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# BUTTERFLY CURVE THEOREMS IN PSEUDO-EUCLIDEAN PLANE

A. **Sliepčević**

*University of Zagreb, Faculty of Civil Engineering, Kačićeva 26, 10000 Zagreb, Croatia*

E. **Jurkin**

*University of Zagreb, Faculty of Mining, Geology and Petroleum Engineering, Pierottijeva 6, 10000 Zagreb, Croatia*

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**Abstract:** Up till now the validity of the Butterfly Theorem has been verified in the Euclidean, isotropic and hyperbolic plane.
In the present paper we prove that the Butterfly Theorem also holds in the pseudo-Euclidean plane. Furthermore, it is shown that an infinite number of butterfly points, located on a conic, is associated with any quadrangle inscribed into a circle. In the Euclidean plane this conic is always a rectangular hyperbola while in the pseudo-Euclidean plane it can also be an ellipse or a special parabola.

## 1. Introduction

The basic object of the Butterfly Theorem is a complete quadrangle inscribed in a circle. For the Euclidean plane this theorem can be traced back to 1815, [2], [8], [12]; numerous proofs, many variants and generalizations of the Butterfly Theorem have been published, see for instance

*E-mail addresses:* anas@grad.hr, ejurkin@rgn.hr

[3] where a long list of references is exhibited. The Butterfly Theorem is also valid in the isotropic and hyperbolic plane, see [1], [4] resp. [10].

The underlying structure of our investigations is a *pseudo-Euclidean (Minkowski) plane* which can be defined as real projective plane $\mathrm{PG}(2,\mathbb{R})$ where the metric is induced with an absolute $\{f, F_1, F_2\}$ in the sense of Cayley–Klein, consisting of a real line $f$ and two real points $F_1$ and $F_2$ incidental with it, [5], [7]; we always assume that $\mathrm{PG}(2,\mathbb{R})$ is embedded into its complexification $\mathrm{PG}(2,\mathbb{R}\subset\mathbb{C})$. The line $f$ is called the *absolute line* and the points $F_1, F_2$ are the *absolute points*. All lines through the absolute points are called *isotropic lines* and all points incidental with $f$ are called *isotropic points*.

An involution of points on the absolute line $f$ having the absolute points for the fixed points is called the *absolute involution*.

Two lines are *perpendicular* if they meet the absolute line in a pair of points corresponding in the absolute involution.

The *midpoint* of the segment $AB$ is the point $P_{AB}$ such that the point pair $AB$ is harmonic with $P_{AB}X_{AB}$, in symbols $(AB, P_{AB}X_{AB})=-1$, where $X_{AB}$ is the isotropic point of the line $AB$.

A classification of the conics in the pseudo-Euclidean plane has been given in [5]. They are classified into: *ellipses* (conics intersecting the absolute line at two imaginary points), *hyperbolas* (conics intersecting the absolute line at two real points), *parabolas* (conics touching the absolute line), *special hyperbolas* (conics passing through one of the absolute points), *special parabolas* (conics touching the absolute line at one absolute point) and *circles* (conics passing through the absolute points).

The pole of the absolute line with respect to the conic $c$ is called the *center* of the conic $c$, [7]. Lines through the center of the conic are its *diameters*. Two lines are conjugate with respect to a circle iff they are perpendicular.

## 2. Butterfly Theorems

Let us begin by stating the Butterfly Theorem in the pseudo-Euclidean plane. This theorem can be proved synthetically as it has been done in [8] for the Euclidean plane.

**Theorem 1.** *Let the complete quadrangle $ABCD$ be inscribed into the circle $c$ of the pseudo-Euclidean plane. Let $l$ be a line perpendicular to the* *diameter $o$ of the circle $c$ and let $P, P'$, $Q, Q'$ and $R, R'$ be the intersections of the line $l$ with the pairs of the opposite sides $AB, CD$, $AC, BD$ and $AD, BC$ of the quadrangle $ABCD$. If $L = o \cap l$ is the midpoint of one of the segments $PP'$, $QQ'$, $RR'$, then it is also the midpoint of the other two.*

