b}\right)}\right) \\ &\overset{(6)}{=} \frac{j(b-j)}{b^2}.\end{aligned} \tag{9}$$

Now we are ready to compute $\mathbb{E}(S_m)$ and $\mathbb{V}(S_m)$.

$$\begin{aligned}\mathbb{E}(S_m) &= \int_0^1 \sum_{w=1}^{m} \psi_b(xb^w)\mathrm{d}x = \frac{1}{b-1}\sum_{w=1}^{m}\sum_{\beta=1}^{b-1}\int_0^{b^w} \psi_b^{\beta}(x)\frac{1}{b^w}\mathrm{d}x = \\ &= \frac{m}{b-1}\sum_{\beta=1}^{b-1}\sum_{j=0}^{b-1}\int_{\frac{j}{b}}^{\frac{j+1}{b}} \frac{b^2(b^2-1)}{12}\left|A_{\beta,j}+iB_{\beta,j}+x-\frac{j}{b}\right|^2\mathrm{d}x = \\ &= \frac{mb^2(b+1)}{12}\sum_{\beta=1}^{b-1}\sum_{j=0}^{b-1}\int_0^{\frac{1}{b}} \left|A_{\beta,j}+iB_{\beta,j}+x\right|^2\mathrm{d}x = \\ &= \frac{mb^2(b+1)}{12}\sum_{j=0}^{b-1}\sum_{\beta=1}^{b-1}\left(\frac{1}{3b^3}+\frac{A_{\beta,j}}{b^2}+\frac{A_{\beta,j}^2+B_{\beta,j}^2}{b}\right) \overset{(8),(9)}{=} \frac{mb(b^2-1)}{72}.\end{aligned}$$

Now we compute $\mathbb{V}(S_m)$.

$$\begin{aligned}\mathbb{E}(S_m^2) &= \int_0^1 \sum_{w=1}^{m}\sum_{v=1}^{m}\psi_b(xb^w)\psi_b(xb^v)\mathrm{d}x = \\ &= \sum_{w=1}^{m}\sum_{v=1}^{w-1}\frac{1}{b^w}\int_0^{b^w}\psi_b(x)\psi_b\left(\frac{x}{b^{w-v}}\right)\mathrm{d}x + \\ &\quad + \sum_{w=1}^{m}\sum_{v=w+1}^{m}\frac{1}{b^v}\int_0^{b^v}\psi_b(x)\psi_b\left(\frac{x}{b^{v-w}}\right)\mathrm{d}x + \sum_{w=1}^{m}\frac{1}{b^w}\int_0^{b^w}\psi_b(x)^2\mathrm{d}x = \\ &= \sum_{w=1}^{m}\sum_{v=1}^{w-1}\frac{1}{b^{w-v}}\int_0^{b^{w-v}}\psi_b(x)\psi_b\left(\frac{x}{b^{w-v}}\right)\mathrm{d}x + \\ &\quad + \sum_{w=1}^{m}\sum_{v=w+1}^{m}\frac{1}{b^{v-w}}\int_0^{b^{v-w}}\psi_b(x)\psi_b\left(\frac{x}{b^{v-w}}\right)\mathrm{d}x + m\int_0^1\psi_b(x)^2\mathrm{d}x.\end{aligned}$$

For $k \geq 0$ we consider now the term

$$\begin{aligned}J_k :=& \frac{1}{b^k}\int_0^{b^k}\psi_b(x)\psi_b\left(\frac{x}{b^k}\right)\mathrm{d}x = \\ =& \frac{1}{b^k}\sum_{j=0}^{b-1}\sum_{l=jb^k}^{(j+1)b^k-1}\frac{1}{(b-1)^2}\sum_{\beta=1}^{b-1}\sum_{\eta=1}^{b-1}\int_{\frac{l}{b}}^{\frac{l+1}{b}}\psi_b^{\beta}(x)\psi_b^{\eta}\left(\frac{x}{b^k}\right)\mathrm{d}x = \\ =& \frac{b^4(b^2-1)^2}{144(b-1)^2b^k}\sum_{j=0}^{b-1}\sum_{l=jb^k}^{(j+1)b^k-1}\sum_{\beta=1}^{b-1}\sum_{\eta=1}^{b-1} \\ & \int_{\frac{l}{b}}^{\frac{l+1}{b}}\left|A_{\beta,l}+iB_{\beta,l}+x-\frac{l}{b}\right|^2\left|A_{\eta,j}+iB_{\eta,j}+\frac{x}{b^k}-\frac{j}{b}\right|^2\mathrm{d}x =\end{aligned}$$

