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CLASSES OF SEMIGROUPS WITH COMPATIBLE NATURAL PARTIAL ORDER I

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Abstract: In this survey we find new semigroups and collect all semigroups known up to now that have a right (two-sided) compatible natural partial order. In this first part trivially resp. totally ordered semigroups are considered and semigroups in special classes – in particular (E-)medial semigroups – with this property are studied. The second part will deal with the classes of E-inversive, eventually regular, and regular semigroups. As far as possible the structure of the semigroups in question is described and methods to construct them are provided.

Introduction

On any semigroup $(S, \cdot)$ the relation

$a \leq_S b$ if and only if $a = xb = by, xa = a,$ for some $x, y \in S^1$

is a partial order, the so-called *natural partial order of* S ([18]). It generalizes that on regular ([11], [23]), in particular inverse ([31]) semigroups. It is easy to see that the following are equivalent for any semigroup S:

(i) $a \leq_S b$;

(ii) $a = xb = by$, $ay = a$, for some $x, y \in S^1$;

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(iii) $a = xb = by$, $xa = a = ay$, for some $x, y \in S^1$.

If $E(S)$ denotes the set of all idempotents in S, then $\leq_S$ restricted to $E(S)$ takes the form: $e \leq_S f$ if and only if $e = ef = fe$. Note that if $a = eb = by$ for some $e \in E(S^1)$, $y \in S^1$, or if $a = xb = bf$ for some $x \in S^1$, $f \in E(S^1)$, then $a \leq_S b$.

A useful property of $\leq_S$ would be right, left, or even two-sided compatibility with multiplication: for any $c \in S$,

$a \leq_S b$ implies $ac \leq_S bc$ ... right compatibility,

$ca \leq_S cb$ ... left compatibility,

$ac \leq_S bc, ca \leq_S cb$ ... two-sided compatibility.

In general, the natural partial order of a semigroup is neither right, nor left compatible with multiplication. In fact, all possibilities may occur:

(1) For any commutative semigroup S, $\leq_S$ is two-sided compatible.

(2) For $S = R^1$, where R is a right zero semigroup, $\leq_S$ is right but not left compatible (since $\leq_R$ is the identity relation). Also, the semigroup $S = (T_2, \circ)$ of all transformations of a two-element set, with respect to composition from the left, has this property ([15]).

(3) For $S = L^1$, where L is a left zero semigroup, $\leq_S$ is left, but not right compatible.

(4) For $S = (T_3, \circ)$, the full transformation semigroup on a three-element set (with composition from the left), $\leq_S$ is neither left, nor right compatible: writing (abc) with $a, b, c \in \{1, 2, 3\}$ for $f \in T_3$ with $1f = a$, $2f = b$, $3f = c$, we have by [15]

$$(111) <_S (112), \text{ but } (333) \circ (111) = (111) \not\leq_S (222) = (333) \circ (112),$$
$$(112) <_S (132), \text{ but } (112) \circ (211) = (221) \not\leq_S (211) = (132) \circ (211).$$

Also the monoid S^1, where $S = \mathcal{M}(I, G, \Lambda; P)$ is a completely simple semigroup with $|I| > 1, |\Lambda| > 1$, has this property: if $e = (j, p_{\mu j}^{-1}, \mu)$, $a = (i, g, \lambda) \in S$, then $e \in E(S)$, hence $e <_{S^1} 1$; but $ea \not\leq_S 1a = a$ and $ae \not\leq_S a1 = a$ (since $\leq_S$ is the identity relation by [27], II.4.2, and $ea \neq a, ae \neq a$).

In the following survey we provide necessary and/or sufficient conditions for the natural partial order on general semigroups and particular classes of semigroups to be right or even two-sided compatible. Also, we describe – as far as possible – the structure of these semigroups and give a method to construct them. At the very least, concrete examples of such semigroups are given.

A considerable simplification of this problem appears if for a semigroup S, in the definition of $\leq_S$, one or both of the elements $x, y \in S$ are idempotent. In this situation the question of compatibility can be treated by properties of $E(S)$. This happens in the following cases:

(a) S is regular: $a \leq_S b \Leftrightarrow a = eb = bf$ for some $e, f \in E(S)$; in particular,

(b) S is inverse: $a \leq_S b \Leftrightarrow a = eb$ $(a = bf)$ for some $e \in E(S)$ $(f \in E(S))$;

(c) S is a band: $e \leq_S f \Leftrightarrow e = ef = fe$.

(d) S is groupbound, called *epigroup* in [29] and *completely* π*-regular* in [17] (i.e., for any $a \in S$ there exists $n > 0$ such that a^n belongs to a subgroup of S): by [12], 1.4.6,

$$a \leq_S b \Leftrightarrow a = eb = bf \quad \text{for some} \quad e, f \in E(S^1);$$

hence this also holds if S is periodic (in particular, finite);

(e) S is E-inversive (i.e., for any $a \in S$ there is $a' \in S$ with $aa' \in \in E(S)$) such that whenever $ax = a$ $(a, x \in S)$ then $ax' = a$ for some $x' \in S$ such that $xx' \in E(S)$: by Result 5.9, Remark (i),

$$a \leq_S b \text{ if and only if } a = xb = bf \text{ for some } x \in S^1, f \in E(S^1).$$

Note that every groupbound semigroup satisfies these conditions (see Example (4) preceding Result 5.9).

It is the aim of this survey to specify in different classes as many semigroups as possible (all those), for which the natural partial order is right (two-sided) compatible. In this first part we consider trivially resp. totally ordered semigroups (Sec. 2), and several special classes (Sec. 3), in particular, (E-)medial semigroups (Sec. 4). Throughout, sufficient conditions for right (two-sided) compatibility with multiplication by particular elements are given. In the second part, the compatibility problem is dealt with for the class of E-inversive (Sec. 5), of eventually regular (Sec. 6), and of regular semigroups (Sec. 7). The references given in part I beginning with 5, 6, or 7, concern results in part II.

The following list gives a sample of semigroups S, for which

(I) $\leq_S$ is *two-sided compatible*: weakly cancellative (in particular, right or left cancellative); right or left simple; completely simple; right or left stratified (see Sec. 2); centric (3.3); commutative (3.3, Cor.); negatively ordered (3.4, Cor.); right and left commutative (3.6, Cor.); externally commutative (4.2); $\mathcal{H}$-commutative (3.8); t-archimedean (3.12); nilextensions of inverse semigroups (3.15, Remark); powerjoined (3.16);

power centralized (3.17); right or left quasicommutative (3.19); residuated (3.22); medial and every element has a right and a left identity (4.1); medial archimedean (4.3); medial without maximal elements (4.4); E-inversive E-unitary with central idempotents (5.7, Cor.); E-inversive rectangular (5.8); eventually regular (groupbound) with central idempotents (6.3); eventually regular unipotent (6.5); locally inverse (7.6); inverse (7.8); regular right (left) commutative (7.15); regular (E-)medial (7.16, 7.18); regular externally commutative (7.17); normal orthogroups (7.19, Cor.); normal cryptogroups (7.22); generalized inverse (7.27).

(II) $\leq_S$ is *right compatible* (besides those in I): right full (3.2); left negatively ordered (3.4); right commutative (3.6); $\mathcal{R}$-commutative (3.7); right archimedean (3.11); left residuated (3.21); medial and every element has a right identity (4.1); stationary on the left (4.8); eventually regular with $E(S)$ a rightzero band (6.4); regular locally $\mathcal{L}$-unipotent (7.3); right inverse (7.4, Cor.); regular with $\mathcal{L}$-majorization (7.9 – see 7.3); regular with $E(S)$ a right seminormal band (7.26).

Most of the proofs are straightforward, so they will be omitted. For some of them we indicate the essential step supposing as first sentence: "Let a $<_S b$, i.e., $a = xb = by$, $xa = a = ay$, for some $x, y \in S$" resp. "$a = eb = bf$ for some $e, f \in E(S)$". Throughout the paper we consider only semigroups with $|S| > 1$.

1. Necessary and/or sufficient conditions

Up to now there is only one criterion for right compatibility of the natural partial order on a general semigroup. It is quite "close" to the definition. However, for many interesting classes of semigroups it makes it possible to answer the question, when $\leq_S$ is right or even two-sided compatible with multiplication.

Result 1.1 ([20]). *For a semigroup S, $\leq_S$ is right compatible if and only if $axb \in abS^1$ for any $a, b, x \in S$ such that $ax = ya = y^2a$ for some $y \in S^1$.*

In case that $\leq_S$ has one of the particular forms considered in the Introduction we have

Corollary. *Let S be a semigroup such that $a \leq_S b$ implies $a = bf$ for some $f \in E(S^1)$. Then $\leq_S$ is right compatible if and only if $aeb \in abS^1$ for any $a, b \in S, e \in E(S)$, such that $ae \in S^1a$.*

Proof. Necessity: Let $a, b \in S$, $e \in E(S)$ be such that $ae \in S^1a$, i.e., $ae = xa$ for some $x \in S^1$. Then $c := xa = ae$ satisfies $c \leq_S a$, whence $cb \leq_S ab$. Thus, $cb = ab \cdot y$ for some $y \in S^1$, so that $aeb = cb = aby \in$ $\in abS^1$.

Sufficiency: Let $a <_S b$ and $c \in S$. Then $a = xb = bf$, $xa = a$, for some $x \in S$, $f \in E(S)$. Thus $bf \in S^1b$, hence $bfc \in bcS^1$, i.e., $bfc = bcy$ for some $y \in S^1$. Therefore, $ac = x \cdot bc = bfc = bc \cdot y$ and $x \cdot ac = ac$, i.e., $ac \leq_S bc$. $\Diamond$

Remark. Note that the particular form of $\leq_S$ is only used in the proof of sufficiency. Thus, Corollary gives a necessary condition for semigroups S with $E(S) \neq \phi$ in order that $\leq_S$ be right compatible. Also, in this condition "$ae \in S^1a$" can be replaced by "$ae \in E(S^1)a$".

A further *necessary* condition for right compatibility is $\mathcal{L}$-majorization: a semigroup S satisfies $\mathcal{L}$-majorization if whenever $a, b, c \in S$ are such that $a \geq_S b$, $a \geq_S c$ and $b\mathcal{L}c$, then $b = c$ (see [27]).

Result 1.2 ([20])**.** *Let S be a semigroup. If $\leq_S$ is right compatible then S satisfies $\mathcal{L}$-majorization.*

Remark. For *regular* semigroups S, $\mathcal{L}$-majorization is also sufficient for right compatibility of $\leq_S$ (see Result 7.9). But in general, $\mathcal{L}$-majorization in a semigroup S does not imply right compatibility of $\leq_S$: Let $Y =$ $= \{0, e, f\}$ be the semilattice with $0 <_Y e <_Y f$ and let $T = \mathcal{M}(I, Y, \Lambda, P)$ be the generalized Rees matrix-semigroup over Y with $|I| = 1$, $|\Lambda| > 1$, and $P = (p_{\nu 1})$ be such that for some λ, $\mu \in \Lambda$, $p_{\lambda 1} = 0$, $p_{\mu 1} = e$. Then $S = T^1$ is $\mathcal{L}$-trivial, i.e., $x\mathcal{L}y$ implies $x = y$; therefore S satisfies $\mathcal{L}$-majorization. But $\leq_S$ is not right compatible: for $(1, e, \mu)$, $(1, f, \lambda) \in S$ we have $(1, e, \mu) \in E(S)$ and thus $(1, e, \mu) <_S 1$; however,

$$(1, e, \mu)(1, f, \lambda) = (1, e, \lambda) \not\leq_S (1, f, \lambda) = 1 \cdot (1, f, \lambda),$$

since $(1, f, \lambda)(1, y, \lambda) = (1, 0, \lambda)$ for any $y \in Y$. Note that S is not regular: $(1, e, \lambda) \notin \mathrm{Reg}\, S$.

