Mathematica Pannonica
22/1 (2011), 73–83

WEAK POLYNOMIAL AUTO-MORPHISMS OF SOLUBLE GROUPS

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Received: March 2011

MSC 2000: 20 F 28, 20 F 16, 20 F 18, 16 Y 30

Keywords: Polynomial automorphism, soluble group, nilpotent group.

Abstract: We extend G. Endimioni's theory of polynomial automorphisms of soluble and nilpotent groups (see [2] and [3]) to a slightly wider class of automorphisms, which leads to proofs that are somewhat shorter and less computational. We also sharpen some of the conclusions.

An automorphism ϕ of a group G is said to be polynomial if there exist finitely many elements $u_1, u_2, \ldots, u_n$ of G and integers $e(1), e(2), \ldots, e(n)$ such that

$$g\phi = \left(u_1^{-1} g^{e(1)} u_1\right)\left(u_2^{-1} g^{e(2)} u_2\right) \ldots \left(u_n^{-1} g^{e(n)} u_n\right)$$

for all g in G. Following [2] let $P_o(G)$ denote the subset (actually submonoid) of the automorphism group $\mathrm{Aut}\, G$ of G of all polynomial automorphisms of G and set $P(G) = \langle P_o(G) \rangle \leq \mathrm{Aut}(G)$.

In [2] Endimioni proves the following. If G is nilpotent of class $c \geq 2$, then $P(G)$ is nilpotent of class $c - 1$. If G is metabelian, then $P(G)$ is metabelian. If G is metabelian and 2-generator, then $IA(G) = C_{\mathrm{Aut}(G)}(G/G')$ lies in $P(G)$ and hence (C. K. Gupta [4]) is metabelian. In [3] he proves that if G is soluble of derived length $d > 1$, then $P(G)$

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is soluble of derived length at most $2(d-1)$. Improving his metabelian results above, he also shows in [3] that if G is (nilpotent of class $c \geq 1$)-by-abelian, then $P(G)$ is (nilpotent of class at most $c-1$)-by-metabelian. His proofs are based on a series of calculations.

Our aim here is, by employing weaker versions of the notion of polynomial automorphism more suited to certain types of inductive proof, to produce shorter proofs of these theorems with little or no calculation. Our methods also give minor generalizations of the above, over and above the obvious one of applying to more automorphisms than just the polynomial ones. For example, we prove that if G is (nilpotent of class c)-by-abelian, then $P(G)$ is (nilpotent of class at most c)-by-abelian. We also weaken his conditions on a nilpotent group G to ensure that $P(G) = P_o(G)$.

The remainder of this paper is divided into three relatively separate sections. The first one is devoted to nilpotent groups, the second to nilpotent-by-abelian groups and the third to soluble groups in general.

1. Nilpotent groups

Let ϕ be a polynomial automorphism of the group G and suppose the u_i and $e(i)$ are as in the above definition of polynomial map. Set $m = \Sigma_i e(i) \in \mathbb{Z}$, the ring of integers. If H/K is any central section of G, then $h\phi \in h^m K$ for all h in H. In particular if G is abelian $P_o(G)$ is just the set of (universal) power automorphisms of G.

Let $\langle 1 \rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_d = G$ be a central series of the nilpotent group G of class $c \leq d$. Let Δ_o denote the set of automorphisms ϕ of G such that $Z_i\phi = Z_i$ for all i and for which there exists an integer m (to be called an associated integer of ϕ) satisfying $g\phi \in g^m Z_{i-1}$ for all $g \in Z_i$ and all $i \geq 1$. Set $\Delta = \langle \Delta_o \rangle \leq \operatorname{Aut} G$. If $\{Z_i\}$ is either the lower or the upper central series of G, then $\Delta_o \supseteq P_o(G)$ and $\Delta \geq P(G)$. Further let Γ_o denote the set of automorphisms ϕ of G such that $Z_i\phi = Z_i$ for all i and for which there exist integers $m(i)$ satisfying $g\phi \in g^{m(i)} Z_{i-1}$ for all $g \in Z_i$ and all $i \geq 1$ and set $\Gamma = \langle \Gamma_o \rangle \leq \operatorname{Aut} G$. In this section Γ plays only a minor role, but in Sections 2 and 3 below it plays very much the dominant role. Example 1 in Section 2 shows that we can have $\Gamma \neq \Delta$ (and hence $\Gamma_o \neq \Delta_o$). Finally set $S = \cap_i C_{\operatorname{Aut} G}(Z_i/Z_{i-1})$. Clearly S is a subgroup of $\operatorname{Aut} G$ with $\Delta_o \supseteq S$, each element of S having 1 as an associated integer.