$$= \frac{b^4(b+1)^2}{144b^k} \sum_{j=0}^{b-1} \sum_{l=jb^k}^{(j+1)b^k-1} \sum_{\beta=1}^{b-1} \sum_{\eta=1}^{b-1}$$

$$\int_0^{\frac{1}{b}} |A_{\beta,l} + iB_{\beta,l} + x|^2 \left| A_{\eta,j} + iB_{\eta,j} + \frac{x}{b^k} + \frac{l-jb^k}{b^{k+1}} \right|^2 \mathrm{d}x =:$$

$$=: \frac{b^4(b+1)^2}{144b^k} \sum_{j=0}^{b-1} \sum_{l=jb^k}^{(j+1)b^k-1} I(j,k,l).$$

Let now $C_{j,k,l} := \frac{l-jb^k}{b^{k+1}}$ and $a \in \{0, \ldots, b-1\}$ such that $l = rb + a$. Note that in the case $k = 0$ we have $l = j = a$ and $C_{j,0,l} = 0$. With (8) and (9) we obtain

$$I(j,k,l) :=$$

$$:= \sum_{\beta=1}^{b-1} \sum_{\eta=1}^{b-1} \Bigg[(A_{\beta,l}^2 + B_{\beta,l}^2)(A_{\eta,j}^2 + B_{\eta,j}^2)\frac{1}{b} +$$

$$+ (A_{\beta,l}^2 + B_{\beta,l}^2)\left(\frac{1}{3b^{2k+3}} + C_{j,k,l}^2 \frac{1}{b} + C_{j,k,l}\frac{1}{b^{k+2}}\right) +$$

$$+ (A_{\beta,l}^2 + B_{\beta,l}^2)A_{\eta,j}\left(\frac{1}{b^{k+2}} + 2C_{j,k,l}\frac{1}{b}\right) + A_{\beta,l}(A_{\eta,j}^2 + B_{\eta,j}^2)\frac{1}{b^2} +$$

$$+ A_{\beta,l}\left(\frac{1}{2b^{2k+4}} + C_{j,k,l}^2\frac{1}{b^2} + C_{j,k,l}\frac{4}{3b^{k+3}}\right) +$$

$$+ A_{\beta,l}A_{\eta,j}\left(\frac{4}{3b^{k+3}} + C_{j,k,l}\frac{2}{b^2}\right) +$$

$$+ (A_{\eta,j}^2 + B_{\eta,j}^2)\frac{1}{3b^3} + A_{\eta,j}\left(\frac{1}{2b^{k+4}} + C_{j,k,l}\frac{2}{3b^3}\right) +$$

$$+ \left(\frac{1}{5b^{2k+5}} + C_{j,k,l}^2\frac{1}{3b^3} + C_{j,k,l}\frac{1}{2b^{k+4}}\right)\Bigg] =$$

$$= \frac{a(b-a)j(b-j)}{b^4}\frac{1}{b} + \frac{a(b-a)(b-1)}{b^2}\left(\frac{1}{3b^{2k+3}} + C_{j,k,l}^2\frac{1}{b} + C_{j,k,l}\frac{1}{b^{k+2}}\right) -$$

$$- \frac{a(b-a)j}{b^3}\left(\frac{1}{b^{k+2}} + 2C_{j,k,l}\frac{1}{b}\right) - \frac{aj(b-j)}{b^3}\frac{1}{b^2} -$$

$$- \frac{a(b-1)}{b}\left(\frac{1}{2b^{2k+4}} + C_{j,k,l}^2\frac{1}{b^2} + C_{j,k,l}\frac{4}{3b^{k+3}}\right) + \frac{aj}{b^2}\left(\frac{4}{3b^{k+3}} + C_{j,k,l}\frac{2}{b^2}\right) +$$

$$+\frac{j(b-j)(b-1)}{b^2}\frac{1}{3b^3}-\frac{j(b-1)}{b}\left(\frac{1}{2b^{k+4}}+C_{j,k,l}\frac{2}{3b^3}\right)+$$
$$+(b-1)^2\left(\frac{1}{5b^{2k+5}}+C_{j,k,l}^2\frac{1}{3b^3}+C_{j,k,l}\frac{1}{2b^{k+4}}\right).$$

