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UNITS OF LOCAL AUTOMORPHISM NEARRINGS

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Abstract: In this paper we determine the unit group of a local endomorphism nearring of a finite p-group G that is generated by a group of automorphisms of G. As a consequence, we then determine the unit group of the endomorphism nearring of G generated by the inner automorphisms of G for any finite nilpotent group G.

1. Introduction

By an automorphism nearring of a group G, we naturally mean an endomorphism nearring of G generated by a group of automorphisms of G. In this paper we will determine the units of local automorphism nearrings of a finite p-group G. These nearrings have been studied in [4], [8], and [9] and include the nearring $I(G)$ for a finite p-group G by [4, Cor. 3.3] where $I(G)$ is the nearring of G generated by the group of inner automorphisms $\mathrm{Inn}(G)$ of G (or equivalently, the zero-symmetric part of the polynomial nearring of the group G). In particular, we shall see that we can use our knowledge of units of local automorphism nearrings to determine of units of $I(G)$ for any finite nilpotent group G. We conclude with the application of this knowledge of units to the case when $I(G)$ is a

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ring. In a sense, this paper is a companion paper to [1] where many of the results focused on the case where any minimal factors of G that are ring modules were self-centralizing. Here we will be considering a situation on the other extreme where centralizers of minimal factors of G are the entire group G. Indeed both [1] and this paper involve generalizations of results that grew out of efforts to determine the units of polynomial nearrings.

Our results on the units of our local automorphism nearrings will involve using ideals that are similar to the augmentation ideals found in the study of group rings to describe the J_2-radical of our nearrings. Before dealing with units in the third and final section of this paper, we first develop some basic facts about these augmentation ideals and their connection with the J_2-radical in the next section.

Throughout this paper, functions will be written on the right and consequently our nearrings will be left nearrings. Our basic reference on nearrings will be [6] which also follows these conventions.

2. Augmentation ideals

We begin this section with the following result:

Proposition 2.1. *Suppose that R is a nearring distributively generated by a multiplicative semigroup S of R. Then the normal subgroup $\Delta(S)$ of the additive group of R generated by the elements of R of the form $1-\alpha, \alpha \in S$, is an ideal of R.*

Proof. By [6, Cor. 9.22], it suffices to show that $\Delta(S)$ is closed under right and left multiplication by elements of S. Since elements of S distribute on both sides, we need only show that $\beta(-r+(1-\alpha)+r)$ and $(-r+(1-\alpha)+r)\beta$ lie in $\Delta(S)$ for all $\alpha, \beta \in S$ and all $r \in R$ to obtain this. Moreover, these will follow if $\beta(1-\alpha)$ and $(1-\alpha)\beta$ lie in $\Delta(S)$. To get the former, observe that $\beta(1-\alpha) = -(1-\beta)+(1-\beta\alpha)$; corresponding work yields the latter. $\Diamond$

The ideal $\Delta(S)$ of Prop. 2.1 is similar to a very important ideal in group rings called the augmentation ideal (see [11], for example), and consequently we shall refer to it as the **augmentation ideal of R with respect to S**. The next result contains a variant of this for normal subgroups when the set of distributive generators is a group which again corresponds to a similar type ideal in group rings.

Proposition 2.2. *Suppose that R is a nearring distributively generated by a multiplicative group A of R. If B is a normal subgroup of A, then the normal subgroup $\Delta(B)$ of the additive group of R generated by the elements of R of the form $(1-\beta)\alpha, \beta \in B, \alpha \in A$, is an ideal of R.*

Proof. Noting that for $\alpha \in A$ and $\beta \in B$, $\alpha(1-\beta) = (1-\alpha\beta\alpha^{-1})\alpha$, it follows that $\Delta(B)$ is an ideal of R using work similar to that in the proof of Prop. 2.1. $\Diamond$

We will call the ideal $\Delta(B)$ of Prop. 2.2 the **augmentation ideal of B in R with respect to A**. These augmentation ideals have a connection with $J_2(R)$ when G is a finite p-group and B is a p-group. The beginnings of this occur in [4], [8], and [9] in the development of the theory of local endomorphism nearrings which we shall now expand upon. In the next result we follow the usual convention of denoting the largest normal p-subgroup of a finite group G by $O_p(G)$ for a prime number p. Also, to save writing we will write p for the element $1 \cdot p$ of R and pR for the R-subgroup $(1 \cdot p)R$.