In the non-regular case the converse of Result 1.2 holds under an additional assumption (sufficiency in the following characterization is due to M. Petrich).

Result 1.3 ([20])**.** *Let S be a semigroup such that S^2 is regular. Then $\leq_S$ is right compatible if and only if S satisfies $\mathcal{L}$-majorization.*

Remark. (i) The condition "S^2 regular" is not necessary for right compatibility of $\leq_S$. For example, on the multiplicative semigroup S of natural numbers, $\leq_S$ is the identity relation (since S is cancellative), hence (right) compatible; but S^2 is not regular. More generally, for any proper inflation S of $(\mathbb{N}, \cdot)$, $\leq_S$ is non-trivial and compatible (see Appendix (C) in part II), but $S^2 \subseteq \mathbb{N}$ is not regular.

(ii) Examples of (non-regular) semigroups S such that S^2 is regular are provided by inflations of regular semigroups $T : ab \in T$ for all $a, b \in S$. Note that $\leq_S$ is right compatible if and only if $\leq_T$ is so (see Appendix (C) in part II).

Another *necessary* condition for two-sided compatibility of $\leq_S$ is given by

Result 1.4 ([28]; see [20])**.** *Let S be a semigroup. If $\leq_S$ is two-sided compatible then every principal order ideal $(a] = \{x \in S | x \leq_S a\}$ of $(S, \leq_S)$ is directed downwards.*

Remark. (i) If S is an inverse semigroup, $\leq_S$ is two-sided compatible (see Result 7.8) and S has the property given in Result 1.4, by [3], Exercise 7.1(4).

(ii) The converse of Result 1.4 does not hold: let $S = (T_2, \circ)$ be the (regular) transformation semigroup on a two-element set, and consider S^0. Then for any $a \in S^0$, (a] is directed downwards (since $0 \in (a]$); but $\leq_{S^0}$ is not left compatible (see Example (2) in the Introduction).

A *sufficient* condition for two-sided compatibility of $\leq_S$ was given by M. Petrich (oral communication):

Result 1.5. *Let S be a semigroup such that any element of S, which is a right identity of some element in S, belongs to the center of S. Then $\leq_S$ is two-sided compatible.*

Proof. Let $a <_S b$, i.e., $a = xb = by$, $xa = a = ay$, for some $x, y \in S$. Then $y \in S$ is a right identity of $a \in S$, hence $y \in Z(S)$. Thus we have for any $c \in S$:

$$ac = x \cdot bc = byc = bcy, \quad x \cdot ac = ac, \quad \text{i.e.,} \quad ac \leq_S bc;$$
$$ca = cb \cdot y = y \cdot cb, \quad ca \cdot y = ca, \quad \text{i.e.,} \quad ca \leq_S cb. \quad \Diamond$$

Remark. The converse of Result 1.5 does not hold: consider any inverse semigroup S that is not a Clifford semigroup. Then $\leq_S$ is two-sided compatible (see Result 7.8) and every idempotent of S is a right identity of itself; but $E(S) \nsubseteq Z(S)$ (otherwise S would be a Clifford semigroup). The same holds for any right zero semigroup. Note that for any semigroup S satisfying the condition in Result 1.5 all idempotents are central. Also, "right" can be replaced by "left".

Next we consider semigroups S such that $E(S)$ forms a subsemigroup (hence $E(S) \neq \phi$). Note that this property is not implied by right or left compatibility of $\leq_S$, in general. For example, any completely simple semigroup S has a trivial, hence a compatible natural partial order (see Sec. 2) – but $E(S)$ is a subsemigroup if and only if S is a rectangular group (see [25], IV.3.3).

Result 1.6. *Let S be a semigroup such that $E(S)$ is a subsemigroup. If $\leq_S$ is right compatible then the following hold:*

(i) *$E(S)$ is a right seminormal band (i.e., $egefg = efg$ for all $e, f, g \in E(S)$);*

(ii) *for any $e \in E(S)$, $E(eSe)$ is a right regular band (i.e., $fgf = gf$ for all $f, g \in E(eSe)$).*

Proof. (i) $E(S)$ is a semilattice Y of rectangular bands $E_\alpha (\alpha \in Y)$, i.e., $efe = e$ for all $e, f \in E_\alpha$ (see [13], 3.1). Let $e, f, g \in E(S)$; then $e \in E_\alpha$, $f \in E_\beta$, $g \in E_\gamma$, for some $\alpha, \beta, \gamma \in Y$; hence $egefg$, $efg \in E_{\alpha\beta\gamma} := B$. Now $ege = e \cdot ege = ege \cdot e$ implies $ege \leq_S e$, so that by hypothesis, $ege \cdot fg \leq_S e \cdot fg$. Since both elements are idempotent we also have $egefg \leq_B efg$. But for the rectangular band B, $\leq_B$ is trivial by 7.1, Cor. Hence it follows that $egefg = efg$.

(ii) Let $e \in E(S)$ and $f, g \in E(eSe) \subseteq E(S)$. Then $f = exe$, $g = eye$, for some $x, y \in S$, and we obtain by (i):
$fgf = exe \cdot eye \cdot exe = e \cdot exe \cdot e \cdot eye \cdot exe = e \cdot eye \cdot exe = eye \cdot exe = fg$. $\Diamond$

Result 1.7. *Let S be a semigroup such that $E(S)$ is a subsemigroup. If $\leq_S$ is two-sided compatible then the following hold:*

(i) *$E(S)$ is a normal band (i.e., $efgh = egfh$ for all $e, f, g, h \in E(S)$);*

(ii) *for any $e \in E(S)$, $E(eSe)$ is a semilattice.*

Proof. (i) By [26] (p. 29), a right and left seminormal band is normal. We give a direct proof. Similarly to the beginning of the proof of Result 1.6(i), we have $efgh$, $egfh \in E_{\alpha\beta\gamma\delta}$ (for $h \in E_\delta$). Since $efghe \leq_S e$ and $hefgh \leq_S h$, we obtain in the rectangular band $B = E_{\alpha\beta\gamma\delta}$:

$$efgh = efgh \cdot egfh \cdot efgh = (efghe) \cdot gf \cdot (hefgh) \leq_B e \cdot gf \cdot h.$$

But B is trivially ordered by 7.1, Cor.; it follows that $efgh = egfh$.

(ii) A right and left regular band, is commutative, hence a semilattice (see Result 1.6). $\Diamond$

In the particular case that S is a monoid, we have

Result 1.8. *Let S be a monoid. If $\leq_S$ is right compatible then*

(i) *$E(S)$ is a right regular band (i.e., $efe = fe$ for all $e, f \in E(S)$);*

(ii) *$E(S)a \subseteq aS$ for every $a \in S$.*

Proof. (i) First, $E(S)$ is a subsemigroup of S : if $e, f \in E(S)$ then $e \leq_S 1_S$ implies $ef \leq_S 1_S f = f$, so that $ef \in E(S)$ by [20]. Hence by Result 1.6 (ii), $E(S) = E(1_S S 1_S)$ is a right regular band.

(ii) Let $e \in E(S)$, $a \in S$. Then $e \leq_S 1_S$ implies that $ea \leq_S 1_S a = a$. Thus $ea = ay$ for some $y \in S$; hence $E(S)a \subseteq aS$. $\Diamond$

For bands with identity this yields the following criterion:

Result 1.9. *Let S be a band with identity. Then $\leq_S$ is right compatible if and only if S is a right regular band. In particular, $\leq_S$ is two-sided compatible if and only if S is a semilattice.*

Proof. Necessity in the first statement holds by Result 1.8(i). Sufficiency: Let $e \leq_S f$, that is, $e = ef = fe$. Then we have for any $g \in S$:
$eg \cdot fg = e \cdot gfg = e \cdot fg = ef \cdot g = eg$, $fg \cdot eg = f \cdot geg = f \cdot eg = fe \cdot g = eg$; therefore $eg \leq_S fg$. Concerning the second statement, observe that a band, which is right- and left regular, is commutative, hence a semilattice. Conversely, for any commutative semigroup S, $\leq_S$ is two-sided compatible. $\Diamond$

Since sufficiency in Result 1.9 holds without the hypothesis of an identity we have the following

Corollary. *Let S be a right regular band; then $\leq_S$ is right compatible. In particular, for any semilattice S, $\leq_S$ is two-sided compatible.*

Generalizing Result 1.9 we have

Result 1.10. *Let S be a semigroup which has a greatest element with respect to $\leq_S$. Then $\leq_S$ is right compatible if and only if $E(S)$ is a right regular band.*

Proof. First by [28], S is either a band B with identity 1_B or an inflation of B at 1_B by only one element $a \notin B$. In the first case, the statement holds by Result 1.9. In the second case, $S = B \cup \{a\}$ where $ea = ae = e$ for every $e \in B$. Since B is a subsemigroup of S with identity and $a \notin E(S) = B$, we have necessity by Result 1.9. Sufficiency: If $e \leq_S f (e, f \in B)$, then $eg \leq_S fg$ for any $g \in B$ (by Result 1.9); also $ea = e \leq_S f = fa$. For $e <_S a$ $(e \in B)$ we have for any $f \in B : ef = fef \leq_S f = af$, and $ea = e \leq_S 1_B = aa$. Therefore, $\leq_S$ is right compatible. $\Diamond$

The converse of Result 1.8 (ii) holds for semigroups, whose natural partial order has a particular form:

Result 1.11. *Let S be a semigroup such that $a \leq_S b$ implies $a = bf$ for some $f \in E(S^1)$. If $E(S)a \subseteq aS^1$ for any $a \in S$ then $\leq_S$ is right compatible.*

Corollary. *Let S be a semigroup such that $a \leq_S b$ implies $a = eb$ $(a = bf)$ for some $e \in E(S^1)(f \in E(S^1))$. If the idempotents of S are central in S then $\leq_S$ is two-sided compatible.*

For monoids, on which the natural partial order has this particular form, Results 1.8(ii) and 1.11 give

Result 1.12. *Let S be a monoid such that $a \leq_S b$ implies $a = bf$ for some $f \in E(S)$. Then $\leq_S$ is right compatible if and only if $E(S)a \subseteq aS$ (equivalently, $E(S)a \subseteq aE(S)$) for any $a \in S$.*

Proof. Concerning the statement in the parentheses, note that in the proof of Result 1.8 (ii), $ea = af$ for some $f \in E(S)$ (by the particular form of $\leq_S$). $\Diamond$

Result 1.13. *Let S be a monoid such that $a \leq_S b$ if and only if $a = eb = bf$ for some $e, f \in E(S)$. Then $\leq_S$ is two-sided compatible if and only if $aE(S) = E(S)a$ for every $a \in S$.*

Remark. In particular, for a *finite* or *regular* monoid S, $\leq_S$ is two-sided compatible if and only if $aE(S) = E(S)a$ for every $a \in S$. In the second case, S is inverse.