For $k>0$ we obtain now

$$J_k=\frac{b^4(b+1)^2}{144b^k}\sum_{j=0}^{b-1}\sum_{r=jb^{k-1}}^{(j+1)b^{k-1}-1}\sum_{a=0}^{b-1}I(j,k,rb+a)=$$
$$=\frac{b^{1-2k}(b^2-1)^2\left(-1-b\left(1+b-5b^{2k}\right)\right)}{25920}.$$

For $k=0$ we obtain

$$J_0=\frac{b^4(b+1)^2}{144b^k}\sum_{j=0}^{b-1}I(j,0,j)=\frac{(b-1)\,b(b+1)^2\,\left(1+b\,(3b-1)\right)}{12960}.$$

Now we have

$$\mathbb{V}(S_m)=\mathbb{E}(S_m^2)-\mathbb{E}(S_m)^2=$$
$$=\sum_{w=1}^{m}\sum_{v=1}^{w-1}J_{w-v}+\sum_{w=1}^{m}\sum_{v=w+1}^{m}J_{v-w}+mJ_0-m^2\frac{(b^2-1)^2b^2}{72^2}=$$
$$=\frac{mb^2(b^4-1)}{25920}+\frac{b^3(1+b+b^2)(1-b^{-2m})}{12960}. \qquad \Diamond$$

Remark 11. For $b=2$ we obtain

$$\mathbb{E}(S_m)=\frac{m}{12}\quad\text{and}\quad\mathbb{V}(S_m)=\frac{m}{432}+\frac{7(1-2^{-2m})}{1620},$$

as in [16, Proof of Cor. 2.4].

The idea of the following proof is the same as in [5, Proof of Th. 2], see also [17].

Proof of Theorem 4. Let $M=b^m$. We will not prove the limit relation of Th. 4 but

(10)

$$\frac{1}{M}\left|\left\{N<M\colon (NF_{b,N}(\omega))^2\le\frac{b^2-1}{6b}\log_b M+y\frac{12}{b^2}\sqrt{\frac{\log_b M(b^4-1)b^2}{25920}}\right\}\right|=$$
$$=\Phi(y)+o(1)\qquad(\text{as } m\to\infty).$$

It is easy to see that both limit relations are equivalent. This follows directly by restricting N with $M/(\log M) \le N \le M$. Of course, N with $N < M/(\log M)$ do not matter in the limit. For the remaining ones we have $|\log M - \log N| \ll \log\log M$ and this difference does not matter in the limit either.

The idea of the proof of (10) is to use (2) and approximate $(NF_{b,N}(\omega))^2$ as a sum of weakly dependent random variables.

We use now the same argument as Drmota et al. in [5, Proof of Th. 2] (for the case $b=2$) for general primes b, see also [17]. Since $M=b^m$, the digits N_j of the b-adic expansion $N = N_0 + bN_1 + \cdots + b^{m-1}N_{m-1}$ can be considered as independent random variables that are uniformly distributed on $\{0,1,\ldots,b-1\}$ (by assuming that all numbers $N < b^m$ have equal probability b^{-m}). Then

$$U_r = \left\{\frac{N}{b^r}\right\} = \frac{N_{r-1}}{b} + \frac{N_{r-2}}{b^2} + \cdots + \frac{N_0}{b^r}$$

is very close to a random variable that is uniformly distributed on $[0,1)$. In fact we can add missing digits $N_{-1}, N_{-2}, \ldots$ that are independent and uniformly distributed on $\{0,1,\ldots,b-1\}$ and get

