Mathematica Pannonica
22/1 (2011), 49–72

CERTAIN RELATIONS OBTAINED STARTING WITH THREE POSITIVE REAL NUMBERS AND THEIR USE IN INVESTIGATION OF BICENTRIC POLYGONS

Mirko **Radić**

University of Rijeka, Department of Mathematics, 51000 Rijeka, Omladinska 14, Croatia

Received: May 2010

MSC 2000: 51 E 12

Keywords: Relation, bicentric polygon.

Abstract: In the article we start with three positive real numbers R_0, r_0, d_0 such that $R_0 > r_0 + d_0$ and establish certain relations which can be obtained by means of these numbers. As will be seen, many interesting properties of bicentric polygons can be relatively easily established using these relations. They are as a key for many problems concerning bicentric polygons. The article is a complement to the article [8].

1. Introduction

In the article we restrict ourselves to the case where conics are circles. Here will be stated one of the main results given in the article. First about some terms and notation which will be used.

A polygon $A_1 \dots A_n$ is called chordal polygon if there is a circle which contains each of the points (vertices) $A_1, \dots, A_n$. A polygon $A_1 \dots A_n$ is called tangential polygon if there is a circle such that segments $A_1A_2, \dots, A_nA_1$ are tangential segments of the circle.

E-mail address: mradic@ffri.hr

A polygon which is both chordal and tangential is shortly called bicentric polygon. If $A_1 \dots A_n$ is a bicentric polygon then it is usually that radius of its circumcircle is denoted by R, radius of incircle by r and distance between centers of circumcircle and incircle by d.

The first one that was concerned with bicentric polygons is German mathematician Nicolaus Fuss (1755–1826). He found relations (conditions) for bicentric quadrilaterals, pentagons, hexagons, heptagons and octagons. Here we list only these for bicentric quadrilaterals, hexagons and octagons

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2), \tag{1.1a}$$

$$3p^4q^4 - 2p^2q^2r^2(p^2 + q^2) - r^4(p^2 - q^2)^2 = 0, \tag{1.1b}$$

$$[r^2(p^2 + q^2) - p^2q^2]^4 - 16p^4q^4r^4(p^2 - r^2)(q^2 - r^2) = 0, \tag{1.1c}$$

where $p = R + d$, $q = R - d$.

The corresponding relation for triangle is given by Euler and it reads as follows

$$R^2 - d^2 - 2Rr = 0. \tag{1.1d}$$

Of course, if $A_1 \dots A_n$ is a given bicentric n-gon then its circumcircle and incircle can be constructed as follows. The intersection of the lines of symmetry of the two consecutive sides (angles) is center C of circumcircle (center I of incircle). Thus $|CA_1| = R$ and distance of I from A_1A_2 is r.

Although Fuss found relation for R, r, d only for bicentric n-gons, $4 \le n \le 8$, it is in his honor to call such relations Fuss' relations also in the case $n > 8$.

The very remarkable theorem concerning bicentric polygons is given by French mathematician Poncelet (1788–1867). This theorem, so called Poncelet's closure theorem for circles, can be stated as follows.

Let C_1 and C_2 be two circles, where C_2 is inside of C_1. If there is a bicentric n-gon $A_1 \dots A_n$ such that C_1 is its circumcircle and C_2 its incircle then for every point P_1 on C_1 there are points $P_1, \dots P_n$ on C_1 such that $P_1, \dots P_n$ is a bicentric n-gon whose circumcircle is C_1 and incircle C_2. Thus, in this case we can construct a bicentric polygon whose circumcircle is C_1 and incircle C_2 and point P_1 is one of its vertices.

Although this famous Poncelet's closure theorem dates from nineteenth century, many mathematicians have been working on number of problems in connection with this theorem. In this article we deal with certain important properties and relations in this connection. The main results refer to Fuss' relations.

Let, for brevity, Fuss' relation for bicentric n-gons be denoted by

$$F_n(R, r, d) = 0. \tag{1.2}$$

Let $(R_0, r_0, d_0) \in \mathcal{R}_+^3$ be a solution of Fuss' relation (1.2) that is, let $F_n(R_0, r_0, d_0) = 0$. Then by Poncelet's closure theorem there is a class

$$C_n(R_0, r_0, d_0) \tag{1.3}$$

of bicentric n-gons such that all n-gons from this class have the same circumcircle and same incircle. Let circumcircle be denoted by C_1 and incircle C_2. Then for every point P_1 on C_1 there are points $P_2, \dots, P_n$ on C_1 such that there exits a bicentric n-gon $P_1 \dots P_n$ from the class (1.3) whose circumcircle is C_1 and incircle C_2.

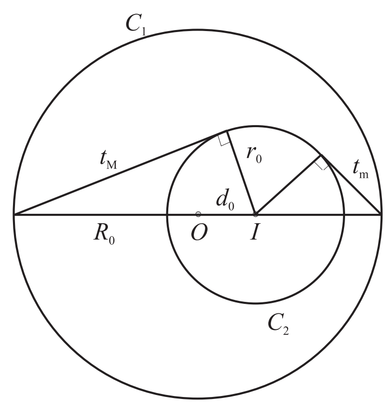

Figure 1. $t_M = \sqrt{(R_0 + d_0)^2 - r_0^2}, \quad t_m = \sqrt{(R_0 - d_0)^2 - r_0^2}$.