Theorem 1. *Assume the notation above.*

a) Δ *is nilpotent of class at most* $\max\{1, d-1\}$ *and at least* $c-1$.

b) *If* G *has finite exponent, then so does* Γ *and* $\Gamma = \Gamma_o$, $\Delta = \Delta_o$ *and* $P(G) = P_o(G)$.

c) $S \cap P_o(G)$ *is a subgroup of* $\mathrm{Aut}\, G$.

d) (Endimioni [2], Th. 1.1) *If* $c > 1$, *then* $P(G)$ *is nilpotent of class exactly* $c-1$.

Suppose in addition to the above that the series $\{Z_i\}$ *is the lower central series of* $G \neq \langle 1 \rangle$ *(so now* $c = d \geq 1$ *and* $Z_i = \gamma^{c+1-i}G$ *for each* i*). Let* e *denote the greatest common divisor of the* $m-1$, *where* m *ranges over all the associated integers* $m \neq 1$ *of all the elements of* Δ_o, *meaning* $e = 0$ *if no such* m *exists. Then:*

e) $(\gamma^i G)^e \leq \gamma^{i+1}G$ *for all* $i \geq 2$ *and* G' *has exponent dividing* e^{c-1}.

f) $[\gamma^i G, \Delta] \leq \gamma^{i+1}G$ *for all* $i \geq 2$.

g) *If* $c \geq 2$ *then* $\Delta/C_\Delta(G')$ *is nilpotent of class at most* $c-2$ *and has exponent dividing* e^{c-2}.

h) *If* $c > 1$ *and if* $\gamma^c G$ *does not have finite exponent, then* $\Delta_o = S = \Delta$ *and* $P_o(G) = P(G)$.

i) (Endimioni [3], Th. 1.5) *If either* G *has finite exponent or* G *is non-abelian and torsion-free, then* $P_o(G) = P(G)$.

It is easy to see that in Th. 1 the group Δ can be strictly larger than $P(G)$. For example, if $G = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| > 1$ and if ϕ is the automorphism of G determined by $x \mapsto xy$ and $y \mapsto y$, then $\phi \in \Delta$ if we take $d = 2$ and $Z_1 = \langle y \rangle$ and yet ϕ is not a power automorphism of G, so $\phi \notin P_o(G)$. If G has finite order then $P(G) = P_o(G)$. If not then here x and y have infinite order and $P_o(G) = \{\pm 1\} = P(G)$. Thus $\phi \in \Delta \backslash P(G)$.

Lemma 1. *Let* G *be a group with* Z *a central subgroup of* G. *Suppose* ϕ *and* ψ *are automorphisms of* G *such that for some integers* m *and* n *we have* $g\phi \in g^m Z$, $z\phi = z^m$, $g\psi \in g^n Z$ *and* $z\psi = z^n$ *for some* $g \in G$ *and all* $z \in Z$. *Then* $g\phi\psi = g\psi\phi$.

Proof. Now $g\phi = g^m a$ and $g\psi = g^n b$ for some a and b in Z. Then
$$g\phi\psi = (g^m a)\psi = (g\psi)^m (a\psi) = g^{mn} b^m a^n = g\psi\phi. \qquad \Diamond$$

Proof of Theorem 1. a) Clearly the inner automorphism group $\mathrm{Inn}\, G$ of G lies in Δ, so if Δ is nilpotent its class must be at least $c-1$. If $d = 1$, since power automorphisms commute, so Δ is abelian. If $d = 2$, then Δ is abelian by Lemma 1. From now on assume that $d \geq 3$.

