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ON REGULARITY OF MATRIX NEAR-RINGS

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Abstract: A classic matrix ring result is that the $n \times n$ full matrix ring over a ring is (von Neumann) regular if and only if the base ring is regular. The near-ring version of the aforementioned property has attracted many near-ring theorists since the introduction of matrix near-rings by Meldrum and van der Walt in the 80's. In this note, we study the transfer of regularity between a base near-ring and its $n \times n$ matrix near-ring extension. A partial answer is provided to the question. We show that if the $n \times n$ full matrix near-ring over a near-ring with identity is regular, so are any $m \times m$ full matrix near-rings over the same base near-ring, $1 \leq m \leq n$.

1. Introduction

In this note, a near-ring is a right zero-symmetric near-ring [5] with identity. For convenience, we shall use R to denote a near-ring. A near-ring (or ring) is called *regular* if for every element a of the near-ring (or ring), there exists an element b such that $aba = a$. An important property of matrix ring is that the $n \times n$ full matrix ring over a ring with identity is regular if and only if the base ring is regular (see Th. 2.14 of [4]). Note the Brown and McCoy [2] showed this is true for rings without identity (see also [6]).

We denote by $M_n(R)$ the $n \times n$ matrix near-ring over R. (We

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refer the readers to [1] and [3] for more on matrix near-rings.) Suppose ρ is a reflexive and transitive relation on $\{1, 2, \cdots, n\}$. Then we let $M_n(\rho, R)$ (see [7]) be the subnear-ring of $M_n(R)$ generated by the elementary matrix functions $\{f_{ij}^r : R^n \to R^n | r \in R, (i,j) \in \rho\}$ where $f_{ij}^r(u_1, u_2, \cdots, u_n) = (0, \cdots, 0, ru_j, 0, \cdots, 0)$ such that ru_j is in the i-th position. In this note, we investigate the transfer of regularity between a near-ring R and the $n \times n$ matrix near-ring over R. We show that if the $n \times n$ matrix near-ring $M_n(R)$ is regular, then each $M_m(R)$ is regular, $1 \leq m \leq n$. However, the author does not know if the converse is true in general.

2. Results

We begin our quest with the following technical result.

Lemma 2.1. *Let $K \subseteq \{1, 2, \cdots, n\}$. Then $e = \sum_{i \in K} f_{ii}^1$ is an idempotent and a distributive element of $M_n(R)$.*

Proof. It is easy to see that if $i \neq j$, we have $f_{il}^r B + f_{jk}^s A = f_{jk}^s A + f_{il}^r B$ for any $r, s \in R$ and any matrices $A, B \in M_n(R)$. Therefore, e is distributive. By Th. 2.2 of [1], we have e an idempotent. $\diamondsuit$

Lemma 2.2. *If e is an idempotent and a distributive element of R, then eRe is a subnear-ring of R. In addition, if R is regular, then so is eRe.*

Proof. The proof is a routine exercise. $\diamondsuit$

Theorem 2.3. *Suppose $M_n(\rho, R)$ is regular where ρ is a reflexive and transitive relation on the set $\{1, 2, \cdots, n\}$. Then R is regular.*

Proof. Let $e = f_{11}^1$. Observe that e is a distributive element of $M_n(\rho, R)$. Then $eM_n(\rho, R)e$ is a regular subnear-ring of $M_n(\rho, R)$ by Lemma 2.2. Let $A \in eM_n(B, R)e$. We have $Ae = f_{11}^{a_1} + \cdots + f_{n1}^{a_n}$ for some $a_i \in R$ by Lemma 3.7 of [3]. Since $Ae = A$, we conclude that $A = f_{11}^{a_1} + \cdots + f_{n1}^{a_n}$. Furthermore, $eA = f_{11}^1(f_{11}^{a_1} + \cdots + f_{n1}^{a_n}) = f_{11}^{a_1}$ by Lemma 3.1(5) of [3]. Hence $A = f_{11}^{a_1}$. Immediately we see that $A = 0$ if and only if $a_1 = 0$. Note also that $f_{11}^a + f_{11}^b = f_{11}^{a+b}$ and $f_{11}^a \cdot f_{11}^b = f_{11}^{ab}$. Therefore R is isomorphic to $eM_n(\rho, R)e$, and we have R is regular. $\diamondsuit$