Important role in the following will play lengths t_m and t_M given by

$$t_m = \sqrt{(R_0 - d_0)^2 - r_0^2}, \quad t_M = \sqrt{(R_0 + d_0)^2 - r_0^2}. \tag{1.4}$$

See Fig. 1, where by C_1 is denoted circumcircle of the polygons from the class (1.3) and by C_2 is denoted incircle of the polygons from this class.

The lengths t_M and t_m can be called *maximal* and *minimal* tangent lengths of the class (1.3).

From Poncelet's closure theorem it is clear that the following holds. If t_1 is any given length such that $t_m \le t_1 \le t_M$, where t_m and t_M are given by (1.4), then there is a bicentric n-gon from the class (1.3) such that its first tangent has the length t_1. In [5, Lemma 1] it is proved that for calculation of tangent lengths of bicentric polygons can be used the following formula

$$(1.5)\qquad (t_2)_{1,2} = \frac{(R_0^2 - d_0^2)t_1 \pm r_0\sqrt{D_1}}{r_0^2 + t_1^2},$$

where $D_1 = (t_M^2 - t_1^2)(t_1^2 - t_m^2)$. If t_1 is given then its consequent is $(t_2)_1$ or $(t_2)_2$.

Concerning signs $+$ and $-$ in expression $\pm\sqrt{D_i}$ it does not matter, since for each integer i such that $1 < i < n$, the following is valid

$$(1.6)\qquad t_{i+1} = \frac{(R_0^2 - d_0^2)t_i + r_0\sqrt{D_i}}{r_0^2 + t_i^2} \iff t_{i-1} = \frac{(R_0^2 - d_0^2)t_i - r_0\sqrt{D_i}}{r_0^2 + t_i^2}.$$

Using this property the following algorithm can be used. Let t_1 be any given length such that $t_m \le t_1 \le t_M$ where t_m and t_M are given by (1.4). Then there exists a bicentric n-gon $A_1 \dots A_n$ from the class $C_n(R_0, d_0, r_0)$ such that its first tangent has the length t_1. The other its tangent length can be calculated as follows.

For t_2 can be used $(t_2)_1$ or $(t_2)_2$ given by (1.5). Depending on which of $(t_2)_1$ and $(t_2)_2$ is taken for t_2 we get ordering of tangent lengths $t_1, \dots, t_n$ clockwise or counterclockwise. Let $(t_2)_1$ be taken for t_2, that is let

$$(1.7)\qquad t_2 = \frac{(R_0^2 - d_0^2)t_1 + r_0\sqrt{D_1}}{r_0^2 + t_1^2}.$$

The following notation will be used

$$(1.8)\qquad D_i = (t_M^2 - t_i^2)(t_i^2 - t_m^2), \quad i = 1, \dots, n.$$

Let t_{i+2}^+ and t_{i+2}^- be given by

$$(1.9)\qquad t_{i+2}^+ = \frac{(R_0^2 - d_0^2)t_{i+1} + r_0\sqrt{D_{i+1}}}{r_0^2 + t_{i+1}^2}, \quad t_{i+2}^- = \frac{(R_0^2 - d_0^2)t_{i+1} - r_0\sqrt{D_{i+1}}}{r_0^2 + t_{i+1}^2}.$$

Then, since by (1.6) it holds

$$(1.10)\qquad \left\{t_{i+2}^+, t_{i+2}^-\right\} = \{t_i, t_{i+2}\},$$

we have the following equality $t^+_{i+2} \cdot t^-_{i+2} = t_i \cdot t_{i+2}$, from which it follows

$$t_{i+2} = \frac{t^+_{i+2} t^-_{i+2}}{t_i}. \tag{1.11}$$

So we have the following sequence

$$t_1, t_2, \frac{t^+_3 t^-_3}{t_1}, \frac{t^+_4 t^-_4}{t_2} \text{ and so on,} \tag{1.12}$$

where for t_2 can be taken t_2 given by (1.7).

Although the closure in Poncelet's closure theorem is a topological property, the formula (1.5), as can be seen, may be very useful in some problems concerning bicentric polygons. So in Th. 2 it plays a very important role.

For brevity in the following expression we shall often use the term n-closure. In short about this. Let (R, r, d) be a triple such that $(R, r, d) \in \in \mathbb{R}^3_+$ and $R > r + d$. Let $n \geq 3$ be an integer. Then it will be said that the triple (R, r, d) has the property that there exists *n-closure* with rotation number 1 for n if there exists a bicentric n-gon $A_1 \ldots A_n$ such that

R: radius of the circumcircle of $A_1 \ldots A_n$,
r: radius of the incircle of $A_1 \ldots A_n$,
d: distance between centers of circumcircle and incircle,
$2 \sum_{i=1}^{n} \arctan \frac{t_i}{r} = 360°$,

where $t_1, \ldots, t_n$ are tangent lengths of the n-gon $A_1 \ldots A_n$.

Now we state one of the main results in the article which will be later proved as Th. 2.

Let (R_0, r_0, d_0) be a triple such that $(R_0, r_0, d_0) \in \mathbb{R}^3_+$ and $R_0 > > r_0 + d_0$. Let (R_1, r_1, d_1) be a triple given by

$$R_1^2 = R_0 \left(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2} \right),$$
$$r_1^2 = (R_0 + r_0)^2 - d_0^2,$$
$$d_1^2 = R_0 \left(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2} \right).$$

Then the following holds good. If the triple (R_0, r_0, d_0) has the property that there exists n-closure with rotation number 1 for n then the triple (R_1, r_1, d_1) has the property that there exists $2n$-closure with rotation number 1 for $2n$. In other words, if $C_n(R_0, r_0, d_0)$ is a class of bicentric n-gons then $C_{2n}(R_1, r_1, d_1)$ is a class of bicentric $2n$-gons. Thus, if (R_0, r_0, d_0) is a solution of Fuss' relation $F_n(R, r, d) = 0$ then (R_1, r_1, d_1) is a solution of Fuss' relation $F_{2n}(R, r, d) = 0$. More about this will be in Th. 2.