We induct on d. Set $\Sigma = C_\Delta(Z_{d-1}) \cap C_\Delta(G/Z_1)$. By induction Δ/Σ is nilpotent of class at most $d-2$. Let $\phi \in \Delta_o$ with m an associated integer and consider $\sigma \in \Sigma$. If $g \in G$, then $g\phi = g^m a$ for some $a \in Z_{d-1}$ and $g\sigma = gb$ for some $b \in Z_1$. Also $b\phi = b^m$, $a\sigma = a$ and b is central in G. Consequently

$$g\phi\sigma = (g^m a)\sigma = (gb)^m a = g^m a b^m = g\phi b\phi = (gb)\phi = g\sigma\phi.$$

Thus Σ lies in the centre of Δ and so Δ is nilpotent of class at most $d-1$.

b) Suppose G has finite exponent r and let s be the order of the automorphism group of the cyclic group of order r. Then any power automorphism of any Z_i/Z_{i-1} has order dividing s. Also S has finite exponent dividing r^{d-1} (e.g. [6] 1.21). Thus any $\phi \in \Gamma_o$ has finite order dividing $t = r^{d-1}s$. Consequently $\phi^{-1} = \phi^{t-1} \in \Gamma_o$ and so $\Gamma = \Gamma_o$. Similarly $\Delta = \Delta_o$. If we choose the series $\{Z_i\}$ to be the lower central series of G, then $P_o(G) \leq \Gamma_o$ and hence $P_o(G) = P(G)$ in the same way.

c) If G has finite exponent the claim follows from Part b). If not there exists i with Z_i/Z_{i-1} not of finite exponent. Let $\phi \in S \cap P_o(G)$, say $g\phi = \prod_j (u_j^{-1} g^{e(j)} u_j)$ for all g in G, where the u_j and the $e(j)$ are as in our introduction. Set $m = \Sigma_j e(j)$. Then modulo Z_{i-1} we have $g \equiv g\phi \equiv g^m$ for all $g \in Z_i$. Since Z_i/Z_{i-1} does not have finite exponent, this yields that $m = 1$. By [1], Th. 1 we have $\phi^{-1} \in P_o(G)$ and Part c) follows.

For the remainder of the proof of Th. 1 assume $\{Z_i\}$ is the lower central series $\{\gamma^{c+1-i}\}$ of G.

d) Then $\operatorname{Inn} G \leq P(G) \leq \Delta$, $c = d$, $\operatorname{Inn} G$ has class $c-1$ and Δ has class at most $\max\{1, d-1\} = c-1$. That $P(G)$ is nilpotent of class $c-1$ follows.

e) Let $\phi \in \Delta_o$ with m an associated integer relative to the series $\{\gamma^{c+1-i}\}$. If $x, y \in G$, then

$$[x,y]^m \in [x,y]\phi\gamma^3 G \leq [x^m\gamma^2 G, y^m\gamma^2 G]\gamma^3 G = [x,y]^{mm}\gamma^3 G.$$

Thus $([x,y]\phi)^{m-1} \in [x,y]^{m(m-1)}\gamma^3 G = \gamma^3 G$. Now there is an inverse automorphism of ϕ, so $[x,y]^{m-1} \in \gamma^3 G$ for all x and y in G and hence $(\gamma^2 G)^{m-1} \leq \gamma^3 G$. Hence it follows that $(\gamma^i G)^{m-1} \leq \gamma^{i+1} G$ for all $i \geq 2$, for the exponent of the $(j+1)$th lower central factor of a group divides that of its jth lower central factor, e.g. see [6] (b) on p. 10. Thus $(\gamma^i G)^e \leq \gamma^{i+1} G$ for all $i \geq 2$ and therefore G' has exponent dividing e^{c-1}.