$$U_r' = U_r + \sum_{j=1}^{\infty} \frac{N_{-j}}{b^{r+j}}.$$

Then U_r' is exactly uniformly distributed on $[0,1)$ and $|U_r - U_r'| \le b^{-r}$. Furthermore, U_r and U_{r+k} get more and more independent as k gets large. More precisely, $U_{r+k} - b^{-k}U_r$ and U_r are independent. This means, that a slight modification of order b^{-k} makes U_r and U_{r+k} independent. Note that

$$\frac{b^2}{12}(NF_{b,N}(\omega))^2 = \sum_{r=1}^{m} \psi_b(U_r) + \mathcal{O}(1)$$

for all $N < b^m$ (see [8, Proof of Cor. 9]). Thus, $\frac{b^2}{12}(NF_{b,N}(\omega))^2$ can be represented (up to a small error term) by a sum of weakly dependent random variables.

We further note that the above smoothing by introducing missing digits can be also obtained by considering the following slight variation of the above probability model. Let $\tilde{N}$ be a random variable that is uniformly distributed on $[0,b^m)$, then $\tilde{U}_r = \{\tilde{N}/b^{r-1}\}$ is uniformly distributed on $[0,1)$ and the common distribution of $(\tilde{U}_r)_{1\le r\le m}$ is the same as that of $(U'_r)_{1\le r\le m}$. Furthermore, if we set $\xi = \tilde{N}/b^m$ and $V_r = \tilde{U}_{m-r+1}$ then ξ is uniformly distributed on $[0,1)$ and

$$V_r = \{\xi b^r\}.$$

Since $|V_r - U_{m-r+1}| \le b^{-(m-r+1)}$ and ψ_b is Lipschitz continuous we also have

$$\begin{aligned}\frac{b^2}{12}(NF_{b,N}(\omega))^2 &= \sum_{r=1}^{m} \psi_b\left(\frac{N}{b^r}\right) + \mathcal{O}(1) = \\ &= \sum_{r=1}^{m} \psi_b\left(\frac{N}{b^{m-r+1}}\right) + \mathcal{O}(1) = \\ &= \sum_{r=1}^{m} \psi_b(\xi b^r) + \sum_{r=1}^{m}\left(\psi_b\left(\frac{N}{b^{m-r+1}}\right) - \psi_b(\xi b^r)\right) + \mathcal{O}(1) = \\ &= \sum_{r=1}^{m} \psi_b(\xi b^r) + \mathcal{O}(1).\end{aligned}$$

Now by Th. 7 it follows that

$$S_m = \sum_{r=1}^{m} \psi_b(\xi b^r)$$

satisfies a central limit theorem. With Lemma 10 we thus have

$$\frac{S_m - m(b^2-1)b/72}{\sqrt{\frac{m(b^4-1)b^2}{25920} + \frac{b^3(1+b+b^2)(1-b^{-2m})}{12960}}} \longrightarrow \mathcal{N}(0,1) \qquad (\text{as } m \to \infty)$$

and also all moments converge. Since $(NF_{b,N}(\omega))^2 = \frac{12}{b^2} S_m + \mathcal{O}(1)$ we get up to the factor $12/b^2$ the same limit relation for $(NF_{b,N}(\omega))^2$ if N is uniformly distributed on $\{0,1,\dots,b^m-1\}$. In the limit the second term of the variance can be neglected and the result follows. $\Diamond$

4. Proof of Proposition 5

Since ψ_b and χ_b^c are on intervals of the form $\left[\frac{k}{b}, \frac{k+1}{b}\right]$, $k \in \{0,...,b-1\}$, translations of the same parabola, it is enough to show that eq. (4) holds for $x = \frac{k}{b}$, $k = 0,1,\dots,b$. In the case where $b=2$, the functions ψ_b and χ_b^c are the same, so we consider henceforward only odd primes b. Since $\psi_b(0) = \chi_b^c(0) = 0$ we assume in the following $k \ge 1$.