We shall use the notions of "molars" and "incisors", introduced in [7] for matrices, to extend the above result. If $\bar{u}, \bar{v} \in R^n$, we define $\bar{u} \sim_i \bar{v}$ if and only if $\pi_j \bar{u} = \pi_j \bar{v}$ for all j such that $(i,j) \in \rho$, where π_j denotes the j-th coordinate projection function. Furthermore, let:

$$W = \{f_{ij}^r | r \in R, 1 \leq i, j \leq n\}.$$

We make the following distinction among the f^r_{ij}'s in an expression:

(1) If $E = a_1 \in W$, then a_1 is an *incisor* in E.

(2) Let $A = a_1 a_2 \cdots a_p$ and $E = a'_1 a'_2 \cdots a'_q$ be expressions of some matrices. If $a_k \in W$, then a_k is a *molar* in

$$(A)(E) = (a_1 a_2 \cdots a_p)(a'_1 a'_2 \cdots a'_q).$$

If $a'_h \in W$, then a'_h is an incisor (resp. a molar) in $(A)(E)$ if a'_h is an incisor (resp. a molar) in E, $1 \leq h \leq q$.

(3) Let A and E be as in (2). If $a_k \in W$, then a_k is an incisor (resp. a molar) in $A + E = a_1 a_2 \cdots a_p + a'_1 a'_2 \cdots a'_q$ if a_k is an incisor (resp. a molar) in A, $1 \leq k \leq p$. If $a'_h \in W$, then a'_h is an incisor (resp. a molar) in $A + E$ if a'_h is an incisor (resp. a molar) in E, $1 \leq h \leq q$.

We proceed the extension by first proving the following important result (Th. 2.4) of structural matrix near-rings. Given ρ on $\{1, 2, \cdots, n\}$. Let $\bar{m}$ be any non-empty subset of $\{1, 2, \cdots, n\}$ of size m satisfying the following condition. For any two elements a, b of $\bar{m}$, we have

$$(a, b) \text{ and } (b, a) \in \rho.$$

Furthermore, we let

$$\rho_{\bar{m}} = \{(a, b) | a, b \in \bar{m}\} \bigcup \{(x, x) | 1 \leq x \leq n\}.$$

Clearly, we have $\rho_{\bar{m}} \subseteq \rho$ an equivalence relation on $\{1, 2, \cdots, n\}$.

For example, if $n = 3$ and

$$\rho = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3)\},$$

then $\bar{m}$ can only be one of the following four sets:

$$\{1\}, \{2\}, \{3\}, \quad \text{and} \quad \{1, 2\}.$$

For each of the first 3 sets (each is of size 1), the equivalence relation $\rho_{\bar{m}}$ is clearly $\{(1, 1), (2, 2), (3, 3)\}$. However, if $\bar{m} = \{1, 2\}$ (of size 2), we have

$$\rho_{\bar{m}} = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}.$$

Theorem 2.4. *Given ρ a reflexive and transitive relation on $\{1, 2, \cdots, n\}$. Suppose $\bar{m}$ is a subset of $\{1, 2, \cdots, n\}$ as described before. Let $e = = \sum_{i \in \bar{m}} f^1_{ii}$. Then $eM_n(\rho, R)e$ is generated as a subnear-ring of $M_n(R)$ by the set*

$$\{f^r_{ij} | r \in R, i \text{ and } j \in \bar{m}\}.$$

Furthermore, $eM_n(\rho, R)e$ is isomorphic to $M_m(R)$.

Proof. We proceed by proving a sequence of claims. We stipulate that $\bar{u}$ and $\bar{v}$ are elements of R^n in the following.

Claim 1. $eM_n(\rho, R)e \subseteq M_n(\rho_{\bar{m}}, R)$ where $\rho_{\bar{m}} = \{(a,b) \in \rho : a, b \in \bar{m}\} \bigcup$ $\bigcup\{(x,x) : 1 \le x \le n\}$.

Proof of Claim 1. In view of [7], it suffices to show that for any $A \in eM_n(\rho, R)e$, we have:

($\pi_j \bar{u} = \pi_j \bar{v}$ for all j such that $(i,j) \in \rho_{\bar{m}}$) implies ($\pi_i A\bar{u} = \pi_i A\bar{v}$) whenever $1 \le i \le n$.