Proposition 2.3. *If R is an automorphism nearring of a finite p-group G generated by a group of automorphisms A of G, then $pR + \Delta(O_p(A)) \subseteq$ $\subseteq J_2(R)$.*

Proof. Suppose that H/K is a minimal factor of R-ideals of G (which is, in fact, a type 2 R-module by [8, Lemma 1.9]). We are going to consider the commutator $[H/K, O_p(A)]$ in the semidirect product of H/K and $O_p(A)$. Upon adapting the typical multiplicative-conjugate description of a commutator $[a,b] = a^{-1}a^b$ used in group theory to our additive-multiplicative setting, the commutator $[H/K, O_p(A)]$ is generated by elements of the form

$$[h+K, \beta] = (-h+K) + (h+K)\beta = (h+K)(-1+\beta), h \in H, \beta \in O_p(A).$$

Since the semidirect product of H/K and $O_p(A)$ is a finite p-group and hence nilpotent, $[H/K, O_p(A)]$ is properly contained H/K. Moreover, since $[h+K, \beta]\alpha = [(h+K)\alpha, \beta^\alpha]$ for any $\alpha \in A$, we get that $[H/K, O_p(A)]$ is an R-module and hence $[H/K, O_p(A)] = 0$ by the minimality of H/K. Since $1 - \beta = 1 - (-1+\beta) - 1$, it follows that $\Delta(O_p(A))$ annihilates the factors of the socle series of G. Now using the socle series of G for the S-socle series in [8, Lemma 1.4], we obtain $\Delta(O_p(A)) \subseteq J_2(R)$. Finally since H/K is an elementary abelian p-group, it also follows that $pR \subseteq J_2(R)$ completing the proof. $\Diamond$

A natural question to ask is when the containment is an equality in Prop. 2.3. As we shall next see, one place this occurs is when the automorphism nearring R of Prop. 2.3 is local. In this setting we know that $O_p(A)$ is the Sylow p-subgroup P of A in [8, Th. 3.2] or [9, Th. 1.4] and that P has a cyclic complement K in A.

Proposition 2.4. *Suppose that R is an automorphism nearring of a finite p-group G generated by a group of automorphisms A of G. If R is local and P is the Sylow p-subgroup of A, then $J_2(R) = pR + \Delta(P)$.*

Proof. As $\rho \equiv 1 \mod \Delta(P)$ for all $\rho \in P$, $R/\Delta(P)$ is distributively generated by the images of the elements of K in this quotient and hence $R/\Delta(P)$ is a commutative ring. Thus $pR + \Delta(P)$ is an ideal of R. Next observe that $R/(pR+\Delta(P))$ is a homomorphic image of the group algebra $\mathbb{Z}_p[K]$. Since this group algebra is semisimple by Maschke's Theorem, $R/(pR + \Delta(P))$ is also semisimple. By [5, Th. 4.2] we know that R has only trivial idempotents and hence $R/(pR + \Delta(P))$ has only trivial idempotents by [7, Thm. 3]. Thus by the Wedderburn Theorem we have that $R/(pR + \Delta(P))$ is a field and hence $pR + \Delta(P)$ is a maximal ideal of R. As $J_2(R)$ is the unique maximal ideal of R by [5, Thms. 2.2 and 2.10], $J_2(R) = pR + \Delta(P)$. $\diamondsuit$

One case in which we are always assured of having a local automorphism nearring is for an automorphism nearring of a finite p-group G generated by a p-group of automorphisms of G [8, Th. 3.5]. In this case, Prop. 2.4 takes on the following form:

Proposition 2.5. *If R is an automorphism nearring of a finite p-group G generated by a p-group of automorphisms A of G, then $J_2(R) = \langle p\rangle+$ $+\Delta(A)$ where $\langle p\rangle$ denotes the additive subgroup of R generated by p.*