**Proof.** Let the line $l$ intersect the absolute line at the point $O$ and the circle $c$ at the points $M$, $M'$. The lines $l$ and $o$ are perpendicular and therefore the point pair $MM'$ is harmonic with $LO$. If we assume that $L$ is the midpoint of the segment $PP'$, it follows that $(PP', LO) = -1$. By $[ABCD]$ we denote the pencil of conics through $A, B, C$, and $D$. Thus, $L$ and $O$ are the fixed points of the involution determined on the line $l$ by the conics of the pencil $[ABCD]$, [6], [11]. The point $L$ is the midpoint of every pair of intersection points of the line $l$ and a conic of the pencil $[ABCD]$. $\Diamond$

The point $L$ that on some line $l$ bisects the segments formed by intersections of the pairs of opposite sides of the quadrangle $ABCD$ with the line $l$ is called *butterfly point* of the quadrangle $ABCD$ and the line $l$ is called the *butterfly line*.

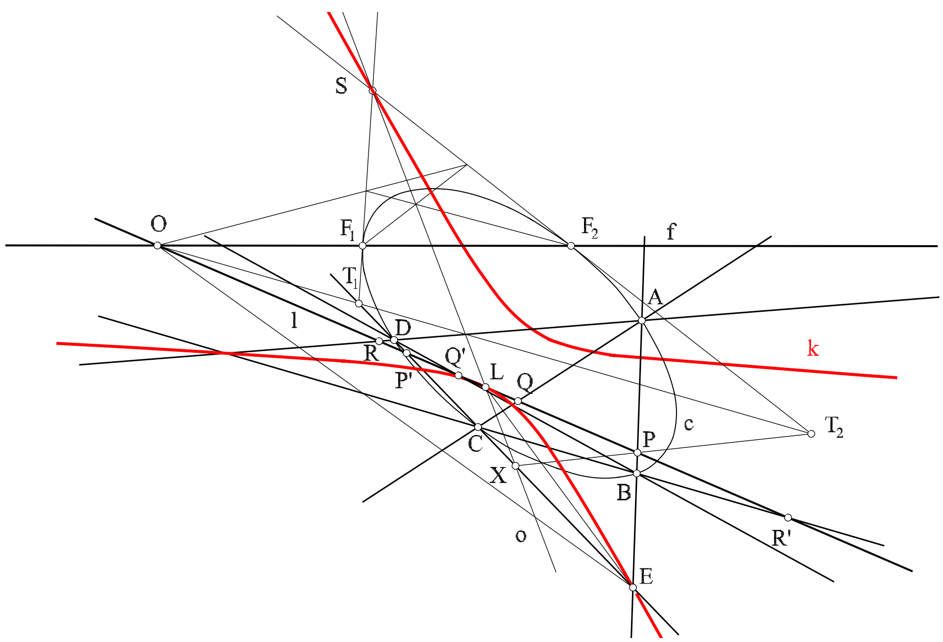


Figure 1.

Fig. 1 presents the quadrangle $ABCD$ inscribed into the circle $c$

and the point $L$ having the property of a butterfly point on the line $l$.

It has been shown in papers [4], [8] and [10] that in the isotropic, Euclidean and hyperbolic plane there is a butterfly point on every diameter of the circle $c$. All of them lie on a conic, the so-called *butterfly points' curve*. To the mentioned theorems there exists the following pseudo-Euclidean version.

**Theorem 2.** *For a given complete quadrangle $ABCD$ inscribed into a circle $c$ of the pseudo-Euclidean plane there is an infinite number of butterfly points. All of these points are located on a conic passing through the center of the circle, the three diagonal points of the quadrangle and the six midpoints of its sides.*