f) Suppose $i \geq 2$ and $g \in \gamma^i G$. Then with $\phi \in \Delta_o$ and associated integer m we have $g\phi \in gg^{m-1}\gamma^{i+1} G = g\gamma^{i+1} G$. Thus $[\gamma^i G, \phi] \leq \gamma^{i+1} G$ and it follows that $[\gamma^i G, \Delta] \leq \gamma^{i+1} G$.

g) This follows from Parts e) and f) and stability theory (e.g. [6], 1.19 and 1.21). (Alternatively the first part of g) also follows from Part a) applied to G'.)

h) Let $\phi \in \Delta_o$ with m as an associated integer. Now $\gamma^c G$ is non-trivial but of infinite exponent and $c \geq 2$. Thus $(\gamma^c G)^{m-1} = \langle 1 \rangle$ by Part e) and $m = 1$. That is, $\phi \in S$, so $\Delta_o = S = \Delta$. Finally, in view of the choice of the series $\{Z_i\}$ we have $S = \Delta_o \supseteq P_o(G)$. Therefore $P_o(G) = P(G)$ by Part c).

i) This is immediate from Parts b) and h). The proof of Th. 1 is complete. $\Diamond$

To illustrate what is happening in Parts e) and f) of Th. 1 and to see that there are no obvious improvements possible to these parts, consider the following example. Let $P = P_{c+1}$ be a Sylow p-subgroup of the symmetric group $\mathrm{Sym}(p^{c+1})$ for some prime p and $c \geq 1$. Then P is nilpotent of class p^c and exponent p^{c+1} (e.g. [5], III.15.3). Also $P = \langle x \rangle D$, where x has order p and D is a direct product of p copies of P_c permuted cyclicly by x. Thus P' is contained in D, has a section isomorphic to P_c, has class p^{c-1} and has exponent p^c.

Set $m = p + 1$ and let Q be a multiplicative copy of the additive group of $\mathbb{Z}[m^{-1}]$ with $y \in Q$ corresponding to $1 \in \mathbb{Z}$. Then Q maps onto $\langle x \rangle$ via $y \mapsto x$. In this way Q acts on P; let $G = QP$, the split extension of P by Q. Then $G' = P'$ and Q centralizes the lower central factors of P, so G is nilpotent of class p^c and $\gamma^i G = \gamma^i P$ for all $i \geq 2$. In particular $\gamma^i G/\gamma^{i+1} G$ has exponent exactly $p = m - 1$ for $2 \leq i \leq c$.

If $a \in Q$ and $u \in P$, define $\phi : G \mapsto G$ by $(au)\phi = a^m u^x$. If also $b \in Q$ and $v \in P$, then

$$\begin{aligned}(aubv)\phi = (abu^b v)\phi = a^m b^m u^{bx} v^x = \\ = a^m b^p b u^{xb} v^x = a^m u^x b^{1+p} v^x = (au)\phi(bv)\phi,\end{aligned}$$

where we have used that the actions of Q and $\langle x \rangle$ on P commute and that b^p is central in G. Also ϕ is bijective on P and Q and normalizes both and $P \cap Q = \langle 1 \rangle$, so $\phi \in \mathrm{Aut}\, G$. Further $g\phi \in g^m \gamma^{i+1} G$ for all $g \in \gamma^i G$ and all i since $m = p + 1$ and each $\gamma^i P/\gamma^{i+1} P$ has exponent p and is centralized by x. Therefore $\phi \in \Delta$, where Δ is computed relative to the lower central series of G. Also $[\gamma^i G, \phi] \leq \gamma^{i+1} G$ for all $i \geq 2$, but not for $i = 1$, since ϕ acts as x on $\gamma^i G = \gamma^i P$ if $i \geq 2$. This is exactly the situation in Parts e) and f) of Th. 1.