From the definition of the functions $\varphi^{\sigma}_{b,h}$ we get

$$\varphi^c_{b,h}\left(\frac{k}{b}\right) = \begin{cases} 0 & h=0, \\ A\left(\left[0,\frac{h}{b}\right), k, \left(0,\frac{c}{b},\dots,\frac{[(b-1)c]}{b}\right)\right) - h\frac{k}{b} & h \le [(k-1)c], \\ (b-h)\frac{k}{b} - A\left(\left[\frac{h}{b},1\right), k, \left(0,\frac{c}{b},\dots,\frac{[(b-1)c]}{b}\right)\right) & [(k-1)c] < h < b, \end{cases} =$$

$$= \begin{cases} 0 & h=0, \\ A\left(\left[0,\frac{h}{b}\right), k, \left(0,\frac{c}{b},\dots,\frac{[(b-1)c]}{b}\right)\right) - h\frac{k}{b} & h \ne 0, \end{cases} :=$$

$$=: \begin{cases} 0 & h=0, \\ A_{h,b,k,c} - \frac{kh}{b} & h \ne 0. \end{cases}$$

With $\chi^c_b = b\sum_{h=0}^{b-1}(\varphi^c_{b,h})^2 - \left(\sum_{h=0}^{b-1}\varphi^c_{b,h}\right)^2$ we get

$$\frac{1}{b-1}\sum_{c=1}^{b-1}\chi^c_b\left(\frac{k}{b}\right) =$$

$$= \frac{1}{b-1}\sum_{c=1}^{b-1} b\sum_{h=1}^{b-1}\left(A_{h,b,k,c} - \frac{kh}{b}\right)^2 - \frac{1}{b-1}\sum_{c=1}^{b-1}\left(\sum_{h=1}^{b-1}\left(A_{h,b,k,c} - \frac{kh}{b}\right)\right)^2$$

$$= \frac{b}{b-1}\sum_{c=1}^{b-1}\sum_{h=1}^{b-1}A^2_{h,b,k,c} - \frac{2k}{b-1}\sum_{h=1}^{b-1}h\sum_{c=1}^{b-1}A_{h,b,k,c} + \frac{k^2}{b}\sum_{h=1}^{b-1}h^2$$

$$- \frac{1}{b-1}\sum_{c=1}^{b-1}\left(\sum_{h=1}^{b-1}A_{h,b,k,c}\right)^2 + k\sum_{h=1}^{b-1}\sum_{c=1}^{b-1}A_{h,b,k,c} - \frac{k^2(b-1)^2}{4}.$$

We will need the following three lemmas to simplify the above expression.

Lemma 12. *For* $h \ne 0$*,* $k \ge 1$*,* b *a prime, we have*

$$\sum_{c=1}^{b-1}A_{h,b,k,c} = b + hk - h - k.$$

Proof. We have

$$\sum_{c=1}^{b-1}A_{h,b,k,c} =$$

$$= \sum_{j=0}^{h-1}\sum_{c=1}^{b-1}\left|\{l \in \{0,\dots,k-1\} : lc \equiv j \bmod b\}\right| =$$

$$= \sum_{c=1}^{b-1} |\{l \in \{0,\dots,k-1\} : \, lc \equiv 0 \bmod b\}| + \sum_{j=1}^{h-1} \sum_{c=1}^{b-1} \sum_{\substack{l=1 \\ lc \equiv j \bmod b}}^{k-1} 1 =$$

$$= b-1+\sum_{j=1}^{h-1} \sum_{l=1}^{k-1} \sum_{\substack{c=1 \\ lc \equiv j \bmod b}}^{b-1} 1 = b-1+\sum_{j=1}^{h-1} \sum_{l=1}^{k-1} 1 = b+hk-h-k. \quad \Diamond$$

Lemma 13. *For $k \geq 1$, b a prime, we have*

$$\sum_{c=1}^{b-1} \sum_{h=1}^{b-1} A_{h,b,k,c}^2 = (b-1)^2 k^2 - \sum_{c=1}^{b-1} \sum_{l=0}^{k-1} \sum_{\tilde{l}=0}^{k-1} \max([lc],[\tilde{l}c]).$$