Note that $f_{ii}^1 \in eM_n(\rho, R)e$ for any $i \in \bar{m}$. Let $A \in eM_n(\rho, R)e$. We fix an integer i where $1 \le i \le n$. Suppose also that $\pi_j \bar{u} = \pi_j \bar{v}$ whenever $(i,j) \in \rho_{\bar{m}}$. If $i \notin \bar{m}$, then $i = j$ and we have

$$\pi_j A\bar{u} = \pi_j eA\bar{u} = 0 \text{ and } \pi_j A\bar{v} = \pi_j eA\bar{v} = 0.$$

If $i \in \bar{m}$, then we have $\pi_j \bar{u} = \pi_j \bar{v}$ for all $j \in \bar{m}$. That is to say $\left(\sum_{h \in \bar{m}} f_{hh}^1\right)\bar{u} = \left(\sum_{h \in \bar{m}} f_{hh}^1\right)\bar{v}$. Therefore,

$$\pi_i A\bar{u} = \pi_i A\left(\sum_{h \in \bar{m}} f_{hh}^1\right)\bar{u} = \pi_i A\left(\sum_{h \in \bar{m}} f_{hh}^1\right)\bar{v} = \pi_i A\bar{v}.$$

This shows that $A \in M_n(\rho_{\bar{m}}, R)$, and thus $eM_n(\rho, R)e \subseteq M_n(\rho_{\bar{m}}, R)$. This proves the claim.

Claim 2. Each matrix of $eM_n(\rho, R)e$ has a formal expression containing only the elements of the form f_{ij}^r where $i, j \in \bar{m}$.

Proof of Claim 2. As an immediate consequence of Claim 1, we have that each matrix of $eM_n(\rho, R)e$ has a formal expression containing only the elements of the forms f_{ij}^r where $(i,j) \in \rho_{\bar{m}}$.

In view of the proof of Th. 2.8 of [7], a matrix $U \in eM_n(\rho, R)e$ can be written as $U = U_1' + \cdots + U_n'$, and each U_i' has a formal expression $E_i' \in \mathbb{E}_n(R)$ where $\mathbb{E}_n(R)$ is the set of all formal expressions representing matrices of $M_n(R)$. (Note a matrix of $M_n(R)$ may have multiple formal expressions.) Furthermore, each E_i' satisfies the following conditions:

(1) if f_{hk}^r is a molar, then every f_{hk}^r in E_i' satisfies that $h = k = i$;

(2) if f_{hk}^r is an incisor, then every f_{hk}^r in E_i' satisfies that $h = i$.

From the fact that $\pi_i U\bar{u} = 0$ for $\bar{u} \in R^n$ and $i \notin \bar{m}$, we have $U_i' = 0$. Therefore, we can assume that U has a formal expression $\sum_{i \in \bar{m}} E_i'$. Furthermore, if f_{ii}^r with $i \in \bar{m}$ is a molar in some E_j', $j \in \bar{m}$, then we must have $i = j$. This is impossible. If f_{ji}^r with $i \notin \bar{m}$ is an incisor in E_j', $j \in \bar{m}$, then since $\pi_i U\bar{u} = 0$ whenever $i \notin \bar{m}$, we can replace f_{ji}^r by f_{ji}^0. As a consequence, we can assume that each E_j', $j \in \bar{m}$, contains only terms of the forms f_{ij}^r with $i, j \in \bar{m}$. This proves Claim 2.

Claim 3. $eM_n(\rho, R)e$ is isomorphic to $M_m(R)$.

Proof of Claim 3. We show that the natural correspondence between formal expressions $\mathbb{E}_m(R)$ and $\mathbb{E}_n(R)$ induces the desired isomorphism. Recall that the set $\bar{m}$ has m elements. Let $\alpha : \{1, 2, \cdots, m\} \to \bar{m}$ such that $1 \le i < j \le m$ if and only if $\alpha(i) < \alpha(j)$. It is clear that α is determined uniquely by the set $\bar{m}$. Define a map $\Phi : \mathbb{E}_m(R) \to \mathbb{E}_n(R)$ such that:

(1) Φ maps $f^r_{ij} \in \mathbb{E}_m(R)$ into $f^r_{\alpha(i)\alpha(j)} \in \mathbb{E}_n(R)$;
(2) if $A, B \in \mathbb{E}_m(N)$, we have $\Phi(A + B) = \Phi(A) + \Phi(B)$; and
(3) if $A, B \in \mathbb{E}_m(N)$, we have $\Phi(AB) = \Phi(A)\Phi(B)$.

