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KILLING AN END POINT WITH AN OPEN MAPPING

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Abstract: We present an example of an end point (in the classical sense) and an open mapping sending this point into a non-end point answering a question posed by J. J. Charatonik and W. J. Charatonik.

1. Introduction

All spaces considered in the paper are assumed to be metric and all mappings are continuous. A mapping $f : X \to Y$ is said to be:

– *interior at a point* $p \in X$ provided that for each open neighbourhood U of p in X the point $f(p)$ is an interior point of the image $f(U)$ in Y;

– *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y.

Obviously, a mapping is open if and only if it is interior at each point of its domain.

A *continuum* means a nonempty compact connected metric space. An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. The symbols $\mathbb{R}$ stands for the set of all real numbers. Let $\mathfrak{m}$ be a cardinal number. By a *simple* $\mathfrak{m}$*-od* with the *vertex* p we mean the union of $\mathfrak{m}$

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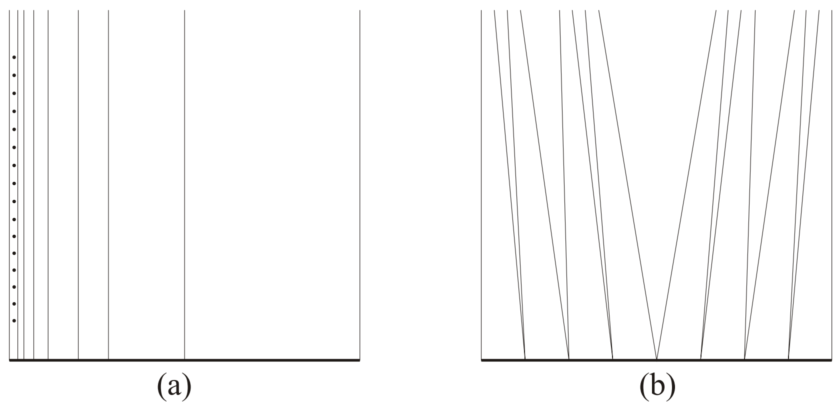

Figure 1. The harmonic comb and the Cantor function comb

arcs every two of which have p as the only common point (the 3-od will be called the *triod*). Let a continuum X and a point $p \in X$ be given. Then p is said to be a *point of order at least* $\mathfrak{m}$ *in the classical sense* provided that p is the center of an $\mathfrak{m}$-od contained in X. We say that p is a *point of order* $\mathfrak{m}$ *in the classical sense* provided that $\mathfrak{m}$ is the minimum cardinality for which the above condition is satisfied (see [1, p. 229]).

Point of order 1 in the classical sense is called an *end point* of a continuum. Point of order 2 in the classical sense is called an *ordinary point* of a continuum. A *branch point* of a continuum is the vertex of a simple triod lying in that continuum.

Notice now that there are many continua without any arc and that the order of a point in classical sense makes no sense for them. Other attempts to the "end" points will be discussed in Sec. 6.

2. Combs

In the following we will use two special continua. The first one is the *harmonic comb* (see Fig. 1a). The horizontal segment on Fig. 1a will be called the *body* of the comb, the remaining segments on Fig. 1a will be called the *teeth* of the comb (the right one is the *first tooth*, the left one is the *last tooth*). The harmonic comb is the union of the body and countable many teeth approaching the last tooth.

The second continuum will be the *Cantor function comb* (see Fig. 1b).

Recall that the *Cantor function* is a non-decreasing continuous mapping $[0,1] \to [0,1]$ sending the middle third to $1/2$ (the middle third of the domain to the middle of the range) and similarly on the remaining thirds. After infinitely many steps we extend the function continuously to $[0,1]$ and we obtain a non-decreasing continuous function φ from $[0,1]$ onto $[0,1]$.

Denote C the Cantor ternary set in $[0,1]$. If $x \in C$ is in the form

$$x = \sum_{n=0}^{\infty} \frac{\alpha_n}{3^n}\,,$$

where $\alpha_n \in \{0,2\}$, then the image $\varphi(x)$ in in the form

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\beta_n}{2^n}\,,$$

where $\beta_n \in \{0,1\},\ \beta_n = \alpha_n/2$.

For each $x \in C$ we construct a tooth in the Cantor function comb as the segment joining $(x,1)$ and $(\varphi(x),0)$. The horizontal segment on Fig. 1b will be called the *body* of the comb, the remaining segments on Fig. 1b will be called the *teeth* of the comb. Notice that the Cantor function comb is an uncountable union of arcs.