Proof. Since $A_{b,b,k,c} = k$ we have

$$\sum_{c=1}^{b-1} \sum_{h=1}^{b-1} A_{h,b,k,c}^2 = -(b-1)k^2 + \sum_{c=1}^{b-1} \sum_{h=1}^{b} A_{h,b,k,c}^2 =$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1} \sum_{h=1}^{b} \left(\sum_{j=0}^{h-1} \sum_{\substack{l=0 \\ lc \equiv j \bmod b}}^{k-1} 1 \right)^2 =$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1} \sum_{h=1}^{b} \sum_{j=0}^{h-1} \sum_{i=0}^{h-1} \sum_{\substack{l=0 \\ lc \equiv j \bmod b}}^{k-1} \sum_{\substack{\tilde{l}=0 \\ \tilde{l}c \equiv i \bmod b}}^{k-1} 1 =$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1} \sum_{l=0}^{k-1} \sum_{\tilde{l}=0}^{k-1} \sum_{\substack{j=0 \\ lc \equiv j \bmod b}}^{b-1} \sum_{\substack{i=0 \\ \tilde{l}c \equiv i \bmod b}}^{b-1} \sum_{h=\max(j,i)+1}^{b} 1 =$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1} \sum_{l=0}^{k-1} \sum_{\tilde{l}=0}^{k-1} \sum_{h=\max([lc],[\tilde{l}c])+1}^{b} 1$$

$$= -(b-1)k^2 + \sum_{c=1}^{b-1} \sum_{l=0}^{k-1} \sum_{\tilde{l}=0}^{k-1} (b - \max([lc],[\tilde{l}c])) =$$

$$= (b-1)^2 k^2 - \sum_{c=1}^{b-1} \sum_{l=0}^{k-1} \sum_{\tilde{l}=0}^{k-1} \max([lc],[\tilde{l}c]). \qquad \Diamond$$

Lemma 14. *For $k \geq 1$, b a prime, we have*

$$\sum_{c=1}^{b-1}\left(\sum_{h=1}^{b-1}A_{h,b,k,c}\right)^2 =$$
$$= (b-1)k(b^2k-b+k-bk) - \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b([lc]+[\tilde{l}c]) - [lc][\tilde{l}c]).$$

Proof. Since $A_{b,b,k,c} = k$ we have

$$\sum_{c=1}^{b-1}\left(\sum_{h=1}^{b-1}A_{h,b,k,c}\right)^2 = \sum_{c=1}^{b-1}\left(-k+\sum_{h=1}^{b}A_{h,b,k,c}\right)^2 =$$
$$= (b-1)k^2 - 2k\sum_{c=1}^{b-1}\sum_{h=1}^{b}A_{h,b,k,c} + \sum_{c=1}^{b-1}\left(\sum_{h=1}^{b}A_{h,b,k,c}\right)^2.$$

We proceed as in the proof of Lemma 13 and get:

$$\sum_{c=1}^{b-1}\left(\sum_{h=1}^{b}A_{h,b,k,c}\right)^2 = \sum_{c=1}^{b-1}\sum_{h=1}^{b}\sum_{m=1}^{b}\sum_{j=0}^{h-1}\sum_{i=0}^{m-1}\sum_{\substack{l=0\\ lc\equiv j \bmod b}}^{k-1}\sum_{\substack{\tilde{l}=0\\ \tilde{l}c\equiv i \bmod b}}^{k-1} 1 =$$
$$= \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}\sum_{h=[lc]+1}^{b}\sum_{m=[\tilde{l}c]+1}^{b} 1 =$$
$$= b^2(b-1)k^2 - \sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b([lc]+[\tilde{l}c]) - [lc][\tilde{l}c]).$$

Together with Lemma 12 this gives the desired result. $\Diamond$

With Lemmas 12, 13 and 14 we have now

$$\frac{1}{b-1}\sum_{c=1}^{b-1}\chi_b^c\left(\frac{k}{b}\right) =$$
$$= \frac{b}{b-1}(b-1)^2k^2 - \frac{b}{b-1}\sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}\max([lc],[\tilde{l}c]) -$$
$$- \frac{2k}{b-1}\sum_{h=1}^{b-1}h(b+hk-h-k) + \frac{k^2}{b}\frac{(b-1)b(2b-1)}{6} -$$
$$- k(b^2k-b+k-bk) + \frac{1}{b-1}\sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b([lc]+[\tilde{l}c]) - [lc][\tilde{l}c]) +$$

$$+\,k\sum_{h=1}^{b-1}(b+hk-h-k)-\frac{k^2(b^2-1)}{4}=$$
$$=\frac{(b+1)k(2b-bk-k)}{12}+\frac{1}{b-1}\sum_{c=1}^{b-1}\sum_{l=0}^{k-1}\sum_{\tilde{l}=0}^{k-1}(b\min([lc],[\tilde{l}c])-[lc][\tilde{l}c])=$$
$$=\frac{(b+1)k(2b-bk-k)}{12}+\frac{1}{b-1}\sum_{c=1}^{b-1}\sum_{l=1}^{k-1}\sum_{\tilde{l}=1}^{k-1}(b\min([c],[\tilde{l}l^{-1}c])-[c][\tilde{l}l^{-1}c]).$$