Since Φ takes a formal expression of $\mathbb{E}_m(R)$ into a formal expression of $\mathbb{E}_n(R)$, it is well-defined. Suppose $U \in M_m(R)$ an $m \times m$ matrix and $E \in \mathbb{E}_m(R)$ a formal expression of U. Note that E consists of expressions of the form $f^r_{ij}, 1 \le i, j \le m$. Then the map Φ induces a map $\phi : M_m(R) \to eM_n(\rho, R)e$ such that $\phi(U)$ is equal to the matrix in $M_n(R)$ realizing the expression $\Phi(E) \in \mathbb{E}_n(R)$. From Claim 2, we have that $\phi(U) \in eM_n(\rho, R)e$ if ϕ is well-defined. We are to show ϕ is well-defined. Let $E_1, E_2 \in \mathbb{E}_m(R)$ representing the same matrix $U \in M_m(R)$. Let also V_1 and $V_2 \in M_n(R)$ be the matrices realizing the expressions $\Phi(E_1)$ and $\Phi(E_2)$, respectively. Suppose $\bar{u} = (u_1, u_2, \cdots, u_n) \in R^n$ and $\bar{u}' = (u_1, u_2, \cdots, u_m) \in R^m$. Then:

$$\pi_j V_1(\bar{u}) = \pi_j U(\bar{u}') \text{ and } \pi_j V_2(\bar{u}) = \pi_j U(\bar{u}')$$

where $j \in \bar{m}$. We also have that:

$$\pi_j V_1(\bar{u}) = 0 \text{ and } \pi_j V_2(\bar{u}) = 0$$

for $j \notin \bar{m}$. Therefore, we have $V_1 = V_2$.

Suppose $A, B \in M_m(R)$ such that $\phi(A) = \phi(B)$, i.e. $\phi(A)(\bar{u}) = = \phi(B)(\bar{u})$. We then have:

$$\pi_i A\bar{u}' = \pi_{\alpha(i)}\phi(A)(u_1, u_2, \cdots, u_m, 0, \cdots, 0)$$

if $1 \le i \le m$. However,

$$\bar{u} \sim_i (u_1, u_2, \cdots, u_m, 0, \cdots, 0) \in R^n$$

if $1 \le i \le m$. This implies:

$$\begin{aligned}\pi_i\phi(A)(\bar{u}) &= \pi_i\phi(A)(u_1, u_2, \cdots, u_m, 0, \cdots, 0)\\ &= \pi_i A(u_1, \cdots, u_m)\end{aligned}$$

if $1 \le i \le m$. Similarly, we have:

$$\pi_i\phi(B)(\bar{u}) = \pi_i B(u_1, \cdots, u_m)$$

if $1 \le i \le m$. Therefore, we have $A = B$. That is to say ϕ is an injection. The surjectivity follows the definition easily. Moreover,

$$\phi(A+B)=\phi(A)+\phi(B) \text{ and } \phi(AB)=\phi(A)\phi(B).$$

In other words, $M_m(R)$ is isomorphic to $eM_n(\rho,R)e$. $\Diamond$

Observed that Th. 2.3 is now an immediate consequence of Lemma 2.2 and the above result by taking the set $\bar{m}=\{1\}$ and ρ the universal relation on $\{1,2,\cdots,n\}$. Furthermore, we have the following result.

Theorem 2.5. *Let $1\le m\le n$. If $M_n(R)$ is regular, then $M_m(R)$ is regular.*

Proof. Let ρ be the universal relation on $\{1,2,\cdots,n\}$ and $\bar{m}$ the set $\{1,2,\cdots,m\}$. Then the result follows by invoking Lemma 2.2 and Th. 2.4. $\Diamond$

The author does not know whether the converse of Th. 2.5 is true. We conclude this note with the following conjecture.

Conjecture 2.6. *There exists a regular zero-symmetric near-ring R with identity that $M_2(R)$ is not regular.*

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