Th. 2 is rather involved and in its proof we shall use some results given in [5] and [7]. From [5] we shall use Th. 1 here written as Th. A which reads as follows.

Theorem A. *Let C_1 and C_2 be any given two circles in the same plane such that C_2 is inside of C_1 and let A_1, A_2, A_3 be any given three different points on C_1 such that there are points T_1 and T_2 on C_2 with the property*

$$|A_1A_2| = t_1 + t_2, \quad |A_2A_3| = t_2 + t_3, \tag{1.13a}$$

where

$$t_1 = |A_1T_1|, \quad t_2 = |T_1A_2|, \quad t_3 = |T_2A_3|. \tag{1.13b}$$

Then

$$|A_1A_3| = k(t_1 + t_3), \tag{1.14a}$$

where

$$k = \frac{2rR}{R^2 - d^2}, \tag{1.14b}$$

R = radius of C_1, r = radius of C_2, $d = |IO|$, I is center of C_2 and O is center of C_1. (See Fig. 2.)

Before stating Th. 1 from [7] we state definition of characteristic point concerning two circles.

Let C_1 and C_2 be any given two circles such that C_2 is (complete) inside C_1. Let R = radius of C_1, r = radius of C_2, $d = |IO|$, where I is center of C_2 and O is center of C_1. Let xOy be co-ordinate system which origin is center O of C_1 and positive part of the x-axis contains center I of C_2.

Let by $S(s, 0)$ be denoted the point where s is given by

$$s = \frac{R^2 + d^2 - r^2 - \sqrt{(R^2 + d^2 - r^2)^2 - 4R^2d^2}}{2d}. \tag{1.15}$$

This point will be called *characteristic point* of the circles C_1 and C_2.

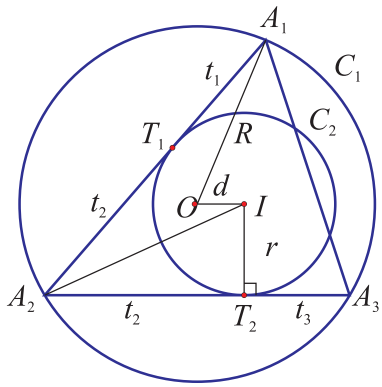

Figure 2

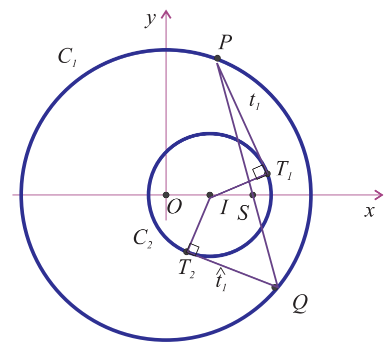

Figure 3

Here let us remark that

$$2(R^2+d^2-r^2)=t_M^2+t_m^2,\quad (R^2+d^2-r^2)^2-4R^2d^2=t_M^2t_m^2, \tag{1.16a}$$

where

$$t_M^2=(R+d)^2-r^2,\quad t_m^2=(R-d)^2-r^2. \tag{1.16b}$$

Thus relation (1.15) can be written as

$$s=\frac{(t_M-t_m)^2}{4d}. \tag{1.17}$$

In the case when $d=0$ can be taken $s=0$ since

$$\frac{\lim_{d\to 0}\frac{\partial}{\partial d}(t_M-t_m)^2}{4}=0.$$

As will be seen the following theorem is trivially valid for $d=0$.

Theorem B. *Let C_1 and C_2 be any given two circles such that C_2 is inside of C_1 and let R = radius of C_1, r = radius of C_2, $d=|IO|$, I is center of C_2 and O is center of C_1. Let xOy be coordinate system as before described. (See Fig. 3.) Let $S(s,0)$ be point such that s is given by* (1.17). *Let PQ be any given chord of the circle C_1 such that it contains point $S(s,0)$. Let PT_1 and QT_2 be tangents drawn from P and Q to C_2 and let*

$$t_1=|PT_1|,\quad \hat{t}_1=|QT_2|. \tag{1.18}$$

Finally, let P, Q, T_1 and T_2 in relation to given coordinate system be given by

$$P(u_1, v_1),\ Q(u_2, v_2),\ T_1(x_1, y_1),\ T_2(x_2, y_2).$$

Then

$$t_1\hat{t}_1 = t_m t_M, \tag{1.19}$$

that is,

$$\left[(u_1 - x_1)^2 + (v_1 - y_1)^2\right]\left[(u_2 - x_2)^2 + (v_2 - y_2)^2\right] - t_m^2 t_M^2 = 0.$$

2. Certain relations obtained starting with three positive real numbers and their use in research of bicentric polygons

First we prove the following theorem in which we state one algorithm relatively very simple and very useful in research of bicentric polygons. It has the key role in proving Th. 2.