3. Elementary example

Our elementary example will be a continuum in a $\mathbb{R}^3$ with coordinates x, y, z. Denote by π the first coordinate projection sending a point (x,y,z) to $(x,0,0)$. For points $a, b \in \mathbb{R}^3$ denote ab the line segment joining a to b.

We will start with a continuum $X \subset \mathbb{R}^3$ formed with two copies of the harmonic comb glued with their bodies. See the left image on Fig. 2. Next we will comb (bend) the teeth in X in such a way that we identify vertically the couples of non-branch points on the first and last teeth on both combs keeping the common body and the first teeth untouched) and obtain a continuum $\widetilde{X}$. See the right image on Fig. 2.

(Technically: we can find $f : [-1,1]\times[0,1]\times\{0\} \to [-1,1]\times[0,1]\times$ $\times[0,1]$ which agrees with π on $cd\setminus\{e\}$, is the identity mapping on $ab\cup oe$, is injective on $D_f\setminus cd$, keeps the first coordinate and is continuous except at e. Then $f(X) = \widetilde{X}$ is a continuum.)

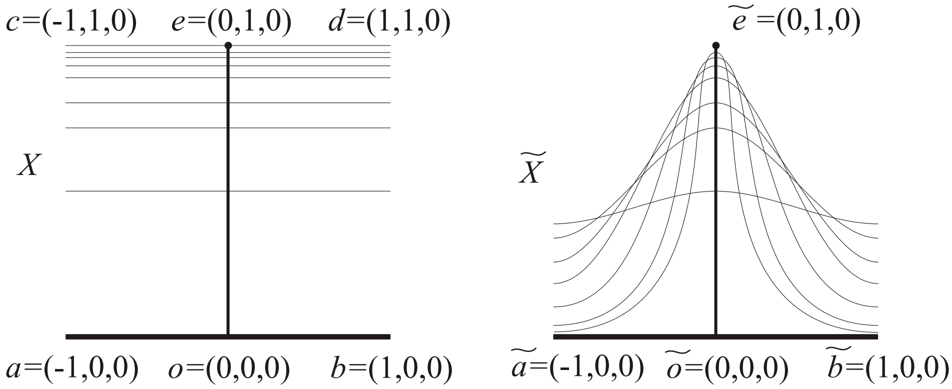

Figure 2. Combing a double harmonic comb

Notice moreover that $\widetilde{X}$ has an end point $e = \widetilde{e}$ (the upper end of the body). We observe that the first coordinate projection $\widetilde{X} \to \pi(\widetilde{X})$ is interior at $\widetilde{e}$. The interiority of this projection fails at those ordinary points of $\widetilde{X}$ which are contained in the common body. The next example will improve the interiority at these points by adding new teeth.

4. Advanced example

We repeat the elementary example approach replacing both the harmonic combs in X with the Cantor function combs forming continua Y and $\widetilde{Y}$. See Fig. 3.

Now the projection of $\widetilde{Y}$ onto the first coordinate is an open mapping $\widetilde{Y}$ to $\pi(\widetilde{Y}) = \widetilde{a}\widetilde{b}$ and the image of an end point $\widetilde{e}$ is not an end point of $\pi(\widetilde{Y})$.

5. Remark

We need some additional notions. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two of its subcontinua is connected. A hereditarily unicoherent and arcwise connected continuum is called a *dendroid.* A dendroid X is said to be *smooth* provided that there exists a point $v \in X$ such that for each point $x \in X$ and each sequence of points x_n tending to x the sequence of arcs vx_n in X tends to the arc vx.

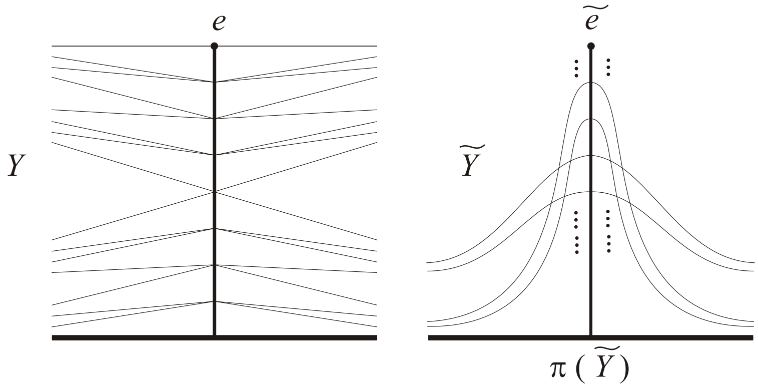

Figure 3. Combing a double Cantor function comb

J. J. Charatonik and W. J. Charatonik in 1997 (see [2, Question 2.3, p. 3730], [3, Question 3.3, p. 103]) asked if the end points of a smooth dendroid are mapped into the end points under an open mapping. The advanced example above answers the question in the negative.