It was noted in [11] that for regular monoids S, $\leq_S$ is right (left) compatible with multiplication by particular elements (see also [36]). This observation holds for numerous semigroups and will be applied in the following sections. First we have the following

Result 1.14. *Let S be a semigroup such that $\leq_S$ is right (left) compatible with multiplication by idempotents. Then $\leq_S$ is right (left) compatible with multiplication by regular elements.*

Proof. Let $a \leq_S b$ and $c = cc'c \in S$. Since $cc' \in E(S)$ we have $a \cdot cc' \leq_S b \cdot cc'$. Hence there are $x, y \in S^1$ such that $acc' = x \cdot bcc' = = bcc' \cdot y$, $x \cdot acc' = acc'$. Multiplying on the right by $c \in S$ we get: $ac = x \cdot bc = bc \cdot c'yc$, $x \cdot ac = ac$, i.e., $ac \leq_S bc$. $\Diamond$

Remark. Let S be a semigroup such that $Se \subseteq eS^1$ for any $e \in E(S)$. Then $\leq_S$ is right compatible with multiplication by idempotents (see Result 1.17 below).

Result 1.15. *Let S be a monoid. If $c \in S$ is right invertible then $a \leq_S b$ $(a, b \in S)$ implies $ac \leq_S bc$.*

Proof. Since $cc' = 1_S$ for some $c' \in S$ we have

$$ac = x \cdot bc = byc = b1_S yc = b \cdot cc' \cdot yc = bc \cdot z \ (z \in S), x \cdot ac = ac;$$

i.e., $ac \leq_S bc$. $\Diamond$

Remark. For a monoid S, every right (left) invertible element $c \in S$ is maximal in $(S, \leq_S)$: if $c \leq_S a$ for some $a \in S$, then $c = xa$, $xc = c$, for some $x \in S$; hence $xcc' = cc'$, that is, $x = 1_S$ and $c = a$. In particular, $1_S \in S$ is maximal in $(S, \leq_S)$.

Result 1.16. *Let S be a semigroup. If $c \in S$ is such that $Sc \subseteq cS^1$, then $a \leq_S b$ $(a, b \in S)$ implies $ac \leq_S bc$. In particular, $\leq_S$ is two-sided compatible with multiplication by elements in the center of S.*

By Result 1.14 we obtain

Result 1.17. *Let S be a semigroup such that $Se \subseteq eS^1$ for any $e \in E(S)$. Then $\leq_S$ is right compatible with multiplication by regular elements.*

Generalizing we have

Result 1.18. *Let S be a semigroup such that for all $a \in S$, $e \in E(S)$, there exists $n > 0$ such that $a^n e \in eS^1$. Then $\leq_S$ is right compatible with multiplication by regular elements.*

Proof. Let $a <_S b$, i.e., $a = xb = by$ and $xa = a$ $(x, y \in S)$. Then for any $k > 0 : a = ay = by \cdot y = by^2 = \cdots = by^k$. If $e \in E(S)$ then for some $n > 0$, $z \in S^1$:

$$ae = x \cdot be = by^n e = be \cdot z, \quad xae = ae, \quad \text{and} \quad ae \leq_S be.$$

Hence the claim follows by Result 1.14. $\Diamond$

Remark. Every periodic semigroup S such that $E(S)$ is a right regular band, satisfies the condition in Result 1.18: if $a \in S$, $e \in E(S)$, then $a^n = f \in E(S)$ for some $n > 0$, whence $a^n e = fe = efe \in eS^1$. Also, any inflation $S = \bigcup_{\alpha \in B} T_\alpha$ of a right regular band B does so (since $a^2 \in B$ for all $a \in S$ and $E(S) = B$); in this case, $\leq_S$ is right compatible with multiplication by *every* element of S because by 1.9, Cor., $\leq_B$ is right compatible (see Appendix (C) in part II); if $S \neq B$ then $\leq_S$ is not trivial. More generally: any nil-extension of a right regular band.

As a particular case of Result 1.17 we get

Result 1.19. *Let S be a semigroup with central idempotents (i.e., $ae = ea$ for all $e \in E(S)$, $a \in S$). Then $\leq_S$ is two-sided compatible with multiplication by regular elements.*

More generally we have

Result 1.20. *Let S be a semigroup such that $aeb = eab$ ($aeb = abe$) for all $a,\ b \in S,\ e \in E(S)$. Then $\leq_S$ is two-sided compatible with multiplication by regular elements.*

Proof. For any $e \in E(S)$, $ae = x \cdot be = bye = b(y \cdot e \cdot e) = b(e \cdot y \cdot e) = be \cdot z$ ($z \in S$) and $x \cdot ae = ae$; also $ea = eb \cdot y = exb = x \cdot eb$ and $eay = ea$; therefore, $ae \leq_S be$ and $ea \leq_S eb$. Thus the claim follows by Result 1.14. $\Diamond$

If the natural partial order has a particular form we have the following

Result 1.21. *Let S be a semigroup such that $a \leq_S b$ if and only if $a = eb = bf$ for some $e, f \in E(S)$, and $efa = fea$ for any $a \in S$, $e, f \in E(S)$. Then $\leq_S$ is two-sided compatible with multiplication by regular elements.*

Proof. For any $c = cc'c \in S$, $ac = e \cdot bc = bfc = b(f \cdot cc' \cdot c) = b(cc' \cdot f \cdot c) = = bc \cdot z$ ($z \in S$); $ca = cb \cdot f = ceb = c(c'c \cdot e \cdot b) = c(e \cdot c'c \cdot b) = w \cdot cb (w \in S)$; hence $ac \leq_S bc$ and $ca \leq_S cb$. $\Diamond$

Remark. (i) Every groupbound semigroup with commuting idempotents satisfies the conditions in Result 1.21.

(ii) The class of semigroups S satisfying $efa = fea$ for any $a \in S$, $e, f \in E(S)$, is properly contained in the class of E-medial semigroups (see Sec. 4.), as any left zero semigroup shows.

Generalizing Result 1.21 we have the following

Corollary. *Let S be a semigroup such that $a \leq_S b$ implies $a = bf$ for some $f \in E(S)$, and $E(S)$ is a right regular band (i.e., $efe = fe$ for any $e, f \in E(S)$). Then $\leq_S$ is right compatible with multiplication by regular elements.*

Proof. $ac = x \cdot bc = bfc = b(f \cdot cc')c = b(cc' \cdot f \cdot cc')c = bc \cdot z$ ($z \in S$) and $x \cdot ac = ac$; hence $ac \leq_S bc$. $\Diamond$

Result 1.22. *Let S be a semigroup such that $abe = bea$ ($eab = bea$) for any $a, b \in S$, $e \in E(S)$. Then $\leq_S$ is two-sided compatible with multiplication by regular elements.*

Proof. Every regular element $a \in S$ is central in S, since for any $x \in S$: $xa = x \cdot a \cdot a'a = a \cdot a'a \cdot x = ax$; hence the statement follows by Result 1.16. (Putting $b = e$ we obtain that very idempotent of S is central in S; thus the statement also follows from Result 1.19.) $\Diamond$

In the following particular case we know more:

Result 1.23. *Let S be a semigroup such that $axy = xya$ for all $a, x, y \in S$.*

Then $\leq_S$ is two-sided compatible with multiplication by any $c \in S$, that has a right or a left identity.

Proof. $ac = x \cdot bc = byc = b(ycc') = b(cc'y) = bc \cdot z$ $(z \in S)$, $x \cdot ac = ac$; $ca = cb \cdot y = cxb = (cc'x)b = (c'xc)b = w \cdot cb$ $(w \in S)$, $ca \cdot y = ca$; therefore, $ac \leq_S bc$ and $ca \leq_S cb$. The proof for "left" is similar. $\Diamond$

Remark. A semigroup S satisfying the identity $axy = xya$ is called (1,2)-commutative in [22]. If every element of S has a right identity then for any $a, x, y, b \in S$: $axyb = a \cdot xyy' \cdot b = a \cdot yy'x \cdot b \in aySxb$. For semigroups with this property we still have

Result 1.24. *Let S be a semigroup such that $axyb \in aySxb$ for all $a, x, y, b \in S$. Then $\leq_S$ is right (left) compatible with multiplication by any $c \in S$, that has a right (left) identity.*

Remark. Note that every regular element $a = aa'a$ of a semigroup S has $a'a \in S$ as a right and $aa' \in S$ as a left identity. Concerning regular semigroups satisfying the condition in Result 1.24, see Result 7.21. and the Remarks following it. *General* observations on semigroups with elements having a right (and left) identity are provided in Appendix (F) of part II.

2. Trivially or totally ordered semigroups

A semigroup S is *trivially ordered* if $\leq_S$ is the identity relation. Evidently, for any trivially ordered semigroup the natural partial order is two-sided compatible. By [4], a semigroup S is trivially ordered if and only if $a^2b = ab = bc$ $(a, b, c \in S)$ implies $ab = b$ (equivalently, $ab^2 = ab = ca$ implies $ab = a$). Examples of trivially ordered semigroups are:

1. Weakly cancellative semigroups, i.e., $xa = xb$ and $ax = bx$ $(a, b, x, y \in S)$ together imply $a = b$; in particular, left or right cancellative semigroups, hence every group ([20]). Furthermore, generalized Rees matrix semigroups $S = \mathcal{M}(I, T, \Lambda, P)$ over a left or right cancellative semigroup T (S is not left or right cancellative, in general; see Appendix (G) in part II).

2. Left or right simple semigroups; in particular, left or right groups, hence every left or right zero semigroup ([20]).

3. Left or right stratified semigroups, i.e., $a \in abS$ resp. $a \in Sba$ for all $a, b \in S$; equivalently, S is simple and contains a minimal left resp.

right ideal ([20]); in particular,

4. Completely simple semigroups (see [27], II.4.2), hence rectangular groups and rectangular bands.

5. Every Rees matrix semigroup over any of the semigroups given in 1. to 4. above ([5]).

Other classes of trivially ordered semigroups are specified in: 2.2; 2.3; 2.4; 2.5; 3.11, Remark (v); 3.12, Remark (ii); 4.3, Remarks (iii) and (iv); 4.8, Remark; 4.9; 5.1, Cor.; 5.2; 5.3; 7.1; 7.12; 7.29.

If a semigroup S contains a zero then $0 <_S a$ for every $a \in S$, $a \neq 0$; hence S is not trivially ordered ($|S| > 1$). In case that $\leq_S$ on $S\backslash 0$ is trivial, $\leq_S$ is two-sided compatible on S. For example, every group with zero or every completely 0-simple semigroup or every nil-semigroup has a nontrivial, two-sided compatible natural partial order. Generally, this holds adjoining a zero to a trivially ordered semigroup.

By contrast, adjoining an identity to a trivially ordered semigroup S does not always yield a compatible natural partial order on S^1: consider the completely simple semigroup S given in Example (4) of the Introduction. The following result gives a necessary and sufficient condition in a more general situation:

Result 2.1. *Let S be a semigroup such that $\leq_S$ is right compatible (in particular, $\leq_S$ is trivial). Then after adjoining an identity, the natural partial order on S^1 is right compatible if and only if $E(S)a \subseteq aS^1$ for every $a \in S$.*

Proof. Necessity holds by Result 1.8(ii). Sufficiency: Let $a \leq_{S^1} b$ ($a, b \in \in S^1$). If $a, b \in S$ then $ac \leq_S bc$ for every $c \in S$, and $a \cdot 1 \leq_S b \cdot 1$. If $a \leq_{S^1} 1$ then $a = e \in E(S)$ by [20]; thus for any $c \in S^1$, $ec = cx$ for some $x \in S^1$, that is, $e \cdot c \leq_{S^1} c = 1 \cdot c$. The case $1 <_{S^1} a$ (with $a \in S^1$) is impossible by the Remark following Result 1.14. $\Diamond$

Remark. It follows, in particular, that for semigroups S without idempotents and with right compatible natural partial order, $\leq_{S^1}$ is also right compatible.