Proof. As $J_2(R) = pR+\Delta(A)$ by Prop. 2.4, the result now follows since $\alpha \equiv 1 \mod \Delta(A)$ for all $\alpha \in A$. $\diamondsuit$

3. Determination of units

From [5, Lemma 2.4 and Th. 2.10] we know that the set of units of a local nearring R, let us denote this as $U(R)$, is the complement of $J_2(R)$ in R. We are now going to use this in conjunction with Prop. 2.4 to describe the units of a local automorphism nearring R of a finite p-group G generated by a group of automorphisms A of G. As we have already partially done in preparation for Prop. 2.4, we will use the notation of [9] where P denotes the normal Sylow p-subgroup of A, K is a complement of P in A, $|K| = k$, and $\alpha \in A$ is a generator for the cyclic group K. By [8, Th. 3.2] or [9, Th. 1.4] and Prop. 2.4, it follows that $R/J_2(R) = R/(pR+\Delta(P))$ is the field extension $\mathbb{Z}_p[\overline{\alpha}]$ of $\mathbb{Z}_p$ where $\overline{\alpha}=\alpha+J_2(R)$. Letting d denote the dimension of $\mathbb{Z}_p[\overline{\alpha}]$ over $\mathbb{Z}_p$ and $1 < k_2 < \ldots < k_d$ denote positive integers between 1 and k so that $\overline{\alpha}, \overline{\alpha}^{k_2}, \ldots, \overline{\alpha}^{k_d}$ form a basis for $\mathbb{Z}_p[\overline{\alpha}]$ over $\mathbb{Z}_p$, the elements $\alpha n_1 + \alpha^{k_2} n_2 + \cdots + \alpha^{k_d} n_d$ where $0 \le n_i < p$ for each i become a set of coset representatives of $J_2(R)$ in R. Since the elements of the nonzero cosets of $J_2(R)$ will be the units of R, we then have the following description of $U(R)$:

Proposition 3.1. *Suppose that R is a local automorphism nearring of a finite p-group G generated by a group of automorphisms A of G. Using the previously introduced notation, $U(R)$ is the disjoint unit of the cosets $\alpha n_1 + \alpha^{k_2} n_2 + \cdots + \alpha^{k_d} n_d + J_2(R)$ where $n_i \neq 0$ for at least one i.*

Combining Prop. 2.5 together with Prop. 3.1, we get the following result:

Proposition 3.2. *If R is an automorphism nearring of a finite p-group generated by a p-group of automorphisms A of G, then the group of units of R is the disjoint union*

$$U(R) = \bigcup_{n=1}^{p-1} (n + \langle p \rangle + \Delta(A)).$$

If G is a finite nilpotent group and $P_1, \ldots, P_m$ are the Sylow subgroups of G, we have either by [3, Ch. 5, Cor. 3.32] or as a consequence of [10, Cor. 4.7] that the unit group of $I(G)$ is the direct product of the unit groups of the $I(P_i)$. Hence we may restrict our attention to p-groups in the determination of the unit group of $I(G)$ when G is a finite nilpotent group thereby putting ourselves into the setting of Prop. 3.2. In fact, if G is a finite p-group and $u = \sum_{i=1}^n \alpha_i n_i$ where $\alpha_i \in \mathrm{Inn}(G)$ and $n_i \in \mathbb{Z}$ is an element of $I(G)$, Prop. 3.2 tells us that u is a unit if and only if the coefficient sum $\sum_{i=1}^n n_i$ is relatively prime to p, which is the result when [3, Ch. 5, Th. 4.41] is applied to $I(G)$. More generally, suppose G is a finite nilpotent group, $P_1, \ldots, P_m$ are the Sylow subgroups of G, $G_i = P_i$ in [10, Cor. 4.7], $e_1, \ldots, e_m$ are the idempotents of this corollary, and $u = \sum_{i=1}^n \alpha_i n_i$ where $\alpha_i \in \mathrm{Inn}(G)$ and $n_i \in \mathbb{Z}$ is an element of $I(G)$. Noting that $e_j u = \sum_{i=1}^n (e_j \alpha_i) n_i$ and $e_j \alpha_i \in \mathrm{Inn}(P_j)$, our preceding observation tells us that u is a unit if and only if $\sum_{i=1}^n n_i$ is relatively prime to $|P_j|$ for each j. Equivalently, u is a unit if and only if $\sum_{i=1}^n n_i$

is relatively prime to $|G|$, which is the result when [3, Ch. 5, Cor. 4.42] is applied to $I(G)$.