Although the map ϕ above lies in Δ it might not lie in $P(G)$. If we now set $n = p^{c+1} + 1$ and define $\psi : G \to G$ by $(au)\psi = a^n u^x$, then we can repeat the above analysis with n and ψ in place of m and ϕ. Define $\sigma : P \to P$ by

$$u^\sigma = \prod\nolimits_{p-1 \geq i \geq 0} x^{-i} u x^i = (x^{1-p} u x^{p-1}) \ldots (x^{-1} u x) u.$$

Then $u^\sigma \in D$ and if $q = p^{c+1}$, then $u^{\sigma q/p} = 1$. Hence $(au)^q = a^q u^{\sigma q/p} = = a^q$, $(au)^x = au^x$ and $(au)\psi = (au)^q (au)^x$. Consequently here we have $\psi \in P_o(G)$.

In Part g) of Th. 1 it is not possible to bound the class of $\Delta/C_\Delta(G')$ just in terms of the class of G', even if we are working with the lower central series of G. For consider P a finite p-group of nilpotency class 2; for example let P be the lower unitriangular group $\mathrm{Tr}_1(3, p)$, e.g. see [6], p. 41. Denote by G the wreath product of P by the cyclic group $\langle g \rangle$ of order p. Then G' is nilpotent of class 2, $C_G(G')$ is the centre of the base group of the wreath product and $G/C_G(G')$ is nilpotent of class at least p (e.g. by [5], III.15.3e again). Now $G/C_G(G')$ embeds into $\Delta/C_\Delta(G')$, where Δ is computed relative to the lower central series of G. Thus here the class of G' is 2, but by choosing p arbitrarily large we see that the class of $\Delta/C_\Delta(G')$ is unboundable.

2. Nilpotent-by-abelian groups

Here we consider a slightly different generalization of the notion of polynomial map. Let

$$\langle 1 \rangle = N_0 \leq N_1 \leq \cdots \leq N_d = N \leq N_{d+1} = G$$

be a normal series of finite length of the group G with $G' \leq N$ and $[N_i, N] \leq N_{i-1}$ for all $i \geq 1$, so in particular G is nilpotent-by-abelian. Let R denote the group ring of G/N over the integers $\mathbb{Z}$; note that R is a commutative ring. Also each N_i/N_{i-1} is an R-module; specifically if $a \in N_i$ and $u = \Sigma_j e(j) u_j N \in R$, where the $e(j)$ are integers and the $u_j \in G$, then

$$(aN_{i-1})^u = \prod\nolimits_j u_j^{-1} a^{e(j)} u_j N_{i-1}.$$

Notice that if $i = d + 1$, then $(aN)^u = a^m N$ for $m = \Sigma_j e(j) \in \mathbb{Z}$.

Let Γ_o denote the set of all automorphisms ϕ of G such that for each $i \geq 1$ there exists $u(i) \in R$ with $N_i \phi = N_i$ and $g\phi \in (gN_{i-1})^{u(i)}$ for all $g \in N_i$. Then set $\Gamma = \langle \Gamma_o \rangle \leq \mathrm{Aut}\, G$. (Notice that if our series $\{N_i\}$

is a central series of G and not just of N, then Γ_o and Γ here are equal to the Γ_o and Γ of Sec. 1.)

Theorem 2. *Assume the notation above.*

a) Γ *is nilpotent-by-abelian and* Γ' *is nilpotent of class at most* d.

b) $P(G)$ *is nilpotent-by-abelian with* $P(G)'$ *nilpotent of class at most the class of* G'. *In particular (Endimioni [2], Th. 1.2) if* G *is metabelian, then* $P(G)$ *is metabelian.*

c) *If* N_1 *is cyclic as* R*-module, then* $C_{\mathrm{Aut}\,G}(G/N_1) \leq \Gamma_o$.

d) (C. K. Gupta [4]) *If* G *is a* 2*-generator metabelian group, then* $IA(G) = C_{\mathrm{Aut}\,G}(G/G')$ *is metabelian.*

Proof. a) If $\phi, \psi \in \Gamma_o$, then ϕ and ψ act on N_i/N_{i-1} as elements of the commutative ring R, and this is for all i. Thus $[\phi, \psi]$ centralizes each N_i/N_{i-1}, so $[N_i, \Gamma'] \leq N_{i-1}$ for each i and stability theory (specifically [6] 1.19 again) yields that Γ' is nilpotent of class at most d. This proves Part a).