We can add three further classes of semigroups with trivial natural partial order.

Result 2.2. *Let S be a semigroup such that no element of S has a right (left) identity. Then $\leq_S$ is the identity relation.*

Proof. If there are $a, b \in S$ with $a <_S b$ then $xa = a = ay$ for some $x, y \in S$. $\Diamond$

Remark. (i) Examples of such semigroups are given by $S = (\mathbb{N}, +)$ where $0 \notin \mathbb{N}$; or Rees matrix semigroups $S = \mathcal{M}(I, T, \Lambda; P)$ over a left (right) cancellative monoid T, where no entry of P is invertible; for instance, $T = (\mathbb{N}, \cdot)$ and $1 \notin P$ – see Appendix (G) in part II.

(ii) By Result 2.2, any element of a semigroup S which has no right or left identity is maximal in $(S, \leq_S)$. The converse does not hold, as any group shows.

Result 2.3. *Let S be a semigroup such that $aS \cap bS = \phi$ or $Sa \cap Sb = \phi$ for any $a, b \in S, a \neq b$. Then $\leq_S$ is the identity relation.*

Proof. Assume that there are $a, b \in S$ such that $a <_S b$. Then $xa = = a = xb$ and $ay = a = by$ for some $x, y \in S$, hence $Sa \cap Sb \neq \phi$ and $aS \cap bS \neq \phi$: contradiction. $\Diamond$

Remark. (i) Examples of semigroups satisfying the condition in Result 2.3 are given by rectangular bands S: assume that there are $e, f \in S$ such that $e \neq f, eS \cap fS \neq \phi$ and $Se \cap Sf \neq \phi$; then $ex = fy, ze = wf$ for some $x, y, w, z \in S$; hence by [13], IV.3.2, $e = exe = fy \cdot e = fe = = f \cdot ze = f \cdot wf = f$: contradiction.

(ii) In particular, let S be a nowhere right (left) reversible semigroup: $aS \cap bS = \phi(Sa \cap Sb = \phi)$ for any $a \neq b$ in S; then $\leq_S$ is the identity relation. Note that the intersection of any two principal right (left) ideals of S is empty and any principal right (left) ideal of S has a unique generator. Hence the partially ordered set of all principal right (left) ideals of S forms an antichain. For example, any left (right) zero semigroup satisfies this condition.

Result 2.4. *Let S be a monoid such that $a \leq_S b$ implies that $a = bu$ or $a = ub$ for some invertible $u \in S$. Then $\leq_S$ is the identity relation.*

Proof. Let $a \leq_S b$ $(a, b \in S)$; then $a = bu$ or $a = ub$ for some $u \in S$ such that $uu' = u'u = 1_S$ $(u' \in S)$. Thus either $b = au'$, whence $a\mathcal{R}b$, or $b = u'a$ whence $a\mathcal{L}b$. In both cases it follows by [20], that $a = b$. $\Diamond$

As examples for monoids having this property we mention the following: a monoid S is called a *right cone* if (i) for any $a, b \in S$ there exists $c \in S$ such that $a = bc$ or $b = ac$, and (ii) $ab = ac$ $(a, b, c \in S)$ implies $b = cu$ for some invertible $u \in S$ ([2]). Note that the principal right ideals of S form a chain.

Result 2.5. *Let S be a right cone; then $\leq_S$ is the identity relation.*

Proof. Let $a \leq_S b$ $(a, b \in S)$, i.e., $a = xb = by$, $xa = a$, for some $x, y \in S$. Thus $xa = xb$, so that $a = bu$ for some invertible $u \in S$. Hence the statement follows from Result 2.4. $\Diamond$

The other extreme for a partially ordered set is the *total* order. By [20], a semigroup S is totally ordered with respect to $\leq_S$ if and only if either $S = E(S)$, where $E(S)$ is a chain, or $S = E(S) \cup \{a\}$, where $E(S)$ is a chain with greatest element, $a \notin E(S)$ and $ea = ae = e$ for every $e \in E(S)$.

Result 2.6. *Let S be a semigroup such that $\leq_S$ is a total order. Then $\leq_S$ is two-sided compatible.*

Proof. First, let $S = E(S)$ where $E(S)$ is a chain. Then for $e, f \in E(S)$, $e \leq_S f$ or $f \leq_S e$, that is, $e = ef = fe$ or $f = fe = ef$. Thus $ef = fe$ for all $e, f \in E(S)$, and $S = E(S)$ is commutative. It follows that also $S = E(S) \cup \{a\}$ is commutative. Hence $\leq_S$ is two-sided compatible in both cases. $\Diamond$

3. Some particular classes

In this section we specify several classes of semigroups, for which the natural partial order is right or even two-sided compatible. First, we obtain immediately from Result 1.1:

Result 3.1. *Let S be a semigroup such that $aSb \subseteq abS^1$ for all $a, b \in S$. Then $\leq_S$ is right compatible.*

Remark. The converse of Result 3.1 does not hold: consider the completely 0-simple semigroup $S = \mathcal{M}^\circ(I, G, \Lambda; P)$ with $p_{\lambda j} = 0$ for some $\lambda \in \Lambda$, $j \in I$. Then for $a = (i, g, \lambda), b = (j, h, \mu) \in S$ we have $ab = 0$ and thus $abS^1 = \{0\}$. But since P is regular, there exist $k \in I$, $\nu \in \Lambda$ such that $p_{\lambda k} \neq 0$, $p_{\nu j} \neq 0$. Hence for $c = (k, g, \nu) \in S$ we have $acb \neq 0$, that is, $aSb \neq \{0\}$. However, $\leq_S$ is (right) compatible since on $S \backslash 0$, $\leq_S$ is the identity relation (see Sec. 2).

The following results are easy consequences of Result 3.1.

Result 3.2. *Let S be a semigroup such that $Sa \subseteq aS (\subseteq aS^1)$ for any $a \in S$ (i.e., S is right full). Then $\leq_S$ is right compatible.*

Result 3.3. *Let S be a semigroup such that $aS = Sa$ for every $a \in S$ (i.e., S is centric). Then $\leq_S$ is two-sided compatible.*

Corollary. *Let S be a commutative semigroup. Then $\leq_S$ is two-sided compatible.*

Remark. (i) The converses of the Results 3.2, 3.3, 3.4 and the Corollary do not hold: consider any left zero semigroup S. Then $\leq_S$ is the identity relation (see Sec. 2), hence (right) compatible, but $Sa = S \nsubseteq \{a\} = aS$ (and S is not commutative).

(ii) A large class of semigroups S satisfying $Sa \subseteq aS$ for every $a \in S$, is that of semilattices Y of right simple semigroups S_α, $\alpha \in Y$ ([25], II.4.9). Note that every right simple semigroup is trivially ordered (see Sec. 2). But if for some $\alpha <_Y \beta$ in Y, $E(S_\alpha) \neq \phi$ then for $a := eb$ with $e \in E(S_\alpha)$, $b \in S_\beta$, we have $a = eb = by$ for some $y \in S$, that is, $a <_S b$ (since $a \in S_\alpha, b \in S_\beta$). Hence $\leq_S$ is not the identity relation, in general.

Another class of semigroups S satisfying $Sa \subseteq aS^1$ for every $a \in S$ is that of *left negatively ordered* semigroups: $ax \leq_S x$ for all $a, x \in S$ (see [6], [7]). For instance, a semigroup S is left negatively ordered if $Sx \subseteq xE(S^1)$ for any $x \in S$; the converse holds for semigroups S such that $a \leq_S b$ implies $a = be$ for some $e \in E(S^1)$. Further examples are given in [6].

Result 3.4 ([6])**.** *Let S be a left negatively ordered semigroup. Then $\leq_S$ is right compatible.*

Proof. Let $a, x \in S$; then $xa \leq_S a$. Hence $xa = ay$ for some $y \in S^1$, that is, $Sa \subseteq aS^1$. Thus S satisfies the condition in Result 3.2. $\Diamond$

Corollary. *Let S be a negatively ordered semigroup (i.e., S is left and right negatively ordered). Then $\leq_S$ is two-sided compatible.*

Remark. (i) By [6], a semigroup S is negatively ordered if and only if S is an inflation of a semilattice. Note that such a semigroup is commutative.

(ii) A semigroup S is negatively ordered if $Sx \subseteq xE(S^1)$ and $xS \subseteq$ $\subseteq E(S^1)x$ for any $x \in S$ (see above). It follows that a semigroup satisfying these conditions is commutative (by (i)).

Result 3.5. *Let S be a semigroup such that $a \in aSb$ ($a \in bSa$) for any $a, b \in S$. Then $\leq_S$ is the identity relation.*

Proof. If $a \leq_S b$ then $a = by = ay$ for some $y \in S^1$. Since $b = bsa$ for some $s \in S$, it follows that $b = bs \cdot a = bs \cdot ay = bsa \cdot y = by = a$. $\Diamond$

Remark. By [17], Th. 1.3.10, a semigroup S satisfies the condition in Result 3.5 if and only if S is a left (right) group.

Result 3.6. *Let S be a right commutative semigroup (i.e., $axy = ayx$ for all $a, x, y \in S$). Then $\leq_S$ is right compatible.*

Proof. S satisfies the condition in Result 3.1. $\Diamond$

Remark. Examples of right commutative semigroups are given by left abelian groups (by [22], Th. 10.5, these are precisely the simple right commutative semigroups). More generally, inflations of a semigroup $T = N \times G$, where N is a left normal band and G is an abelian group, are right commutative. Note that by [26], V.5.6(8), a semigroup S is right commutative and S^2 is a band if and only if S is an inflation of a left normal band B (i.e., $efg = egf$ for all $e, f, g \in B$). For further examples see [26], IV.6 (dual).

Corollary. *Let S be a right and left commutative semigroup. Then $\leq_S$ is two-sided compatible.*

Remark. (i) Examples of semigroups which are right and left commutative: commutative semigroups; semigroups S containing an element $s_o \in S$ such that the principal ideal of S generated by $s_o \in S$ is a commutative subsemigroup of S, endowed with the new operation: $a * b = as_0b$ – note that $(S, *)$ is commutative if and only if $as_ob = bs_oa$ for any $a, b \in S$. This is satisfied, for instance, if $s_o \in S$ is also regular, since by (iv) below, $a * x * b = b * x * a$, i.e., $as_oxs_ob = bs_oxs_oa$ for any $a, b, x \in S$, so that for any inverse $x \in S$ of $s_o \in S$ the above condition is satisfied.

(ii) Every regular, resp. weakly cancellative, semigroup S, that is right and left commutative, is commutative: in the first case, S is externally commutative (see (iv) below), hence S is a strong semilattice of abelian groups (by [22], Th. 11.4); in the second, $a \cdot xy = a \cdot yx$ and $xy \cdot a = yx \cdot a$ imply $xy = yx$ for all $x, y \in S$.