We conclude with some consequences of Prop. 3.2 for $R = I(G)$ when $I(G)$ is a ring for a finite group G. Here too we may restrict to the case of p-groups since we know from [2] that G is nilpotent of class at most 3. If $I(G)$ is a ring for a finite p-group G, Prop. 3.2 becomes the following:

Corollary 3.3. *If G is a finite p-group for which $I(G)$ is a ring, then the group of units of $I(G)$ is the disjoint union*

$$U(I(G)) = \bigcup_{n=1}^{p-1}(n + \langle p, 1-\alpha|\alpha \in \mathrm{Inn}(G)\rangle)$$

where $\langle p, 1-\alpha|\alpha \in \mathrm{Inn}(G)\rangle$ denotes the additive subgroup of $I(G)$ generated by p and $1-\alpha, \alpha \in \mathrm{Inn}(G)$.

Proof. Since $I(G)$ is a ring, we get in Prop. 2.1 that $\Delta(\mathrm{Inn}(G))$ is additively generated by the elements $1-\alpha, \alpha \in \mathrm{Inn}(G)$, so that $\langle p\rangle+$ $+\Delta(\mathrm{Inn}(G))$ is additively generated by these elements and p. The result now follows from Prop. 3.2. $\Diamond$

A special case where the result of Cor. 3.3 is especially easy to work with is when the socle series of G relative to $I(G)$ has length at most 2. In this case G has nilpotency class at most 2 by [8, Lemma 1.9] and hence $I(G)$ is a commutative ring by [2]. Here we have:

Corollary 3.4. *Suppose that G is a finite p-group for which the socle series of G relative to $I(G)$ has length at most 2. If $g_1, \ldots, g_k \in G$ form a set of generators for $G/Z(G)$ where $Z(G)$ is the center of G, then the group of units of $I(G)$ is the disjoint union*

$$U(I(G)) = \bigcup_{n=1}^{p-1}(n + \langle p, 1-\tau_{g_1}, \ldots, 1-\tau_{g_k}\rangle)$$

where τ_x denotes the inner automorphism of G induced by $x \in G$.

Proof. From the proof of [6, Cor. 10.38] we know that $J_2(R)^2 = 0$. Thus for any $\alpha, \beta \in \mathrm{Inn}(G)$,

$$(1) \quad 1-\alpha\beta = (1-\alpha) + (1-\beta) - (1-\alpha)(1-\beta) = (1-\alpha) + (1-\beta)$$

as $(1-\alpha)(1-\beta) \in J_2(R)^2$. Since the τ_{g_i} generate $\mathrm{Inn}(G)$, our corollary now follows from Cor. 3.3 by eq. (1). $\Diamond$

As an illustration of the use of Cor. 3.4, consider the dihedral group of order 8:

$$D_4 = \langle a, b | 4a = 2b = 0, b + a + b = 3a \rangle.$$

This is a group whose socle series has length 2. Here a and b form a set of generators for $D_4/Z(D_4)$ and it is easily verified that $\langle 2, 1 - \tau_a, 1 - \tau_b \rangle$ is a group of exponent 2 that is the direct sum of the additive subgroups of $I(D_4)$ generated by each of 2, $1 - \tau_a$, and $1 - \tau_b$. Thus it follows that the elements of $U(I(D_4))$ are

$$1 + 2i_1 + (1 - \tau_a)i_2 + (1 - \tau_b)i_3$$

where each i_j is either 0 or 1. In particular, $|U(I(D_4))| = 8$.

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