b) Suppose G' is nilpotent of class c. Choose

$$\langle 1 \rangle = \gamma^{c+1}G' \leq \gamma^{c}G' \leq \cdots \leq \gamma^{2}G' \leq G' \leq G$$

as our series $\{N_i\}$. Note that this is a characteristic series of G with $d = c$. Then $P(G) \leq \Gamma$ and b) follows from Part a).

c) Suppose $N_1 = \left\langle a^R \right\rangle$ and consider $\phi \in C_{\mathrm{Aut}\,G}(G/N_1)$. Now $a\phi =$ $= a^u$ for some $u \in R$. Also if $b \in N_1$, then $b = a^w$ for some $w \in R$. Then for $g \in G$ we have $b^g\phi = (b\phi)^{g\phi} = (b\phi)^g$, since $g^{-1}(g\phi) \in N_1 \leq C_G(N_1)$. Thus

$$b\phi = a^w\phi = (a\phi)^w = a^{uw} = (a^w)^u = b^u.$$

Thus $\phi \in \Gamma_o$.

d) We use the characteristic series $\langle 1 \rangle \leq G' \leq G$ of G, so $d = 1$. Now G' is cyclic as G-module; specifically if $G = \langle x, y \rangle$, then G' is G-generated by $[x, y]$. Thus $IA(G) \leq \Gamma$ by Part c) and hence $IA(G)$ is metabelian by Part a). $\Diamond$

Endimioni [3], Th. 1.2 follows at once from Th. 2b), namely that if G' is nilpotent of class $c \geq 1$, then $P(G)''$ is nilpotent of class at most $c - 1$. Continuing with the notation above, suppose N is nilpotent of class $c \geq 1$ and set $N_i = \gamma^{c+1-i}N$ for each $i \leq d$, so here $d = c$. Let Φ_o denote the set of all automorphisms ϕ of G such that $N\phi = N$ (and hence $N_i\phi = N_i$ for all i) and for each $i = c$, $c + 1$ there exists $u(i) \in R$ with $g\phi \in (gN_{i-1})^{u(i)}$ for all $g \in N_i$ and set $\Phi = \langle \Phi_o \rangle \leq \mathrm{Aut}\,G$. Clearly $\Gamma_o \leq \Phi_o$ and $\Gamma \leq \Phi$ and by Th. 2a) the subgroup Γ' is nilpotent of class at most c.

Proposition. *Assume the notation above.*

a) *The subgroup* Φ *is (nilpotent of class at most c)-by-abelian.*

b) *If* G/G'' *is* 2*-generator, then* $IA(G)$ *is (nilpotent of class at most c)-by-abelian.*

This is an improvement on [3], Prop. 2.4, which states that $IA(G)$ is (nilpotent of class at most $c-1$)-by-metabelian whenever G is 2-generator and (nilpotent of class $c \geq 1$)-by-abelian.

Proof. a) Φ acts as an abelian group on G/N and N/N'. Thus Φ' centralizes G/N, $\gamma^1 N/\gamma^2 N$ and hence $\gamma^i N/\gamma^{i+1}N$ for each $i \geq 1$, e.g. by [6] (b) on p. 10 again. Thus Φ' stabilizes the series $\{N_i\}$ of length $c+1$ and therefore is nilpotent of class at most c (e.g. [6], 1.19). Part a) follows.

b) Choose $N = G'$. Suppose for the moment that $G'' = \langle 1 \rangle$. Then clearly $\Phi = \Gamma$. If $G = \langle x, y \rangle$, then $G' = \left\langle [x,y]^G \right\rangle$ and Th. 2c) yields that $C_{\operatorname{Aut} G}(G/G') \leq \Gamma$. In the general case, note that by definition Φ is the inverse image in $\operatorname{Aut} G$ of the Γ subgroup of $\operatorname{Aut}(G/G'')$ computed from the series $\langle 1 \rangle \leq G'/G'' \leq G/G''$. Thus the special case above applied to the group G/G'' yields that $C_{\operatorname{Aut} G}(G/G') \leq \Phi$. The claim now follows from Part a) and the definition of $IA(G)$. $\Diamond$

We now present some examples that limit any possible generalizations of Th. 2.