(iii) There are right and left commutative semigroups which are *not* commutative: let X, Y be disjoint sets, $0 \in Y$, and $f : X \times X \to Y$ a function such that $f(x_1, x_2) \neq f(x_2, x_1)$ for some $x_1, x_2 \in X$; on $S = X \cup Y$ define a multiplication by $a * b = f(a, b)$ if $a, b \in X$, and $a * b = 0$ otherwise. Then by [26], V.3.8(2), $(S, *)$ is a semigroup such that $S^3 = \{0\}$. Hence $r * s * t = 0 = r * t * s, s * t * r = 0 = t * s * r$ for any $r, s, t \in S$, but S is not commutative since $x_1 * x_2 \neq x_2 * x_1$. Note that $0 \in S$ is the zero of S (since $0 \notin X$) and that on $S \setminus \{0\}, \leq_S$ is the identity relation (since $0 \neq a <_S b$ implies $a = x * b, x * a = a$, whence $a = x * a = x * x * b = 0$: contradiction). See also the semigroup given in [22], p. 27.

(iv) Let S be a right and left commutative semigroup. Then S is externally commutative (i.e., $axb = bxa$ for any $a, b, x \in S$): $axb = = a \cdot bx = ab \cdot x = b \cdot ax = bxa$. In particular, $x^n y = yx^n$ for any $x, y \in S$, $n \geq 2$ (put $a = x^{n-1}$). Furthermore, the idempotents of S are central (put $x = e \in E(S)$). For S regular see (ii) above. Concerning externally commutative semigroups in general, see Result 4.2 below.

Result 3.7. *Let S be an $\mathcal{R}$-commutative semigroup (i.e., $ab \in baS^1$ for all $a, b \in S$). Then $\leq_S$ is right compatible.*

Proof. S satisfies the condition in Result 3.2. $\diamondsuit$

Remark. $\mathcal{R}$-commutative semigroups S are also called right c-semigroups. By [22], Th. 5.2, a semigroup S is $\mathcal{R}$-commutative if and only if Green's relation $\mathcal{R}$ is a commutative congruence on S. Furthermore, by [22], Th. 5.3, S is a semilattice of archimedean semigroups. (See also [24].)

Result 3.8. *Let S be an $\mathcal{H}$-commutative semigroup (i.e., $ab \in bSa$ for all $a, b \in S$). Then $\leq_S$ is two-sided compatible.*

Proof. S satisfies the condition in Result 3.2 and its dual: $aS \subseteq Sa$ for any $a \in S$.

Remark. By [22], Th. 5.1 and Th. 5.2, a semigroup S is $\mathcal{H}$-commutative if and only if S is both $\mathcal{R}$- and $\mathcal{L}$-commutative (equivalently, Green's relation $\mathcal{H}$ is a commutative congruence on S).

Result 3.9. *Let S be a semigroup satisfying the identity $aba = ba$. Then $\leq_S$ is right compatible.*

Proof. S satisfies the condition in Result 3.7. $\diamondsuit$

Remark. Examples are given by inflations S of a right regular band B; if $S \neq B$, then $\leq_S$ is not trivial (see Appendix (C) in part II).

Several of the next results depend on the following (see Result 1.18)

Result 3.10. *Let S be a semigroup such that for all $a, b \in S$ there exists $n > 0$ such that $a^n b \in bS^1$. Then $\leq_S$ is right compatible.*

Proof. Let $a <_S b$, i.e., $a = xb = by$, $xa = a = ay$ $(x, y \in S)$. Hence for any $k > 0$, $a = by^k$. If $c \in S$, then $ac = x \cdot bc = by^n c = b \cdot y^n c = bc \cdot z$ for some $n > 0$, $z \in S^1$. Since $x \cdot ac = ac$ it follows that $ac \leq_S bc$. $\diamondsuit$

Result 3.11. *Let S be a right archimedean semigroup (i.e. for all $a, b \in$ $\in S$ there exists $n > 0$ such that $a^n \in bS$). Then $\leq_S$ is right compatible.*

Proof. S satisfies the condition in Result 3.10. $\diamondsuit$

Remark. (i) A semigroup S is right archimedean if and only if for all $a, b \in S$ there exists $n > 0$ such that $a^n \in bS^1$ (if $a^n = b$ then $a^{n+1} = ba \in bS$). Examples of right archimedean semigroups, which are not right simple, are provided by nil-extensions S of a right simple semigroup T, in particular, proper inflations of a right simple semigroup T (if $a, b \in S$ then $a^m, b^n \in T$ for some $m, n > 0$, hence $a^m = b \cdot b^{n-1}x$

($= bx$ if $n = 1$) for some $x \in T$; note that T is a non-trivial right ideal of S).

(ii) Every right (left) archimedean semigroup S contains at least one maximal element: Assume that there is no maximal element in S. Let I be any right ideal of S, $c \in I$ and $a \in S$. Then there exists $b \in S$ with a $<_S b$, hence $xa = a$ for some $x \in S$ and $x^k a = a$ for any $k > 0$. Since $x^n = cs$ for some $n > 0$, $s \in S$, we get $a = x^n a = csa \in I$. Therefore $I = S$ and S is right simple. It follows by Example 2 of Sec. 2, that $\leq_S$ is the identity relation, a contradiction. In case that every element of S is maximal we have

(iii) A right archimedean semigroup S with $E(S) \neq \phi$ is trivially ordered if and only if S is a right group:

Sufficiency holds by Example 2 in Sec. 2.

Necessity: Let $a \in S$, $e \in E(S)$; then $e = e^n = ax$ for some $x \in S$. Hence S is E-inversive, and thus by Result 5.1 below, S is completely simple: $S = \mathcal{M}(I, G, \Lambda; P)$. Since S is right archimedean it follows that $|I| = 1$. Normalizing P at $i = 1$, i.e., $p_{\lambda 1} = 1_G$ for any $\lambda \in \Lambda$, and giving Λ the multiplication of a right zero semigroup, it is easily seen that S is isomorphic with the right group $T = G \times \Lambda$.

(iv) By [17], Th. 1.4.6 (dual), a semigroup is right archimedean with $E(S) \neq \phi$ if and only if S is a nil-extension of a right group. (Note that in the cited result the statement in the parentheses of item (iv) does not hold, as every left group shows.)

(v) A semigroup S is a right group if and only if S is right archimedean and every element of S has a left identity: Necessity holds by [3], Th. 1.27 and Lemma 1.26. Sufficiency: By the proof of (ii) above (starting with $xa = a$), S is right simple. Furthermore $E(S) \neq \phi$: let $a \in S$; then $a = a'a$ for some $a' \in S$, hence $a = (a')^k a$ for every $k > 0$; now $(a')^n = ax$ for some $n > 0$, $x \in S$, so that $a = (a')^n a = ax \cdot a$ and $ax \in E(S)$. It follows by [3], Th. 1.27, that S is a right group. In particular, a semigroup S is a group if and only if S is a right (left) archimedean monoid.

Result 3.12. *Let S be a t-archimedean semigroup (i.e., S is right and left archimedean). Then $\leq_S$ is two-sided compatible.*

Remark. (i) A semigroup S is t-archimedean if and only if for all $a, b \in S$ there exists $n > 0$ such that $a^n \in bS \cap Sb$. For example, any group and every nil-semigroup S (i.e., for any $a \in S$ there is $n > 0$ with $a^n = 0$, the zero of S) is t-archimedean.

(ii) By [4],5.3, for any t-archimedean semigroup S the following hold:

(1) $|E(S)| \leq 1$;

(2) if $E(S) = \phi$, then $\leq_S$ is the identity relation;

(3) if $|E(S)| = 1$, then S is trivially ordered if and only if S is a group;

(4) if S has a zero, then S is a nil-semigroup and $S\backslash\{0\}$ is trivially ordered.

(iii) A semigroup S is a group if and only if S is t-archimedean and every element of S has a right and a left identity – this follows from Remark (v) above and its dual.

Result 3.13. *Let S be a semigroup such that for all $a, b \in S$ there exists $k > 0$ with $a^k \in b^{k+1}S$. Then $\leq_S$ is right compatible.*

Proof. S satisfies the condition in Result 3.11. $\Diamond$

Remark. By [17], Th. 1.4.4, a semigroup S satisfies the condition in Result 3.13 if and only if S is a nil-extension of a right simple semigroup (note that $E(S) = \phi$ is possible). As a particular case we get

Result 3.14. *Let S be a semigroup such that for all $a, b \in S$ there exists $k > 0$ with $a^k = b^k a^k$. Then $\leq_S$ is right compatible.*

Proof. S satisfies the condition in Result 3.11. $\Diamond$

Remark. By [17], Th. 1.4.9, a semigroup S satisfies the condition in Result 3.14 if and only if S is a nil-extension of a periodic right group.

Result 3.15. *Let S be a semigroup such that for all $a, b \in S$ there exists $k > 0$ with $a^k \in b^k S b^k$. Then $\leq_S$ is two-sided compatible.*

Proof. S satisfies the condition in Result 3.11 and its dual: $a^m \in Sb$ for some $m > 0$. $\Diamond$

Remark. By [17], Th. 1.4.7, a semigroup S satisfies the condition in Result 3.15 if and only if S is a nil-extension of a group. More generally, for any nilextension S of an inverse semigroup, $\leq_S$ is two-sided compatible by [36].

Result 3.16. *Let S be a powerjoined semigroup (i.e., for all $a, b \in S$ there exist $m, n > 0$ such that $a^m = b^n$). Then $\leq_S$ is two-sided compatible.*

Proof. S satisfies the condition in Result 3.11 and its dual (if $n = 1$ then $a^{m+1} = ba \in bS$). $\Diamond$

Examples of powerjoined semigroups:

1. Finite groups; note that $\leq_S$ is the identity relation (see Sec. 2). More generally: periodic groups.

2. Proper inflations $S = \bigcup_{g \in G} T_g$ of a group G of order n: if $a, b \in S$, $a \in T_g$ and $b \in T_h$ say, then $a^2 = g^2$, $b^2 = h^2 \in G$, hence $a^{2n} = 1_G = b^{2n}$. Note that $\leq_S$ is not trivial and that $a \leq_S b$ implies $ac = bc$ and $ca = cb$ for any $c \in S$ (see Appendix (C) in part II).

3. Nil-semigroups: for $a, b \in S$ exist $m, n > 0$ such that $a^m = 0 = b^n$. Note that $\leq_S$ is the identity relation on $S \backslash 0$.

4. Monogenic semigroups: if $S = \langle c \rangle$ and $a = c^m$, $b = c^n$, then $a^n = b^m$. Note that $\leq_S$ is the identity relation if S is infinite. If S is finite, then $\leq_S$ is trivial if and only if S is a cyclic group (necessity: $c^n = e \in E(S)$ for some $n > 0$, thus $c^{n+1} = e \cdot c \leq_S c$ and $c^{n+1} = c$; since $c^n = e \in K_c$, the group part of S, $c = c^{n+1} \in K_c$ and $S = \langle c \rangle = K_c$, i.e., S is a cyclic group; for sufficiency see Sec. 2).

Remark. Let S be a powerjoined semigroup; then the following hold:

(i) S is t-archimedean (see the Remarks following Result 3.12).

(ii) $|E(S)| \leq 1$; if $|E(S)| = 1$ then S is periodic.