Theorem 1. *Let* (R_0, r_0, d_0) *be a triple such that* $(R_0, r_0, d_0) \in \mathbb{R}_+^3$ *and* $R_0 > r_0 + d_0$. *Let* (R_1, r_1, d_1) *and* (R_2, r_2, d_2) *be triples given by*

$$R_1^2 = R_0\left(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \tag{2.1a}$$

$$r_1^2 = (R_0 + r_0)^2 - d_0^2, \tag{2.1b}$$

$$d_1^2 = R_0\left(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \tag{2.1c}$$

and

$$R_2^2 = R_0\left(R_0 - r_0 + \sqrt{(R_0 - r_0)^2 - d_0^2}\right), \tag{2.1d}$$

$$r_2^2 = (R_0 - r_0)^2 - d_0^2, \tag{2.1e}$$

$$d_2^2 = R_0\left(R_0 - r_0 - \sqrt{(R_0 - r_0)^2 - d_0^2}\right). \tag{2.1f}$$

Then

$$R_1 > r_1 + d_1, \quad R_2 > r_2 + d_2, \tag{2.2a}$$

$$R_1 d_1 = R_2 d_2 = R_0 d_0, \tag{2.2b}$$

$$R_1^2 + d_1^2 - r_1^2 = R_2^2 + d_2^2 - r_2^2 = R_0^2 + d_0^2 - r_0^2, \tag{2.2c}$$

$$\frac{R_1^2-d_1^2}{2r_1}=\frac{R_2^2-d_2^2}{2r_2}=R_0, \tag{2.2d}$$

$$\frac{2R_1d_1r_1}{R_1^2-d_1^2}=\frac{2R_2d_2r_2}{R_2^2-d_2^2}=d_0, \tag{2.2e}$$

$$\begin{aligned} -\left(R_1^2+d_1^2-r_1^2\right)+\left(\frac{R_1^2-d_1^2}{2r_1}\right)^2+\left(\frac{2R_1d_1r_1}{R_1^2-d_1^2}\right)^2 \\ =-\left(R_2^2+d_2^2-r_2^2\right)+\left(\frac{R_2^2-d_2^2}{2r_2}\right)^2+\left(\frac{2R_2d_2r_2}{R_2^2-d_2^2}\right)^2=r_0^2. \end{aligned} \tag{2.2f}$$

where

$$r_1r_2=t_Mt_m, \quad t_M^2=(R_0+d_0)^2-r_0^2, \quad t_m^2=(R_0-d_0)^2-r_0^2. \tag{2.2g}$$

Proof. First we prove that $R_1 > r_1+d_1$. Using relations (2.1) we can write

$$(R_1-d_1)^2=R_1^2+d_1^2-2R_1d_1=2R_0(R_0+d_0)-2R_0d_0.$$

Thus

$$(R_1-d_1)^2>2R_0(R_0+r_0)-2R_0d_0-\left((R_0-d_0)^2-r_0^2\right)=(R_0+r_0)^2-d_0^2=r_1^2.$$

So from $(R_1-d_1)^2>r_1^2$ it follows $R_1>r_1+d_1$.

In the same way can be proved that $R_2>r_2+d_2$.

The other relations given by (2.2) can be also straightforwardly obtained from relations (2.1). ◊

Corollary 1.1. *Let R_i, r_i, d_i, $i=0,1,2$, be as in Th. 1. Then (R_1,r_1,d_1) and (R_2,r_2,d_2) are two solutions of the system in R, r, d given by*

$$Rd=R_0d_0, \tag{2.3a}$$

$$R^2+d^2-r^2=R_0^2+d_0^2-r_0^2, \tag{2.3b}$$

$$R^2-d^2=2R_0r. \tag{2.3c}$$

Corollary 1.2. *Let R_i, r_i, d_i, $i=0,1,2$, be as in Th. 1. Then*

$$(R_0-d_0)^2-r_0^2=(R_1-d_1)^2-r_1^2=(R_2-d_2)^2-r_2^2, \tag{2.4a}$$

$$(R_0+d_0)^2-r_0^2=(R_1+d_1)^2-r_1^2=(R_2+d_2)^2-r_2^2. \tag{2.4b}$$

In other words, minimal and maximal tangent length are the same for each triple (R_0,r_0,d_0), (R_1,r_1,d_1) and (R_2,r_2,d_2).

Now it will be shown how we can proceed using the following algorithm.

Let $i_1, \dots, i_k, i_{k+1}$ be any given integers from the set $\{1, 2\}$. Let for brevity, the sequence $i_1, \dots, i_k$ be denoted by u and the sequence $i_1, \dots, i_k, i_{k+1}$ be denoted by v. Then, if $i_{k+1} = 1$,

$$R_v^2 = R_u \left(R_u + r_u + \sqrt{(R_u + r_u)^2 - d_u^2} \right), \tag{2.5a}$$

$$r_v^2 = (R_u + r_u)^2 - d_u^2, \tag{2.5b}$$

$$d_v^2 = R_u \left(R_u + r_u - \sqrt{(R_u + r_u)^2 - d_u^2} \right). \tag{2.5c}$$

But if $i_{k+1} = 2$, then

$$R_v^2 = R_u \left(R_u - r_u + \sqrt{(R_u - r_u)^2 - d_u^2} \right), \tag{2.6a}$$

$$r_v^2 = (R_u - r_u)^2 - d_u^2, \tag{2.6b}$$

$$d_v^2 = R_u \left(R_u - r_u - \sqrt{(R_u - r_u)^2 - d_u^2} \right). \tag{2.6c}$$

For example, we have

$$\begin{aligned}
R_{1,1}^2 &= R_1 \left(R_1 + r_1 + \sqrt{(R_1 + r_1)^2 - d_1^2} \right), \\
R_{1,2}^2 &= R_1 \left(R_1 - r_1 + \sqrt{(R_1 - r_1)^2 - d_1^2} \right), \\
R_{1,2,1}^2 &= R_{1,2} \left(R_{1,2} + r_{1,2} + \sqrt{(R_{1,2} + r_{1,2})^2 - d_{1,2}^2} \right), \\
R_{1,2,2}^2 &= R_{1,2} \left(R_{1,2} - r_{1,2} + \sqrt{(R_{1,2} - r_{1,2})^2 - d_{1,2}^2} \right).
\end{aligned}$$

Let for brevity, instead of sequences $i_1, \dots, i_k$ and $i_1, \dots, i_k, i_{k+1}$ be written integers $i_1 \dots i_k$ and $i_1 \dots i_k i_{k+1}$. So, instead of $R_{1,1}$ and $R_{1,2}$ can be written R_{11} and R_{12}.