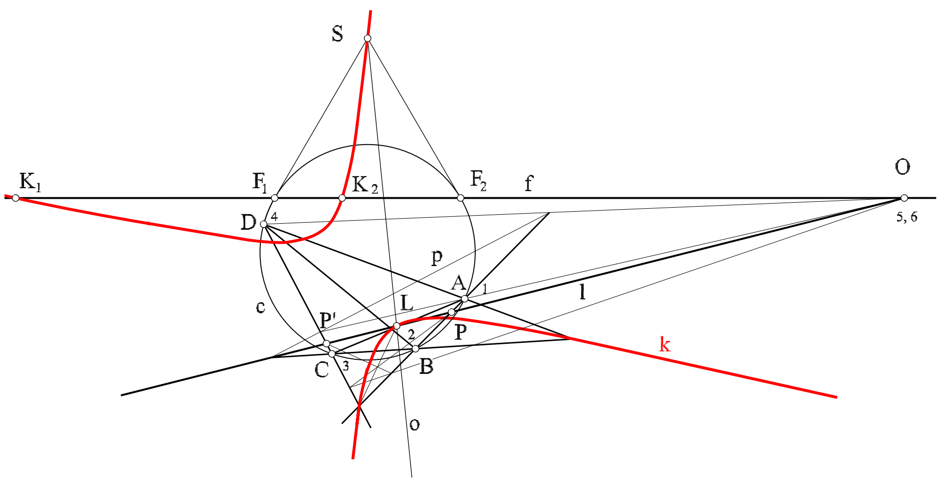


Figure 2.

**Proof.** Let $o$ be a diameter of the circle $c$ and let $O$ be its pole with respect to $c$, Fig. 1. Let $CD$ intersect $o$ at the point $X$ and $SF_1$ at the point $T_1$ and let $T_2$ be the intersection of $SF_2$ with $OT_1$. Obviously, $SF_1$, $SF_2$, $SX$ and $SO$ form a harmonic quadruple of lines and, therefore, $XT_1$, $XT_2$, $XS$ and $XO$ also form a harmonic quadruple of lines. If we denote the intersection point of $AB$ and $XT_2$ by $P$, then the line $l = OP$ meets the lines $o$ and $CD$ at the points $L$, $P'$, respectively, such that $(PP', LO) = -1$. It follows from Th. 1 that $L$ is a butterfly point of the quadrangle $ABCD$. We conclude that there is a butterfly point on each diameter of the circle $c$.

It is obvious from the construction that the midpoints of the sides of the quadrangle $ABCD$ play the roles of the butterfly points on those sides.

By connecting the diagonal point $E = AB \cap CD$ with $P, P', L, O$ a harmonic quadruple of lines is obtained. Thus, for every diameter $o$ the line $EL$ is a harmonic conjugate of $EO$ with respect to $AB$ and $CD$. It follows that all butterfly points lie on the conic $k$ generated by a projectivity between the pencils of lines with vertex $E$ and $S$. The conic $k$ passes through $S$ and $E$, as well as through the other two diagonal points (since the same construction can be repeated for them). $\Diamond$

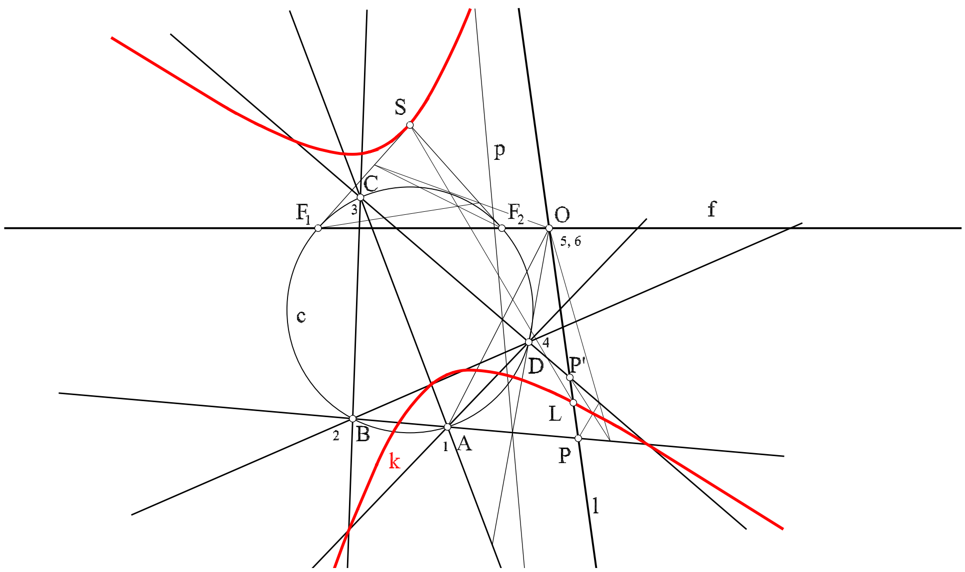


Figure 3.