(iii) If $a \in S$ has a right (left) identity then $a^n = a$ for some $n > 1$: $a = ax$ for some $x \in S$ implies $a = ax^k$ for any $k > 0$; since $x^m = a^l$ for some $m, l > 0$, it follows that $a = a \cdot a^l = a^{l+1}$. In particular, if $a \in S$ is not maximal in $(S, \leq_S)$ then $a \in S$ is regular. Therefore, if S has a non-maximal element then by (ii), $|E(S)| = 1$ and S is periodic (see Examples 2, 3 above). If every element of S is maximal then $\leq_S$ is the identity relation; more precisely, if $E(S) \neq \phi$ then S is a periodic group by Remark (ii)(2) following Result 3.12; for $E(S) = \phi$ see Example 4 following Result 3.16.

(iv) S has at least one maximal element, by (i) and Remark (ii) following Result 3.11. Note that a maximal element of S is not necessarily regular (see Examples 2.,3.,4. above).

(v) A semigroup S is a periodic group if and only if S is powerjoined and every element of S has a right and a left identity, by (i) and Remark (iii) following Result 3.12.

(vi) If $a <_S b$ then $a = b^n$ for some $n > 1$: $a = xb = xa$ for some $x \in S$ implies that $a = x^k b$ for any $k > 0$; since $x^m = b^l$ for some $m, l > 0$ we obtain $a = b^l \cdot b = b^{l+1}$. In particular, if $a \in S$ is not minimal in $(S, \leq_S)$ then $a^n <_S a$ for some $n > 1$.

A somewhat larger class is that of *power centralized* semigroups S: for any $a, b \in S$ there exists $n > 0$ such that $a^n b = ba^n$. This class was introduced by P. Moravec, Power centralized semigroups, *Semigroup*

Forum **73** (2006), 143–155. There, the powerjoined semigroups and the periodic semigroups with central idempotents are mentioned as examples. Also, any externally commutative semigroup belongs to this class – see Remark (iv) preceding Result 3.7.

Result 3.17. *Let S be a power centralized semigroup. Then $\leq_S$ is two-sided compatible.*

Proof. S satisfies the condition in Result 3.10 and its dual: $ab^m \in S^1a$ for some $m > 0$. $\Diamond$

Result 3.18. *Let S be a semigroup such that for all $a, b \in S$ there exists $n > 0$ with $ab \in b^nS^1$. Then $\leq_S$ is right compatible.*

Proof. S satisfies the condition in Result 3.10. $\Diamond$

Result 3.19. *Let S be a left quasi-commutative semigroup (i.e., for all $a, b \in S$ there exists $n > 0$ such that $ab = b^na$. Then $\leq_S$ is two-sided compatible.*

Proof. First, $\leq_S$ is right compatible, since S satisfies the condition in Result 3.18. Also, $ca = cb \cdot y = cx \cdot b = x^nc \cdot b = z \cdot cb$ for some $n > 0$, $z \in S$, $ca \cdot y = ca$; thus $ca \leq_S cb$. $\Diamond$

Result 3.20. *Let S be a σ-reflexive semigroup (i.e., for all $a, b \in S$ there exists $n > 0$ such that $ab = (ba)^n$). Then $\leq_S$ is two-sided compatible.*

Proof. S satisfies the condition in Result 3.7 and its dual: $ab \in S^1ba$ $\forall a, b \in S$. $\Diamond$

Remark. For a detailed treatment of several classes of semigroups mentioned above, see [17] and [22]. In particular, for a semigroup S the concepts: σ-reflexive, left quasi-commutative, and right quasicommutative are equivalent ([22], Th. 8.2).

As a further class of semigroups with right compatible natural partial order we mention the following: for a semigroup S and $a, b \in S$ the left residual $a \cdot\!. b \in S$ of a by b with respect to $\leq_S$ (if it exists) is defined by the condition that $x \leq_S a \cdot\!. b$ if and only if $xb \leq_S a$ (see [7]). If $a \cdot\!. b$ exists for all $a, b \in S$, S is called a semigroup *with left residuals (S is left residuated).*

Result 3.21. *Let S be a semigroup such that the left residuals $ab \cdot\!. b$ exist for all $a, b \in S$. Then $\leq_S$ is right compatible.*

Proof. Let $a \leq_S b$ $(a, b \in S)$ and $c \in S$. Since $bc \leq_S bc$, we have $b \leq_S bc \cdot\!. c$. Hence $a \leq_S b \leq_S bc \cdot\!. c$, so that by definition, $ac \leq_S bc$. $\Diamond$

Result 3.22. *Let S be a residuated semigroup (i.e., S is left and right* *residuated). Then $\leq_S$ is two-sided compatible.*

Finally, we consider the class of $\mathcal{U}$-*semigroups* S defined by the condition that the subsemigroup generated by any two subsemigroups of S is equal to their set theoretic union (see [26]). They appear in the study of the lattice of all subsemigroups of a semigroup – including the empty set – under set inclusion. For such semigroups, $\leq_S$ is "almost" two-sided compatible:

Result 3.23. *Let S be an $\mathcal{U}$-semigroup. If $a <_S b$ and $c \in S$ then the following hold:*

(i) *$ac \leq_S bc$ whenever $ac \neq a$, and* (ii) *$ca \leq_S cb$ whenever $ca \neq a$.*

Proof. (i) We have $a = xb = by$, $xa = a = ay$, hence $a = by^k$ for any $k > 0$ and $ac = x \cdot bc = b \cdot yc$, $x \cdot ac = ac$ $(x, y \in S)$. By [26], V.2.17(2), there exists $n > 0$ such that $yc = y^n$ or $yc = c^n$. Hence either: $ac = by^n = a$, or: $ac = bc^n = bc \cdot z$ $(z \in S^1)$ and $ac \leq_S bc$.

(ii) is proved in a similar way. $\Diamond$

4. (E-)Medial semigroups

A semigroup S is called *medial* if it satisfies the identity $axyb = = ayxb$ (see [22], [26]). Note that if $E(S) \neq \phi$ then $E(S)$ is a normal band. Examples:

(1) commutative semigroups;

(2) right (left) commutative semigroups (see Result 3.6);

(3) normal bands B, i.e., $efgh = egfh$ for all $e, f, g, h \in B$ (see [27], IV.1.2); more generally, inflations of normal bands (note that by [26], V.5.6(7), a semigroup S is an inflation of a normal band if and only if S is medial and S^2 is a band).

(4) rectangular abelian groups, i.e., $S = R \times G$, where R is a rectangular band ($efg = eg$ $\forall e, f, g \in R$ – see [13], IV.3.2) and G is an abelian group (by [22], Th. 9.8, these are precisely the simple medial semigroups); more generally,

(5) $S = N \times T$ where N is a normal band and T is any commutative semigroup (note that S is commutative if and only if N is a semilattice).

(6) $S = C \times T$ where C is a semilattice and T is a rectangular abelian group (note that S is commutative if and only if T is an abelian group – a commutative rectangular band has only one element: see [13], IV.3.2).

(7) Generalized Rees matrix semigroups $S = \mathcal{M}(I, T, \Lambda; P)$ over a commutative semigroup T with $P = (p_{\lambda i})$ such that for any $\lambda \in \Lambda$, $p_{\lambda i} = p_{\lambda j} \forall i, j \in I$ ($i \in I, p_{\mu i} = p_{\nu i} \forall \mu, \nu \in \Lambda$). Note that S is not commutative if $|I| > 1$ or $|\Lambda| > 1$ – see Appendix (G) in part II.

(8) $S = \langle Y, S_\alpha; \varphi_{\alpha,\beta}\rangle$, i.e., strong semilattices of medial semigroups $S_\alpha (\alpha \in Y)$, are again medial, because for all $a \in S_\alpha$, $b \in S_\beta$, $x \in S_\gamma$, $y \in S_\delta$, say, and $\varepsilon = \alpha\beta\gamma\delta$: $axyb = (a\varphi_{\alpha,\varepsilon})(x\varphi_{\gamma,\varepsilon})(y\varphi_{\delta,\varepsilon})(b\varphi_{\beta,\varepsilon}) = ayxb$ since the product is formed in S_ε.

Remark. (i) In general, for a medial semigroup $S, \leq_S$ is not right or left compatible: consider the generalized Rees matrix semigroup $S = \mathcal{M}(I, T, \Lambda; P)$ where $I = \Lambda = \{1, 2\}, T$ is the semilattice $0 < e < f$, and P is given by $p_{11} = e, p_{12} = f, p_{21} = p_{22} = 0$. Then $(1, e, 1) <_S (1, f, 1)$, but $(1, e, 1)(2, f, 2) = (1, e, 2) \nleq_S (1, f, 2) = (1, f, 1)(2, f, 2)$, since $(1, f, 2)(k, x, \lambda) = (1, 0, \lambda) \neq (1, e, \lambda)$ for any $(k, x, \lambda) \in S$.

(If $p_{11} = e, p_{21} = f, p_{12} = p_{22} = 0$, then $\leq_S$ is not left compatible.) Note that S is medial:

$$(i, a, \lambda)(k, x, \kappa)(l, y, \nu)(j, b, \mu) = (i, ap_{\lambda k}xp_{\kappa l}yp_{\nu j}b, \mu),$$
$$(i, a, \lambda)(l, y, \nu)(k, x, \kappa)(j, b, \mu) = (i, ap_{\lambda l}yp_{\nu k}xp_{\kappa j}b, \mu).$$

If one of $p_{\lambda k}, p_{\kappa l}, p_{\nu j}$ is 0 then also one of $p_{\lambda l}, p_{\nu k}, p_{\kappa j}$ is 0, and equality holds. If none of $p_{\lambda k}, p_{\kappa l}, p_{\nu j}$ is 0 then $\lambda = \kappa = \nu = 1$, so that p_{1k}, p_{1l}, p_{1j} are equal to e or f. If one of these is e then both products are equal to $(i, axybe, \mu)$. If none of these is e then all are equal to f and we obtain $(i, axybf, \mu)$ – see Appendix (G) in part II.

(ii) By [26], IV.3.1, a medial semigroup S is a semilattice of archimedean semigroups; see also Result 4.3.

Result 4.1. *Let S be a medial semigroup. Then $\leq_S$ is right (left) compatible with multiplication by any $c \in S$, which has a right (left) identity.*

Remark. (i) For an inflation S of a normal band B (see Example (3) above) an element $a \in S$, $a \notin B$, has no right and no left identity. In spite of that, $\leq_S$ is two-sided compatible, since by Result 1.7, Cor., for any normal band B, $\leq_B$ is two-sided compatible (see Appendix (C) in part II).

(ii) For semigroups $S = N \times T$ given in Example (4) above with T a commutative monoid, every element has a right and a left identity. It follows that $\leq_S$ is two-sided compatible. Note that $\leq_S$ is not trivial if $\leq_N$ or $\leq_T$ is not trivial. More generally,

(iii) Let $S = \langle Y, S_\alpha; \varphi_{\alpha,\beta}\rangle$ be such that $S_\alpha = N_\alpha \times T_\alpha$ where N_α is a normal band and T_α is a commutative monoid for any $\alpha \in Y$. Then S is a medial semigroup (see Example (6) above), in which every element has a left and a right identity. Hence $\leq_S$ is two-sided compatible – see Appendix (B) in part II.