Concerning indices, let us remark that the situation is in some way connected with the fact that there are 2^k integers with k digits from the set $\{1, 2\}$. So, if $k = 3$, we have indices

$$111, 112, 121, 122, 211, 212, 221, 222$$

and we have

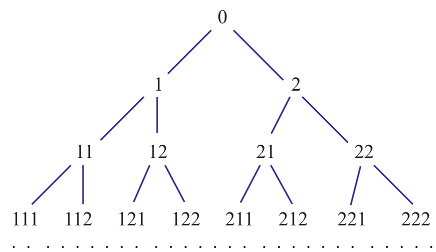

Figure 4

$$R_{111}^2 = R_{11}\left(R_{11} + r_{11} + \sqrt{(R_{11} + r_{11})^2 - d_{11}^2}\right)$$

and so on. See Fig. 4.

Now we can state the following corollary of Th. 1.

Corollary 1.3. *Let R_0, r_0, d_0 be as in Th. 1 and let R_v, r_v, d_v be given by* (2.5) *or* (2.6). *Then for every $v \in \{1, 2, 11, 12, 21, 22, 111, 112, \dots\}$ we have*

$$R_v > r_v + d_v, \tag{2.7a}$$

$$R_v d_v = R_0 d_0, \tag{2.7b}$$

$$R_v^2 + d_v^2 - r_v^2 = R_0^2 + d_0^2 - r_0^2, \tag{2.7c}$$

$$\frac{R_v^2 - d_v^2}{2r_v} = R_u, \quad \frac{2R_v d_v r_v}{R_v^2 - d_v^2} = d_u, \tag{2.7d}$$

$$-\left(R_v^2 + d_v^2 - r_v^2\right) + \left(\frac{R_v^2 - d_v^2}{2r_v}\right)^2 + \left(\frac{2R_v d_v r_v}{R_v^2 - d_v^2}\right)^2 = r_u^2. \tag{2.7e}$$

$$r_{u1} r_{u2} = t_M t_m, \quad t_M^2 = (R_u + d_u)^2 - r_u^2, \quad t_m^2 = (R_u - d_u)^2 - r_u^2. \tag{2.7f}$$

The proof is in the same way as the proof of relations (2.2).

Now we state the following conjecture.

Conjecture 1. *Let (R_0, r_0, d_0) and (R_1, r_1, d_1) be as in Th. 1. Then the following holds good. If the triple (R_0, r_0, d_0) has the property that there exists n-closure with rotation number 1 for n then the triple (R_1, r_1, d_1) has the property that there exists $2n$-closure with rotation number 1 for $2n$.*

This conjecture will be shown as a true one. First we prove the following theorem where using some algebraic procedures can be, by the way, obtained many interesting relations very useful in investigation of bicentric n-gons.

Theorem 2. *Conj. 1 is a true one for each integer n, $3 \leq n \leq 9$.*
Proof. Let t_1 be any given length (in fact positive number) such that

$$t_m \leq t_1 \leq t_M, \tag{2.8}$$

where

$$t_M^2 = (R_0 + d_0)^2 - r_0^2, \quad t_m^2 = (R_0 - d_0)^2 - r_0^2. \tag{2.9}$$

Let starting from t_1 and using triple (R_0, r_0, d_0) and formula (1.5), we get tangent lengths

$$t_1, t_2, t_3, \ldots, t_n, t_{n+1}, \tag{2.10}$$

where $t_{n+1} = t_1$. Then starting from t_1 and using triple (R_1, r_1, d_1) and formula (1.5), we get tangent lengths

$$\hat{t}_1, \hat{t}_2, \hat{t}_3, \ldots, \hat{t}_{2n-1}, \hat{t}_{2n}, \hat{t}_{2n+1}, \tag{2.11a}$$

where $\hat{t}_{2n+1} = \hat{t}_1$ and

$$\hat{t}_{2i-1} = t_i, \quad i = 1, 2, 3, \ldots, n. \tag{2.11b}$$

Here let us remark that we can take $\hat{t}_1 = t_1$ also in starting from (R_1, r_1, d_1) since by Cor. 1.2 it is valid
$(R_0 - d_0)^2 - r_0^2 = (R_1 - d_1)^2 - r_1^2, \quad (R_0 + d_0)^2 - r_0^2 = (R_1 + d_1)^2 - r_1^2.$

For brevity and simplicity of calculation we can take $t_1 = t_M$ since by Poncelet's closure theorem the following is valid. If there exists a tangent length t such that $t_m \leq t \leq t_M$ with property that there exists n-closure then there exists n-closure for every tangent length t_1 such that $t_m \leq t_1 \leq t_M$.