Example 1. Even if the series $\{N_i\}$ is central in G, we cannot conclude in general that Γ is nilpotent; that is, it is critical in Sec. 1 that the $u(i)$ for ϕ are all equal (to m there).

For consider the lower unitriangular group $G = \operatorname{Tr}_1(c+1, F)$, where F is a field and $c \geq 2$. Then G is nilpotent of class c. Set

$$t = \operatorname{diag}(m^c, m^{c-1}, \ldots, m, 1) \in GL(c+1, F),$$

where m is an integer with $m \neq 0$, 1 if $\operatorname{char} F = 0$ and $1 < m < \operatorname{char} F$ otherwise. Then t acts on G via conjugation and for each $i \geq 1$ it raises each element of $\gamma^i G/\gamma^{i+1}G$ to its m^i-th power. Set $N_i = \gamma^{c+1-i}G$ for $0 \leq i \leq c$, with $N = N_{c+1} = G$ and $d = c$. Then with $T = \langle t \rangle\, G$, the factor $T/\gamma^c G$ embeds into Γ. Now T/G' is easily seen not to be nilpotent, indeed t acts fixed-point freely on G/G', and $c \geq 2$. Therefore Γ is not nilpotent and in particular, in the notation of Sec. 1, $\Delta < \Gamma$. Clearly with suitable choices of F we can arrange for G to be finite or torsion-free and c is arbitrary subject only to $c \geq 2$.

Example 2. In Part a) of Th. 2 we cannot deduce that Γ' has class less than d (compare Th. 1, where Δ is always nilpotent of class less than d).

Continue with the notation of Example 1, but now consider the series

$$\langle 1\rangle = \gamma^{c+1}G \leq \gamma^{c}G \leq \cdots \leq \gamma^{2}G \leq G \leq T$$

for T, so again $d = c \geq 2$. It is easily seen that $T' = G$, so T' is nilpotent of class d. Also T has trivial centre, so $T \cong \operatorname{Inn} T \leq \Gamma$ and T' embeds into Γ'. The latter therefore has class exactly d (using Part a) of Th. 2.

Example 3. Part d) of Th. 2 does not extend to 3-generator metabelian groups. Specifically there exists a 3-generator metabelian group G such that $IA(G)$ is not soluble and hence does not embed into Γ for any allowable choice of the series $\{N_i\}$ of G.

For let A be an infinite cyclic group, J an image ring of $\mathbb{Z}$, M a free JA-module of rank 2 and G the split extension of M by A. Thus G is metabelian and 3-generator. $\operatorname{Aut}_A M$ embeds into $IA(G) \leq \operatorname{Aut} G$ by mapping $\phi \in \operatorname{Aut}_A M$ to the map $ax \mapsto a(x\phi)$ for all $a \in A$ and $x \in M$. (It is easy to check that this map is an automorphism of G.) Thus the insoluble group $GL(2, JA)$ embeds into $IA(G)$.

3. Soluble groups

Consider the normal series

$$\langle 1\rangle = N_0 \leq N_1 \leq \cdots \leq N_e = G$$

of finite length of the group G with all its factors abelian. Let Γ_o denote the set of all automorphisms ϕ of G such that for each $i \geq 1$ we have $N_i\phi = N_i$ and for some $w(i)$ in the group ring over $\mathbb{Z}$ of G/N_i we have $g\phi \in g^{w(i)}N_{i-1}$ for all $g \in N_i$. Set $\Gamma = \langle \Gamma_o \rangle \leq \operatorname{Aut} G$. (Note that if the series $\{N_i\}$ is as in Sec. 2, that is, if also $d = e - 1$ and $[N_i, N_d] \leq N_{i-1}$ for $i \geq 1$, then the Γ_o and Γ above are exactly the Γ_o and Γ of Sec. 2.)