As a particular class of medial semigroups we consider semigroups S satisfying $axb = bxa$ for all $a, b, x \in S$: by [22], Lemma 11.1, such semigroups S are medial. They are called *externally commutative*, also *completely symmetrical* (see [22]). For such semigroups, Result 4.1 can be sharpened obtaining as a particular case of Result 3.17:

Result 4.2. *Let S be an externally commutative semigroup (i.e., $axb =$ $= bxa$ for any $a, b, x \in S$). Then $\leq_S$ is two-sided compatible.*

Proof. For any $a, b \in S$, $a^2b = ba^2$. Hence the statement follows by Result 3.17. $\Diamond$

Remark. (i) For the particular case of a right and left commutative semigroup S see the Cor. of Result 3.6 and Remark (iv) following it. Note that in general, S is not commutative (see Remark (iii) preceding Result 3.7).

(ii) There are externally commutative semigroups which are not right nor left commutative. The following is an adaptation of the construction of $\mathcal{N}$-semigroups due to T. Tamura (see [25], II.7.3). Let $(S, +)$ be a commutative cancellative semigroup and $f : S \times S \to S$ be a function satisfying for any $a, b, c \in S$:

$$f(a,b) + f(a+b+f(a,b),c) = f(b,c) + f(a,b+c+f(b,c)) = \\ = f(b,a) + f(c,a+b+f(b,a)).$$

Define a new operation on S by: $a * b = a + b + f(a,b)$; then $(S, *)$ is an externally commutative semigroup (associativity holds by the first equality, external commutativity by the second). If in addition:

(1) $f(x,y) \neq f(y,x)$ for some $x, y \in S$ then $(S, *)$ is not commutative;

(2) $f(x,y) + f(a, x+y+f(x,y)) \neq f(y,x) + f(a, x+y+f(y,x))$ for some $a, x, y \in S$, then $(S, *)$ is not right commutative (hence also not commutative);

(3) $f(x,a) + f(y, x+a+f(x,a)) \neq f(y,a) + f(x, y+a+f(y,a))$ for some $a, x, y \in S$, then $(S, *)$ is not left commutative (hence again not commutative).

(iii) By contrast, if S is an externally commutative semigroup such that every $a \in S$ has a right (left) identity, then S is commutative: for any $a, b \in S$ and any right identity $a' \in S$ of $a \in S$, $ab = aa'a' \cdot b = = b \cdot a'a' \cdot a = b \cdot aa'a' = ba$. In particular, any externally commutative semigroup without maximal elements is commutative (see Appendix (F) in part II).

Result 4.3. *Let S be a medial semigroup which is archimedean (i.e., for any $a, b \in S$ there is $n > 0$ such that $a^n \in SbS$). Then $\leq_S$ is two-sided compatible.*

Proof. Let $a <_S b$ and $c \in S$. Then $a = xb = by$, $xa = a = ay$, for some $x, y \in S$. Hence $a = by^k$ for any $k > 0$. Since $y^n = sct$ for some $n > 0$, $s, t \in S$, we have

$$ac = x \cdot bc = by^n c = b(s \cdot ct)c = b(ct \cdot s)c = bc \cdot z \ (z \in S), x \cdot ac = ac;$$

that is, $ac \leq_S bc$. Similarly, $ca \leq_S cb$. $\Diamond$

Examples of medial archimedean semigroups:

(1) Rectangular abelian groups: S is medial by example (4) above; S is archimedean since S is (completely) simple. More generally, inflations $S = \bigcup_{\alpha \in T} T_\alpha$ of a rectangular abelian group T: for any $a, b \in S, b \in T_\beta$ say, $a^2 \in T$ and thus $a^2 = x\beta y = xby$ for some $x, y \in T$, since T is (completely) simple; S is medial since T is so. Note that $\leq_S$ is not trivial if $S \neq T$ (see Appendix (C) in part II).

(2) Generalized Rees matrix semigroups $S = \mathcal{M}(I, T, \Lambda; P)$ where T is a commutative, archimedean semigroup and P is such that for any $\lambda \in \Lambda, p_{\lambda i} = p_{\lambda j}\ \forall i, j \in I$ (for any $i \in I$, $p_{\mu i} = p_{\nu i} \forall \mu, \nu \in \Lambda$): indeed, if (i, a, λ), $(j, b, \mu) \in S$ then for $ap_{\lambda i}$, $bp_{\lambda j}p_{\mu i} \in T$ there are $n > 0$, $x \in T$, such that $(ap_{\lambda i})^n = bp_{\lambda j}p_{\mu i}x$, whence $(i, a, \lambda)^{n+1} = = (i, x, \lambda)(j, b, \mu)(i, a, \lambda)$; S is medial by example (7) above. Note that $\leq_S$ is non-trivial if $\leq_T$ is non-trivial (see Appendix (G) in part II); for example if T is a commutative nil-semigroup.

Remark. (i) Let S be a right commutative archimedean semigroup without idempotents. Then by [26], IV.5.1, $a \neq xa$ for all $a, x \in S$; hence $\leq_S$ is the identity relation, by Result 2.2.

(ii) Also, every medial weakly cancellative semigroup S is trivially ordered (see Sec. 2). By [26], III.4.7, these semigroups are exactly the subdirect products of a rectangular band and a commutative cancellative semigroup.

A further sufficient condition for compatibility of $\leq_S$ is given in

Result 4.4. *Let S be a medial semigroup. If $(S, \leq_S)$ does not contain maximal elements then $\leq_S$ is two-sided compatible.*

Proof. Every element of S has a right and a left identity (see the proof of Result 2.2). Thus the statement follows from Result 4.1 and its dual. $\Diamond$

Examples of semigroups satisfying both conditions in Result 4.4:

(i) Chains without greatest element (considered as particular semilattices).

(ii) $S = C \times T$ where C is a chain without greatest element and T is a rectangular abelian group (see example (6) above): if $(\alpha, a) \in S$ then $(\alpha, a) <_S (\beta, a)$ for any $\beta \in C$ with $\alpha <_C \beta$.

(iii) $S = N \times T$ where N is a normal band and T is a commutative semigroup without maximal elements (see example (5) above): if $(\alpha, a) \in \in S$ then $(\alpha, a) <_S (\alpha, b)$ for any $b \in T$ with $a <_T b$.

(iv) Strong semilattices Y of rectangular abelian groups $S_\alpha, S = = \langle Y, S_\alpha; \varphi_{\alpha,\beta}\rangle$, such that $(Y, \leq_Y)$ has no maximal element and each $\varphi_{\alpha,\beta}$ is surjective: S is medial by [26], IV. 3.10 (8) (see also [22], Th. 9.10); $(S, \leq_S)$ has no maximal elements, by Appendix (F) in part II.

(v) Iterated inflations $S = \bigcup_{n=0}^{\infty} S_n$ of a medial trivially ordered semigroup S_o (for example, S_o a rectangular abelian group) – see Appendix (D) in part II: at each step, every maximal element is inflated.

Next we will consider the larger class of *E-medial* semigroups S: $aefb = afeb$ for all $a, b \in S$, $e, f \in E(S)$. Note that if $E(S) \neq \phi$ then $E(S)$ is a normal band. Examples:

(1) Medial semigroups.

(2) Semigroups S with $E(S) = \phi$ ([4], [5]) or with commuting idempotents; in particular

(3) Inverse semigroups; more generally, inflations of inverse semigroups (note that these are not regular). An inverse semigroup S is medial if and only if S is commutative: if S is medial then for all $x, y \in S$,

$$xy = xx^{-1} \cdot xy \cdot y^{-1}y = xx^{-1} \cdot yx \cdot y^{-1}y = (xx^{-1} \cdot y)(x \cdot y^{-1}y) \leq_S y \cdot x$$

since $xx^{-1}y \leq_S y$, $xy^{-1}y \leq_S x$, and $\leq_S$ is two-sided compatible by 7.8; similarly $yx \leq_S xy$, hence $xy = yx$.

(4) Rectangular groups $S = R \times G$, where R is a rectangular band and G is a non-abelian group (note that S is not medial); more generally, inflations of such semigroups; in particular, non-abelian groups.

Remark. Let S be a semigroup such that $E(S)$ is an ideal of S; then S is E-medial if and only if $aefa = afea$ for any $a \in S$, $e, f \in E(S)$ (for S

a band, see [13], Ex. IV.13, or [27], IV.1.2): for the proof of sufficiency let $a, b \in S, e, f \in E(S)$; then

$$\begin{aligned} aefb &= aef \cdot aef \cdot b = (aefa) \cdot efb = (afea) \cdot efb \cdot efb = \\ &= a(f \cdot ea \cdot e \cdot f) \cdot befb = a(f \cdot e^2 a \cdot f) \cdot bfeb = \\ &= af(e \cdot af \cdot bf \cdot e)b = af(e \cdot bf \cdot af \cdot e)b = \\ &= af(eb \cdot f \cdot af \cdot eb) = af(eb \cdot af^2 \cdot eb) = \\ &= afeb \cdot afeb = afeb. \end{aligned}$$

Result 4.5. *Let S be an E-medial semigroup such that $a \leq_S b$ implies $a = bf$ for some $f \in E(S^1)$. Then $\leq_S$ is right compatible with multiplication by regular elements.*

There are several particular classes of medial resp. E-medial semigroups of special interest.

Result 4.6. *Let S be a semigroup satisfying the identity $axb = ab$. Then $\leq_S$ is two-sided compatible; more precisely, $a \leq_S b$ implies $ac = bc$ and $ca = cb$ for every $c \in S$.*

Remark. By [25], III.4.10 (4), a semigroup S satisfies $axb = ab$ for all $a, b, x \in S$ if and only if S is an inflation of a rectangular band B. For example, see the six-element semigroup in [13], Ex. IV.9. If $S \neq B$ then $\leq_S$ is not the identity relation (see Appendix (C) in part II). Generalized Rees-matrix semigroups satisfying the condition in Result 4.6 are given in Example (iii) following Result 8.6 (see Appendix (G) in part II).

More generally, we consider semigroups S such that $aeb = ab$ for all $a, b \in S$, $e \in E(S)$ ([33]). In case that $E(S) \neq \phi$, $E(S)$ is a rectangular band and S is E-medial. If the natural partial order has a particular form, we have

Result 4.7. *Let S be a semigroup such that $aeb = ab$ for all $a, b \in S$, $e \in E(S)$. If $a \leq_S b$ implies $a = bf$ for some $f \in E(S^1)$, then $\leq_S$ is right compatible; more precisely, $a \leq_S b$ implies $ac = bc$ for any $c \in S$.*

Remark. (i) By [25], IV. 3.12 (3), a semigroup S satisfies $aeb = ab$ for all $a, b \in S$, $e \in E(S)$, *and* S^2 is regular if and only if S is an inflation of a rectangular group T. For regular semigroups satisfying this condition, see Result 7.12.

(ii) Let $S = R \times G$ be a rectangular group, where R is a rectangular band and G is a nontrivial group. Then $\leq_S$ has the indicated form (since S is regular) and S satisfies $aeb = ab$ for all $a, b \in S$, $e \in E(S)$, but not $axb = ab$ for all $a, b, x \in S$.

(iii) Let S be a *rectangular semigroup*, i.e., if $a, b, x, y \in S$ are such that three of $ax, ay, bx,\ by \in S$ are equal then all four are equal. Such a semigroup satisfies $aeb = ab$ for all $a, b \in S$, $e \in E(S)$ (by [3], Ex. 3.2(7)), but in general not $axb = ab$ for all $a, b, x \in S$ (see (ii)).