Moreover notice that another example answering this question was constructed by L. G. Oversteegen already in 1980 (see [5, Example 3.2, p. 118]). The open mapping in this example is moreover *monotone* (all inverse-images of points are connected). The Oversteegen's example is nice and clever, unfortunately it is not simple a we will not write its definition here. (Hint. Use a "book" of Cantor function combs, control their shapes, including the length of teeth and their placements on their bodies. The identification of points on the teeth with their attaching points on the body gives the desired open monotone mapping of a smooth dendroid onto an arc. Notice that our advanced example uses a two leaves book and projects onto the bottom of the book erasing only one end point while the Oversteegen's example uses uncountable leaves book, projects on the common body and erases uncountable many end points.)

6. Other ends

We have seen that the notion of an end point defined by the order in the classical sense does not recognize the nature of the continuum in the neighbourhood of a point very well. The "better end point" in the sense that there is only "one way" to it can be defined using small neighbourhoods and points on their boundaries.

We can define an *order* of a point p in a continuum X (in the sense of Menger–Urysohn), written $\operatorname{ord}(p, X)$, this way. Let $\mathfrak{n}$ stand for a cardinal number.

We write:

$\operatorname{ord}(p, X) \leq \mathfrak{n}$ provided that for every $\epsilon > 0$ there is an open neighbourhood U of p such that $\operatorname{diam} U \leq \epsilon$ and $\operatorname{card} \operatorname{bd} U \leq \mathfrak{n}$;

$\operatorname{ord}(p, X) = \mathfrak{n}$ provided that $\operatorname{ord}(p, X) \leq \mathfrak{n}$ and for each cardinal number $\mathfrak{m} < \mathfrak{n}$ the condition $\operatorname{ord}(p, X) \leq \mathfrak{m}$ does not hold;

$\operatorname{ord}(p, X) = \omega$ provided that the point p has arbitrarily small open neighbourhoods U with finite boundaries $\operatorname{bd} U$ and $\operatorname{card} \operatorname{bd} U$ is not bounded by any $n \in \mathbb{N}$.

Thus, for any continuum X we have

$$\operatorname{ord}(p, X) \in \{1, 2, \dots, n, \dots, \omega, \aleph_0, 2^{\aleph_0}\}$$

(convention: $\omega < \aleph_0$); see [4, Sec. 51, I, p. 274].

Point of order 1 in a continuum X is called an *end point* of X (in the sense of Menger–Urysohn).

The following classical result is well known.

Theorem ([6, Chapter 8, (7.31), p. 147])**.** *The order (in the sense of Menger–Urysohn) of a point is never increased under an open mapping.*

From the theorem we see that end points in the sense of Menger–Urysohn are mapped again into such points.

Finally, we recall several other notions describing the "ends":

- A point p of a continuum X is called an *end point* of X if for each two subcontinua of X both containing p, one of the subcontinua contains the other.
- A point p of a continuum X is called an *absolute end point* of X if there is a point $q \in X$ such that $X \setminus \{p\}$ is the union of all proper subcontinua of X which contain q.
- A proper subcontinuum K of a continuum X is said to be a *terminal continuum* of X provided that if whenever A and B are proper subcontinua of X having union equal to X such that $A \cap K \neq \emptyset \neq$ $\neq B \cap K$, then either $X = A \cup K$ or $X = B \cup K$.

- A proper subcontinuum K of a continuum X is said to be an *absolutely terminal continuum* of X provided that K is a terminal continuum of each subcontinuum L of X which properly contains K.

These notions (and some other as well) and their properties are studied in [3]. General problem in that paper is for what classes of mappings which classes of "end" continua are preserved.

7. Questions

We finish with natural questions:

Question 1. Is there a planar smooth dendroid and an open mapping not preserving the end points in the classical sense?

Question 2. Is the advanced example presented above the simplest possible smooth dendroid having an open mapping erasing an end point in the classical sense?

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