We will need the following lemma, which will be proved afterwards.

Lemma 15. *Let b be a prime and $a\in\{1,\ldots,b-1\}$. We have*
$$\sum_{c=1}^{b-1}(b\min([c],[ac])-[c][ac])=\begin{cases}\frac{b(b^2-1)}{6} & a=1,\\ \frac{b(b^2-1)}{12} & a\neq 1.\end{cases}$$

With this last lemma we get

$$\frac{1}{b-1}\sum_{c=1}^{b-1}\chi_b^c\left(\frac{k}{b}\right)=\frac{(b+1)k(2b-bk-k)}{12}+\frac{k(k-1)}{b-1}\frac{b(b^2-1)}{12}=$$
$$=\frac{(b+1)k(b-k)}{12}=\psi_b\left(\frac{k}{b}\right),$$

where we have used [8, Lemma 13] for the last equality. This finishes the proof of Prop. 5. ♢

Proof of Lemma 15. The case $a=1$ is easy:
$$\sum_{c=1}^{b-1}(b\min([c],[c])-[c][c])=\sum_{c=1}^{b-1}(bc-c^2)=\frac{b(b^2-1)}{6}.$$
From now on let $a\in\{2,\ldots,b-1\}$ be fixed and let $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and ceiling function respectively. We have

$$\Sigma:=\sum_{c=1}^{b-1}(b\min([c],[ac])-[c][ac])=$$
$$=\sum_{c=1}^{b-1}\min([c][-ac],[-c][ac])=$$
$$=\sum_{l=0}^{a-1}\sum_{\frac{lb}{a}\le c<\frac{(l+1)b}{a}}\min(c(b-ac+lb),(b-c)(ac-lb))=$$

$$= \sum_{l=0}^{a-1} \left(\sum_{\frac{lb}{a} \le c < \frac{lb}{a-1}} (b-c)(ac-lb) + \sum_{\frac{lb}{a-1} \le c < \frac{(l+1)b}{a}} c(b-ac+lb) \right) =$$

$$= -\sum_{c=1}^{b-1} ac^2 + \sum_{l=0}^{a-1} \left(\sum_{\frac{lb}{a} \le c < \frac{lb}{a-1}} \left((l+a)bc - lb^2\right) + \sum_{\frac{lb}{a-1} \le c < \frac{(l+1)b}{a}} (l+1)bc \right).$$

We set now $x_l := \lceil \frac{lb}{a} \rceil$ and $y_l := \lceil \frac{lb}{a-1} \rceil$ and note that $x_0 = 0, y_0 = 0$ and $x_a = b, y_{a-1} = b$. We have

$$\Sigma = -\frac{a(b-1)b(2b-1)}{6} + \sum_{l=0}^{a-1} \left(\sum_{c=x_l}^{y_l-1} (l+a)bc - lb^2 + \sum_{c=y_l}^{x_{l+1}-1} (l+1)bc \right) =$$

$$= -\frac{a(b-1)b(2b-1)}{6} + \sum_{l=0}^{a-1} \left((l+a)b\frac{(x_l+y_l-1)(y_l-x_l)}{2} - lb^2(y_l-x_l) + \right.$$

$$\left. +(l+1)b\frac{(x_{l+1}+y_l-1)(x_{l+1}-y_l)}{2} \right) =$$

$$= -\frac{a(b-1)b(2b-1)}{6} + \sum_{l=0}^{a-1} \left(\frac{(a-1)b}{2} y_l(y_l-1) - lb^2(y_l-x_l) - \right.$$