Another construction of butterfly points was given in [9] for the Euclidean plane. The same construction can be applied to the pseudo-Euclidean plane: Let us choose a point $O$ on the absolute line. By using Pascal's theorem we can construct the tangent $l$ of the conic determined by $A, B, C, D, O$ at $O$, Fig. 2. Let $P, P'$ be the intersections of $l$ with $AB, CD$, respectively. The point $O$ is the touching point of one of the conics of the pencil $[ABCD]$, while the midpoint $L$ of the segment $PP'$ is the touching point of the other conic. Therefore, $L$ is a butterfly point of the quadrangle $ABCD$ and $l$ is a butterfly line. The line $o = SL$ is the polar line of $O$ with respect to the circle $c$.

If the butterfly conic $k$ intersects the absolute line $f$ at the points $K_1$ and $K_2$, they are the fixed points of the involution determined on $f$ by the conics of the pencil $[ABCD]$ and the touching points of $f$ with

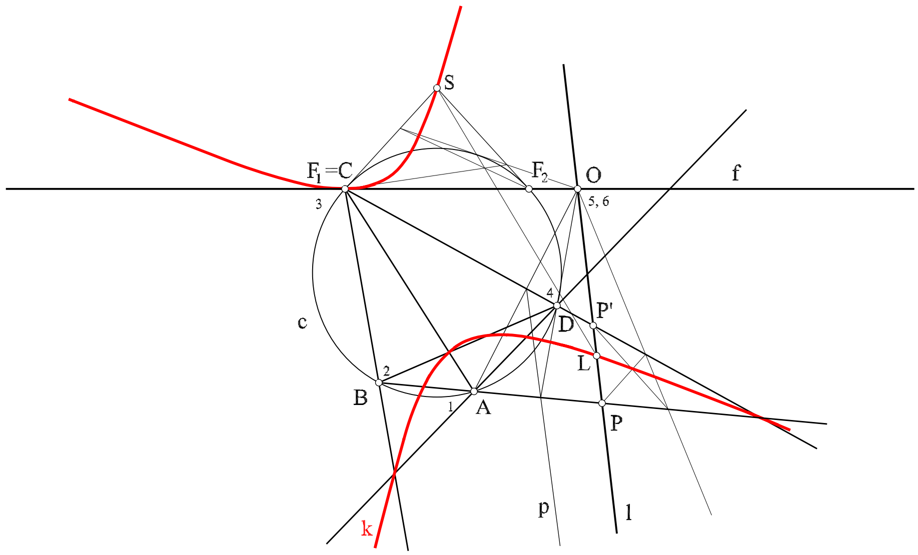


Figure 4.

two parabolas of that pencil. The circle $c$ is one of the conics of the same pencil and therefore its intersections $F_1$, $F_2$ with $f$ together with the fixed points $K_1$, $K_2$, form a harmonic quadruple of points, $(F_1F_2, K_1K_2) = -1$. Thus, $k$ is a rectangular hyperbola, Fig. 2. The described situation happens in the case when the pencil $[ABCD]$ contains ellipses, hyperbolas and two parabolas. In the Euclidean plane every pencil containing a circle has this property. This does not hold in the pseudo-Euclidean plane. If the pencil consists only of hyperbolas and one circle, as it is shown in Fig. 3, then the butterfly curve has no real isotropic points and it is an ellipse. If the pencil $[ABCD]$ contains one circle, one special parabola and all other conics are special hyperbolas, then the butterfly curve is a special parabola, Fig. 4. In a special case, when two basic points, e.g. $C, D$, of the pencil coincide with the absolute points, the butterfly curve splits into the absolute line $f$ and the line on which lie the centers of all circles passing through $A, B$.

We can conclude our study with the following theorem:

**Theorem 3.** *Let the complete quadrangle $ABCD$ be inscribed into the circle $c$ of the pseudo-Euclidean plane. The butterfly curve $k$ of the quadrangle $ABCD$ is a conic the type of which depends on the type of the pencil $[ABCD]$ of conics as follows:*

- *$k$ is a rectangular hyperbola if the pencil $[ABCD]$ contains the circle*

*$c$, ellipses, hyperbolas and two parabolas.*

- *$k$ is an ellipse if the pencil $[ABCD]$ contains, apart from the circle $c$, only hyperbolas.*
- *$k$ is a special parabola if the pencil $[ABCD]$ contains, apart from the circle $c$ and one special parabola, only special hyperbolas.*

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