(**i**$_1$) *The proof that* $t_2 = \hat{t}_3$. Starting from $t_1 = t_M$ and using formula (1.5) it can be found that

$$t_2 = \frac{R_0 - d_0}{R_0 + d_0} t_M, \tag{2.12a}$$

$$\hat{t}_2 = \frac{R_1 - d_1}{R_1 + d_1} t_M. \tag{2.12b}$$

Now by the rule given by sequence (1.12) we have

$$\hat{t}_3 = \frac{(R_1^2 - d_1^2)^2 \hat{t}_2^2 - r_1^2 D_2}{\left(r_1^2 + \hat{t}_2^2\right)^2 t_M}, \tag{2.13}$$

where $D_2 = (t_M^2 - \hat{t}_2^2)(\hat{t}_2^2 - t_m^2)$, that is, notation (1.8) is used. It is easy to show that $\hat{t}_3$ can be written as

$$\hat{t}_3 = \frac{\left(R_1^2 - d_1^2\right)^2 - 4R_1 d_1 r_1^2}{\left(R_1^2 - d_1^2\right)^2 + 4R_1 d_1 r_1^2} t_M. \tag{2.14}$$

Since by (2.2d), (2.2e) and (2.2f) hold relations

$$R_0 = \frac{R_1^2 - d_1^2}{2r_1}, \quad d_0 = \frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}, \tag{2.15a}$$

$$r_0 = \sqrt{-\left(R_1^2 + d_1^2 - r_1^2\right) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 d_1 r_1}{R_1^2 - d_1^2}\right)^2}, \tag{2.15b}$$

it is easy to see that t_2 can be written as

$$t_2 = \frac{\left(R_1^2 - d_1^2\right)^2 - 4R_1 d_1 r_1^2}{\left(R_1^2 - d_1^2\right)^2 + 4R_1 d_1 r_1^2} t_M.$$

This proves that $\hat{t}_3 = t_2$.

Here let us remark that the proof that $\hat{t}_3 = t_2$ is not difficult even by hand (without using computer algebra). It will not be so in the proofs that $\hat{t}_5 = t_3$, $\hat{t}_7 = t_4$ and so on.

For brevity, in the expressions of tangent lengths t_3, t_4, t_5 we use quantities p_0 and q_0 given by

$$p_0 = \frac{R_0 + d_0}{r_0}, \quad q_0 = \frac{R_0 - d_0}{r_0}.$$

Also, for tangent lengths $\hat{t}_2, \ldots, \hat{t}_9$ we use

$$p = \frac{R_1 + d_1}{r_1}, \quad q = \frac{R_1 - d_1}{r_1}.$$

($\mathbf{i}_2$) *The proof that* $\hat{t}_5 = t_3$. Starting from t_2 given by (2.12a) and using rule (1.12) we get

$$t_3 = \frac{p_0^2 q_0^2 - p_0^2 + q_0^2}{p_0^2 q_0^2 + p_0^2 - q_0^2} t_M. \tag{2.16a}$$

Now starting from $\hat{t}_3$ given by (2.14) and using rule (1.12) we get

$$\hat{t}_4 = \frac{q\left(p^4 q^4 + 2p^4 q^2 - 3p^4 - 2p^2 q^4 + 2p^2 q^2 + q^4\right)}{p\left(p^4 q^4 - 2p^4 q^2 + p^4 + 2p^2 q^4 + 2p^2 q^2 - 3q^4\right)} t_M, \tag{2.16b}$$

$$\hat{t}_5 = \frac{\nu_5}{\delta_5} t_M, \tag{2.16c}$$

where

$$\nu_5 = p^8q^8 - 4p^8q^6 + 6p^8q^4 - 4p^8q^2 + p^8 + 4p^6q^8 + 4p^6q^6 - 4p^6q^4 - \\ - 4p^6q^2 - 10p^4q^8 + 4p^4q^6 + 6p^4q^4 + 4p^2q^8 - 4p^2q^6 + q^8,$$

$$\delta_5 = p^8q^8 + 4p^8q^6 - 10p^8q^4 + 4p^8q^2 + p^8 - 4p^6q^8 + 4p^6q^6 + 4p^6q^4 - \\ - 4p^6q^2 + 6p^4q^8 - 4p^4q^6 + 6p^4q^4 - 4p^2q^8 - 4p^2q^6 + q^8.$$

Replacing R_0, d_0, r_0 in t_3 by expressions for R_0, d_0, r_0 given by (2.15) we find that $\hat{t}_5 = t_3$.

In quite the same way can be proceeded and found that $\hat{t}_7 = t_4$, $\hat{t}_9 = t_5$, ..., $\hat{t}_{17} = t_9$. Thus for $n = 9$ and $2n = 18$ the sequences (2.10) and (2.11a) can be written as

$$t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9$$
$$\hat{t}_1, \hat{t}_3, \hat{t}_5, \hat{t}_7, \hat{t}_9, \hat{t}_{11}, \hat{t}_{13}, \hat{t}_{15}, \hat{t}_{17}$$

where

$$\hat{t}_i = t_{i - \frac{i-1}{2}}, \quad i = 1, 3, 5, \ldots, 17$$

and $t_1, t_2, \ldots, t_9$ are tangent lengths of bicentric 9-gon and $\hat{t}_1, \hat{t}_3, \ldots, \hat{t}_{17}$ are tangent lengths of bicentric 18-gon which have odd indices. (Of course, for every integer $n \geq 3$ each of the sequence $t_1, t_2, \ldots, t_n$ and $\hat{t}_1, \hat{t}_3, \ldots, \hat{t}_{2n-1}$ has n member.)