(iv) A semigroup S is called *stationary on the left* if $ac = bc$ $(a, b, c \in S)$ implies $ax = bx$ for any $x \in S$. By [30], also [3], Ex. 3.2(9), such semigroups are rectangular, hence satisfy $aeb = ab$ for all $a, b \in S$, $e \in E(S)$. Their natural partial order has the desired property:

Result 4.8. *Let S be a semigroup, which is stationary on the left. Then $\leq_S$ is right compatible; more precisely, $a \leq_S b$ implies $ac = bc$ for every $c \in S$.*

Proof. If $a <_S b$ then $ay = a = by$ for some $y \in S$, hence $ac = bc$ for any $c \in S$. $\lozenge$

Remark. If S is a rectangular *monoid*, then $\leq_S$ is the identity relation: let $a \leq_S b$, i.e., $a = xb = by$, $xa = a = ay$ for some $x, y \in S$. Then $a \cdot 1_S = ay = by$, and three of $a1_S$, ay, $b1_S$, $by \in S$ are equal; therefore $a = b$. If $\leq_S$ has a particular form we have

Result 4.9. *Let S be a monoid such that $a \leq_S b$ implies $a = eb$ or $a = bf$ for some $e, f \in E(S)$. Then the following are equivalent:*

(i) *S satisfies $aeb = ab$ for any $a, b \in S$, $e \in E(S)$;*

(ii) *$E(S) = \{1_S\}$;*

(iii) *$\leq_S$ is the identity relation.*

Proof. (i) implies (ii): Let $e \in E(S)$; for $a = b = 1_S$ we obtain $e = 1_S$. (ii) implies (iii), by the particular form of $\leq_S$.

(iii) implies (i): $e = 1_S$ ($0 \notin S$ since $|S| > 1$) and (i) is trivial. $\lozenge$

Remark. Let S be a regular or groupbound monoid. Then $\leq_S$ has the above form (see the Introduction). Therefore, $\leq_S$ is the identity relation if and only if $E(S) = \{1_S\}$. In fact, in this case S is a group since S is E-inversive (see Result 5.3 below).

For another particular class of E-medial semigroups we have

Result 4.10. *Let S be a semigroup such that $axeb = aexb$ for all $a, b, x \in S$, $e \in E(S)$. Then $\leq_S$ is two-sided compatible with multiplication by regular elements.*

Proof. For any $c = cc'c \in S$,

$$ac = x \cdot bc = byc = b \cdot ycc' \cdot c = b \cdot cc'y \cdot c = bc \cdot z \ (z \in S), \quad xac = ac;$$

$$ca = cb \cdot y = cxb = c \cdot c'cx \cdot b = c \cdot xc'c \cdot b = w \cdot cb \ (w \in S), \quad ca \cdot y = ca;$$

therefore, $ac \leq_S bc$ and $ca \leq_S cb$. $\lozenge$

Remark. (i) Any semigroup with central idempotents satisfies the condition in Result 4.10 (see Result 1.19).

(ii) The class $\mathcal{V}$ of semigroups considered in Result 4.10 is properly contained in between the class of medial and that of E-medial semigroups: (1) any non-abelian group S is non-medial, but $S \in \mathcal{V}$; (2) any inverse monoid S containing a non-central idempotent is E-medial, but $S \notin \mathcal{V}$ (for $ea \neq ae$, also $aa^{-1}ea \neq aea^{-1}a$). Regular semigroups belonging to the class $\mathcal{V}$ are considered in Result 7.19 below.

With respect to the converse of Result 4.10 we have (for S a band, see [13], Ex. IV.12):

Result 4.11. *Let S be a semigroup such that $E(S)$ is a subsemigroup of S satisfying: if $a, b, x \in S$, $e \in E(S)$, then $axeb, aexb \in E_\alpha$ for some $\alpha \in Y$, where $E(S)$ is the semilattice Y of rectangular bands E_α ($\alpha \in Y$). If $\leq_S$ is two-sided compatible with multiplication by idempotents then $axeb = aexb$ for all $a, b, x \in S$, $e \in E(S)$.*

Proof. Note that $E(S)$ is an ideal of S. Let $a, b, x \in S$, $e \in E(S)$; then we have for some $\alpha \in Y$ that $axeb, aexb \in E_\alpha$, hence

$$axeb = axeb \cdot aexb \cdot axeb = (axeba) \cdot ex \cdot (baxeb) \leq_S aexb,$$

because $axeba = axeb \cdot a = a \cdot xeba$ with $axeb \in E(S)$, i.e., $axeba \leq_S a$, and also $baxeb \leq_S b$; since $exbaxeb$, $aex \in E(S)$, it follows that $axeba \cdot exbaxeb \leq_S a \cdot exbaxeb$, and $aex \cdot baxeb \leq_S aex \cdot b$; thus the inequality holds. Since by 7.1, Cor., $\leq_S$ is the identity relation on E_α, equality prevails. $\Diamond$

Remark. Any inflation $S = \bigcup_{\alpha \in B} T_\alpha$ of a rectangular band B satisfies the conditions in Result 4.11, since by 7.1, Cor., B is trivially ordered (see Appendix (C) in part II).

A generalization of medial semigroups in another direction is given by the class of *E-externally medial* semigroups S, i.e., $exyf = eyxf$ for any $e, f \in E(S)$, $x, y \in S$. If $E(S) \neq \phi$ then $E(S)$ forms a normal band. Examples:

(1) Medial semigroups.

(2) Semigroups S in which every idempotent is a left (resp. right) zero of S. In particular, (non commutative) semigroups with zero as unique idempotent; for example, semigroups with zero multiplication or idempotentfree semigroups with a zero adjoined.

(3) Generalized Rees matrix semigroups $S = \mathcal{M}(I, T, \Lambda; P)$, where T is a semigroup with zero as unique idempotent and P is arbitrary; hence

$E(S) = \{(i, 0, \lambda) \in S | i \in I, \lambda \in \Lambda\}$. Note that in general, S is not medial: let T be a semigroup as above which is also right 0-cancellative; consider $T^1, |I| > 1$, and P such that $1 \in T^1$ is not an entry of P and there are $p_{\lambda i}$, $p_{\lambda j} \in P$ which do not commute (hence $p_{\lambda i}, p_{\lambda j} \neq 0$) – then $(i, 1, \lambda) \notin E(S)$ and $(i, 1, \lambda)(i, 1, \lambda)(j, 1, \lambda)(i, 1, \lambda) \neq (i, 1, \lambda)(j, 1, \lambda)(i, 1, \lambda)(i, 1, \lambda)$.

(4) Semigroups S such that $E(S)$ is a right (left) ideal of S and $E(S)$ is a rectangular band: if $e, f \in E(S)$, $x, y \in S$, then $exyf$, $eyxf \in E(S)$ and

$$exyf = exyf \cdot eyxf \cdot exyf = (exyfe) \cdot yx \cdot (fexyf) = e \cdot yx \cdot f,$$

because $exyfe \leq_{E(S)} e$, $fexyf \leq_{E(S)} f$, hence $exyfe = e$, $fexyf = f$ (since a rectangular band is trivially ordered, by 7.1, Cor.). For instance, let $S =< Y; S_o, S_1; \varphi >$ be the strong semilattice $Y : 0 <_Y 1$ of a rectangular band S_o and an idempotentfree semigroup S_1 with arbitrary linking homomorphism $\varphi : S_1 \to S_o$. In general, S is not medial: take for S_1 any non-medial, idempotentfree semigroup (for example, $S_1 = \mathbb{N} \times G$ where $\mathbb{N}$ is the additive semigroup of natural numbers without zero and G is any non-abelian group).

(5) Let S be a semigroup such that $E(S)a \subseteq aS^1$, $aE(S) \subseteq S^1a$ for any $a \in S$, and every element of S has both an idempotent left and right identity; then S is E-externally medial if and only if S is commutative. Concerning necessity, let $x, y \in S$; then $x = x'x$, $y = yy'$ for some $x', y' \in E(S)$ and $xy = x'xyy' = x'y \cdot xy' \leq_S yx$, since $x'y = yz (z \in S^1)$ and $xy' = wx (w \in S^1)$ imply that $x'y \leq_S y$ and $xy' \leq_S x$, whence the inequality follows by Result 4.12 (and its dual) below; similarly, $yx \leq_S xy$ and thus $xy = yx$. (Note that every inverse semigroup satisfies the given conditions – see Example (3) for E-medial semigroups; also every semigroup with central idempotents in which any element has an idempotent left (right) identity – see Appendix (F) in part II).

Result 4.12. *Let S be an E-externally medial semigroup (i.e., $exyf = eyxf$ for any $e, f \in E(S)$, $x, y \in S$). If for any $a \in S$ there is some $g \in E(S)$ with $a = ag$ ($a = ga$), then $\leq_S$ is right (left) compatible.*

Proof. Let $a <_S b$; then $a = xb = by$, $xa = a$ for some $x, y \in S$, and there is $g \in E(S)$ such that $b = bg$. Hence, if $c \in S$, $c = ch$ with $h \in E(S)$, then

$$ac = x \cdot bc = byc = b(g \cdot yc \cdot h) = b(g \cdot cy \cdot h) = bc \cdot z \ (z \in S),$$

$x \cdot ac = ac$; i.e., $ac \leq_S bc$. $\Diamond$

Remark. (i) If in an E-externally medial semigroups S every element has both an idempotent left *and* right identity then $\leq_S$ is two-sided compatible; note that in this case S is medial (see Result 4.1).

(ii) Let S be a strong semilattice Y of monoids $S_\alpha(\alpha \in Y)$. Then S satisfies the conditions of Result 4.12 if and only if S is commutative (necessity: since every S_α $(\alpha \in Y)$ is commutative, so is S). In this case, $\leq_S$ is two-sided compatible by Result 3.3, Cor.

(iii) A regular semigroup S is E-externally medial (satisfies the conditions of Result 4.12) if and only if S is medial (see Remark (i)). In this case, $\leq_S$ is two-sided compatible (see Result 7.16, Remark (i),(ii) following it). In particular, any completely simple semigroup $S = \mathcal{M}(I, G, \Lambda; P)$ over an abelian group G such that in each row of P all entries are equal (i.e., for any $\lambda \in \Lambda$, $p_{\lambda i} = p_{\lambda j}$ for all $i, j \in I$) is E-externally medial. In this case, $\leq_S$ is the identity relation; note that S is a rectangular abelian group since $E(S)$ is a subsemigroup of S (see [25], IV.3.3).

(iv) More generally, semigroups satisfying the conditions in Result 4.12 are given by "rectangular commutative monoids", i.e., $S = R \times T$, where R is a rectangular band and T is a commutative monoid (note that in general, S is not "rectangular" in the sense of Remark (iii) following Result 4.7). S is again medial, and for any $(\alpha, a) \in S$ there is $(\alpha, 1_T) \in$ $\in E(S)$ such that $(\alpha, 1_T)(\alpha, a) = (\alpha, a) = (\alpha, a)(\alpha, 1_T)$; hence $\leq_S$ is two-sided compatible (by Remark (i)). Note that $\leq_S$ is the identity relation if and only if $\leq_T$ is so (since $(\alpha, a) \leq_S (\beta, b) \Leftrightarrow \alpha = \beta,\ a \leq_T b$) – see Appendix (G) in part II.

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