Theorem 3. *Assume the notation above.*

a) *If* $e = 1$ *then* Γ *is abelian; if* $e \geq 2$ *then* Γ *is soluble of derived length at most* $2(e-1)$ *and at least that of* $\operatorname{Inn} G$.

b) (Endimioni [3], Th. 1.3) *If* G *is soluble of derived length* $d \geq 2$, *then* $P(G)$ *is soluble of derived length at most* $2(d-1)$ *and at least* $d-1$.

Proof. a) If $e = 1$ clearly Γ is abelian. Let $e \geq 2$. Since $\operatorname{Inn} G$ is a subgroup of Γ, the lower bound is immediate. For the remainder we induct on e. If $e = 2$ then G and Γ are metabelian by Th. 2a). Suppose

$e \geq 3$. By induction we may assume that $\Gamma/C_\Gamma(G/N_1)$ is soluble of derived length at most $2(e-2)$. Set $\Sigma = C_\Gamma(G/N_1) \cap C_\Gamma(N_1)$. Then Σ stabilizes the series $\langle 1 \rangle \leq N_1 \leq G$ and therefore is abelian. Provided $C_\Gamma(G/N_1)/\Sigma$ is abelian, we have that Γ is soluble of derived length at most $2(e-2)+2 = 2(e-1)$, and the proof of a) will be complete.

Let $\phi \in \Gamma_o$, so $x\phi = x^w$ for all x in N_1 for some fixed $w = = \Sigma_j n(j)g(j)$ in $\mathbb{Z}G$, where the $n(j)$ are integers and the $g(j)$ lie in G. If $\psi \in C_\Gamma(G/N_1)$, then

$$x\phi\psi = \prod_j \left(x^{n(j)g(j)}\psi\right) = \prod_j (x\psi)^{n(j)g(j)} = x\psi\phi,$$

where we have used that $g(j)\psi \in g(j)N_1$, that $x \in N_1$ and that N_1 is abelian. Thus $\left[C_\Gamma(G/N_1), \Gamma\right] \leq C_\Gamma(N_1)$, so $C_\Gamma(G/N_1)/\Sigma$ is abelian (it is even a central section of Γ), as required.

b) Choose $\{N_i\}$ to be the derived series of G, so $\{N_i\}$ is characteristic in G. Then $e = d$ and $\operatorname{Inn} G \leq P(G) \leq \Gamma$. Clearly $\operatorname{Inn} G$ has derived length at least $d-1$ and Γ has derived length at most $2(d-1)$ by Part a). Thus b) follows. $\Diamond$

Remarks. As recorded in Endimioni [3], Th. 1.4, it follows from Th. 3b) that if G is polycyclic, then so is $P(G)$ since any soluble group of automorphisms of a polycyclic group is polycyclic (e.g. [6], 5.2). A similar remark applies to (torsion-free by finite) soluble minimax groups, for example, since a soluble group of automorphisms of a (torsion-free by finite) soluble minimax group is again a (torsion-free by finite) soluble minimax group.

The proof above of Th. 3a) actually shows for $e > 2$ that Γ has a normal series

$$\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_{2(e-1)} = \langle 1 \rangle$$

with abelian factors such that $[\Gamma_{2i}, \Gamma] \leq \Gamma_{2i+1}$ for each $i = 1, 2, \ldots, e-2$.

In Th. 3 if G has trivial centre then clearly Γ and $P(G)$ have derived lengths at least d. This can also happen in the nilpotent case. If $d \geq 2$ and F is any field, then $G = \mathrm{Tr}_1(2^d, F)$ and $H = \mathrm{Tr}_1(2^d - 1, F)$ both have derived length d (e.g. [6], p. 42) and H is an image of the centre factor group of G. Thus G is nilpotent and is of derived length d, and Γ and $P(G)$ have derived lengths at least d.

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