We found that for the proof that, say, $\hat{t}_{19} = t_{10}$ needs a computer with larger capacity than usual (standard) computer has. Here let us remark that Conj. 1 will be later proved generally using Th. 3 and Th. A and Th. B. ◊

Now, concerning sequences (2.10) and (2.11a), let us point out some of n-closures. Let the triple (R_0, r_0, d_0) has 3-closure, that is after t_1, t_2, t_3 appears $t_4 = t_1$. Then the triple (R_1, d_1, r_1) has 6-closure since $\hat{t}_7 = t_4 = t_1$. Also can be easily seen that for $n = 4, 5, 6, 7, 8, 9$ it is valid. If the triple (R_0, r_0, d_0) has n-closure then the triple (R_1, r_1, d_1) has $2n$-closure.

Here are some important corollaries of Th. 2 where we restrict ourselves to $3 \leq n \leq 9$.

Corollary 2.1. *Let (R_0, r_0, d_0) and (R_1, r_1, d_1) be as in Th. 1. Let the triple (R_1, r_1, d_1) has the property that there exists $2n$-closure with rotation number 1 for $2n$. Then the triple (R_0, r_0, d_0) has the property that there exists n-closure with rotation number 1 for n.*

Thus, if and only if the triple (R_0, r_0, d_0) has the property that there exists n-closure with rotation number 1 for n then the triple (R_1, r_1, d_1) has the property that there exists $2n$-closure with rotation number 1 for $2n$.

Corollary 2.2. *Let $\hat{t}_2, \hat{t}_4, \ldots, \hat{t}_{2n}$, be tangent lengths with even indices in the sequence* (2.11a)*. Then these tangent lengths can be obtained starting from $\hat{t}_2$ and using triple* (R_0, r_0, d_0)*.*

Proof. Starting from $\hat{t}_2$ instead of $\hat{t}_1$ we get tangent lengths

$$\hat{t}_2, \hat{t}_3, \hat{t}_4, \ldots, \hat{t}_{2n-1}, \hat{t}_{2n}, \hat{t}_1$$

which are the same as tangent lengths given by (2.11a), but now $\hat{t}_2, \hat{t}_4$, $\ldots$, $\hat{t}_{2n}$ are first, third, $\ldots$,$(2n-1)$-th (as before $\hat{t}_1, \hat{t}_3, \ldots, \hat{t}_{2n-1}$). $\Diamond$

Thus there are bicentric n-gons $A_1 \ldots A_n$ and $B_1 \ldots B_n$ from the class $C_n(R_0, r_0, d_0)$ such that $\hat{t}_1, \hat{t}_3, \ldots, \hat{t}_{2n-1}$ are tangent lengths of the n-gon $A_1 \ldots A_n$ and $\hat{t}_2, \hat{t}_4, \ldots, \hat{t}_{2n}$ are tangent lengths of the n-gon $B_1 \ldots B_n$. Such n-gons can be called *conjugate* bicentric n-gons. (More about this will be later.)

Before stating some other corollaries of Th. 2 here are some examples concerning tangent lengths in the sequence (2.11a).

Example 1. Let $R_0 = 5$, $r_0 = 2.1$, $d_0 = 2$. Then the triple (R_0, r_0, d_0) is a solution of Euler's relation (1.1d). Since in this case $t_m = 2.142428529\ldots$ and $t_M = 6.67757441\ldots$ we can take for t_1, say, $t_1 = 4$. Using formula (1.5) we get

$$t_2 = 2.257285250\ldots, \quad t_3 = 5.973973936\ldots.$$

For bicentric hexagon, where $R_1 = 8.340410221\ldots$, $r_1 = 6.812488532\ldots$, $d_1 = 1.198981792\ldots$, we can also take $\hat{t}_1 = t_1 = 4$ (since holds (2.4)). Using formula (1.5) we get

$$\hat{t}_2 = 2.394758676\ldots, \quad \hat{t}_3 = t_2, \quad \hat{t}_4 = 3.576556479\ldots,$$

$$\hat{t}_5 = t_3, \quad \hat{t}_6 = 6.337801531\ldots.$$

Example 2. Let $R_0 = 7$, $r_0 = 4.8$, $d_0 = 1$. Then the triple (R_0, r_0, d_0) is a solution of Fuss' relation (1.1a). Since in this case $t_m = 3.6$, $t_M = 6.4$ we can take $t_1 = 5$. Using formula (1.5) we get

$$t_2 = 3.610778912\ldots, \quad t_3 = 4.608, \quad t_4 = 6.380894692\ldots.$$

For bicentric octagon, where $R_1 = 12.841450671\ldots$, $r_1 = 11.757550765\ldots$, $d_1 = 0.545109752\ldots$, we can also take $\hat{t}_1 = t_1 = 5$. Using formula (1.5) we get

$$\hat{t}_2 = 4.043395991\ldots,\quad \hat{t}_3 = t_2,\quad \hat{t}_4 = 3.814401072\ldots,\quad \hat{t}_5 = t_3$$
$$\hat{t}_6 = 5.698180452\ldots,\quad \hat{t}_7 = t_4,\quad \hat{t}_8 = 6.040266757\ldots.$$

Corollary 2.3. *If and only if $F_n(R_0, r_0, d_0) = 0$ then $F_{2n}(R_1, r_1, d_1) = 0$.*

Proof. Let R_1, r_1, d_1 in $F_{2n}(R_1, r_1, d_1) = 0$ be replaced, respectively by

$$\sqrt{R_0\left(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2}\right)},\quad \sqrt{(R_0 + r_0)^2 - d_0^2},$$

$$\sqrt{R_0\left(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2}\right)}$$

then we get relation $F_n(R_0, r_0, d_0) = 0$.