$$\left. -\frac{(l+a)b}{2} x_l(x_l-1) + \frac{(l+1)b}{2} x_{l+1}(x_{l+1}-1) \right) =$$

$$= -\frac{a(b-1)b(2b-1)}{6} +$$

$$+ \sum_{l=0}^{a-1} \left(\frac{(a-1)b}{2} y_l(y_l-1) - lb^2(y_l-x_l) - \frac{(l+a)b}{2} x_l(x_l-1) \right) +$$

$$+ \sum_{l=1}^{a} \frac{lb}{2} x_l(x_l-1) =$$

$$= \frac{ab(b^2-1)}{6} + \sum_{l=1}^{a-1} \left(\frac{(a-1)b}{2} y_l(y_l-1) - lb^2(y_l-x_l) - \frac{ab}{2} x_l(x_l-1) \right).$$

We define now

$$g(b,a) := \sum_{l=1}^{a-1} -\frac{ab}{2} \left\lceil \frac{lb}{a} \right\rceil \left(\left\lceil \frac{lb}{a} \right\rceil - 1 \right) + lb^2 \left\lceil \frac{lb}{a} \right\rceil =$$

$$= \sum_{l=1}^{a-1} -\frac{ab}{2}\left\lfloor\frac{lb}{a}\right\rfloor\left(\left\lfloor\frac{lb}{a}\right\rfloor+1\right)+lb^2\left\lfloor\frac{lb}{a}\right\rfloor+lb^2.$$

Note that $\frac{lb}{a}$ is not an integer for $l \in \{1,\dots,a-1\}$ and therefore we can write $\lceil\cdot\rceil = \lfloor\cdot\rfloor + 1$.

So far we have

$$\Sigma = -\frac{b(b+1)(a-3b+2ab)}{6} + g(b,a) - g(b,a-1). \tag{11}$$

Our goal is now to find a closed formula for $g(b,a)$. We will need the following lemma.

Lemma 16. *Let m, n be positive and co-prime and let $i(l,m,n) \in \in \{0,\dots,n-1\}$ such that $i(l,m,n) \equiv lm \mod n$. Then we have*

1. $\displaystyle\sum_{l=1}^{n-1}\left\lfloor\frac{lm}{n}\right\rfloor = \frac{(m-1)(n-1)}{2},$
2. $\displaystyle\sum_{l=1}^{n-1}\left\lfloor\frac{lm}{n}\right\rfloor^2 = \frac{(m^2-1)(n-1)(2n-1)}{6n} - \frac{2}{n}\sum_{l=1}^{n-1} i(l,m,n)\left\lfloor\frac{lm}{n}\right\rfloor,$
3. $\displaystyle\sum_{l=1}^{n-1} lm\left\lfloor\frac{lm}{n}\right\rfloor = \frac{(m^2-1)(n-1)(2n-1)}{6} - \sum_{l=1}^{n-1} i(l,m,n)\left\lfloor\frac{lm}{n}\right\rfloor.$

Proof. Use that $\{i(l,m,n) : l = 1,\dots,n-1\} = \{1,\dots,n-1\}$, since m and n are co-prime, and $\frac{lm}{n} = \left\lfloor\frac{lm}{n}\right\rfloor + \frac{i(l,m,n)}{n}$. Then the above results follow by easy calculations. $\Diamond$

With the above lemma we get now

$$\begin{aligned} g(b,a) &= -\frac{b(b^2-1)(a-1)(2a-1)}{12} + b\sum_{l=1}^{a-1} i(l,b,a)\left\lfloor\frac{lb}{a}\right\rfloor - \\ &- \frac{ab(a-1)(b-1)}{4} + \frac{b(b^2-1)(a-1)(2a-1)}{6} - \\ &- b\sum_{l=1}^{a-1} i(l,b,a)\left\lfloor\frac{lb}{a}\right\rfloor + \frac{b^2(a-1)a}{2} = \\ &= \frac{b(b+1)(a-1)(1+a-b+2ab)}{12} \end{aligned}$$

and hence from eq. (11) it follows that

$$\sum_{c=1}^{b-1}(b\min([c],[ac]) - [c][ac]) = \frac{b(b^2-1)}{12}.$$

This finishes the proof of Lemma 15. ◊

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