Conversely, if R_0, r_0, d_0 in $F_n(R_0, r_0, d_0)$ be replaced by expressions for R_0, r_0, d_0 given by (2.15) we get Fuss; relations $F_{2n}(R_1, r_1, d_1) = 0$.

It can be easily check using computer algebra. In the case when n is small it is not difficult to check even by hand, without using computer algebra. ♢

Notice 1. As can be seen, Cor. 2.3 can be very useful concerning Fuss' relations.

Corollary 2.4. *Let $n \geq 4$ be an even integer and let the triple (R_0, r_0, d_0) has the property that for every bicentric n-gon $A_1 \ldots A_n$ from the class $C_n(R_0, r_0, d_0)$ it is valid*

$$t_i t_{i+\frac{n}{2}} = t_m t_M,\quad i = 1, \ldots, \frac{n}{2}, \tag{2.17}$$

where $t_1, \ldots, t_n$ are tangent lengths of the n-gon $A_1 \ldots A_n$ and t_m and t_M are given by (2.9)*. Then triple (R_1, r_1, d_1) also has this property, that is, for every bicentric $2n$-gon $B_1 \ldots B_{2n}$ from the class $C_{2n}(R_1, r_1, d_1)$ it is valid*

$$u_i u_{i+n} = t_m t_M,\quad i = 1, \ldots, n, \tag{2.18}$$

where $u_1, \ldots, u_{2n}$ are tangent lengths of the $2n$-gon $B_1 \ldots B_{2n}$.

Proof. This corollary follows from Cor. 2.1 and Cor. 2.2. See Fig. 5a. If there are $\frac{n}{2}$ vertices between A_i and $A_{i+\frac{n}{2}}$ then there are n vertices between B_i and B_{i+n}. See also Fig. 5b where $2n = 8$. ♢

Here are some examples.

Example 3. Let $n = 4$. Then, as it is well known, relations (2.17) are valid. Thus, for $2n = 8, 16, 32, \ldots$ relations (2.18) are also valid.

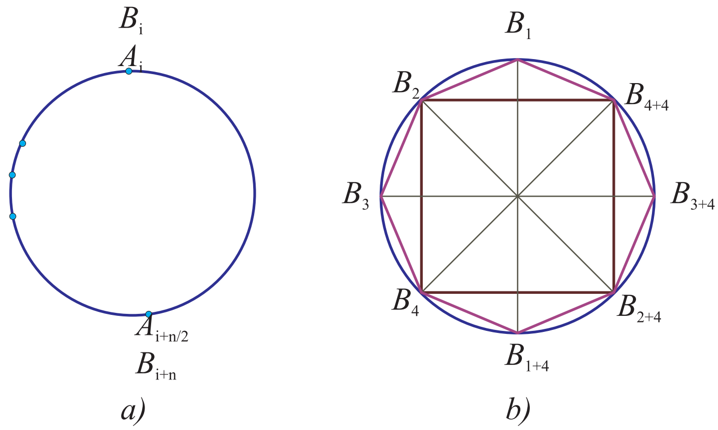

Figure 5

Example 4. Let $n=6$. The proof that holds (2.17) can be as follows.

In the case when $n=6$ it is not difficult to show that tangent lengths are given by

$$t_m \le t_1 \le t_M, \quad t_3 = \frac{(R_0^2-d_0^2)t_1 - r_0\sqrt{D_1}}{r_0^2+t_1^2}, \qquad t_5 = \frac{(R_0^2-d_0^2)t_1 + r_0\sqrt{D_1}}{r_0^2+t_1^2},$$

$$t_2 = \frac{(R_1^2-d_1^2)t_1 - r_1\sqrt{D_1}}{r_1^2+t_1^2}, \qquad t_4 = \frac{(R_1^2-d_1^2)t_3 + r_1\sqrt{D_3}}{r_1^2+t_3^2},$$

$$t_6 = \frac{(R_1^2-d_1^2)t_1 + r_1\sqrt{D_1}}{r_1^2+t_1^2}.$$

To prove that, say $t_2t_5 = t_mt_M$, we need to prove that

$$t_2t_5 = \frac{(R_1^2-d_1^2)t_1 - r_1\sqrt{D_1}}{r_1^2+t_1^2} \cdot \frac{(R_0^2-d_0^2)t_1 + r_0\sqrt{D_1}}{r_0^2+t_1^2} = t_mt_M.$$

Since hold relations $R_1^2 - d_1^2 = 2R_0r_1$ and $2R_0r_0 = R_0^2 - d_0^2$ given by (2.3c) and (1.1d) we can write

$$\begin{aligned} t_2t_5 &= \frac{r_0r_1\,(4R_0^2t_1^2 - D_1)}{(r_1^2+t_1^2)(r_0^2+t_1^2)} \\ &= \frac{r_0r_1\,(r_1^2+t_1^2)\,(r_0^2+t_1^2)}{(r_1^2+t_1^2)(r_0^2+t_1^2)} = r_0r_1 = t_mt_M. \quad \text{(See (2.7f).)} \end{aligned}$$

Now, let for brevity in the following expression, tangent length $\hat{t}_1, \hat{t}_3, \ldots, \hat{t}_{2n-1}$ and $\hat{t}_2, \hat{t}_4, \ldots, \hat{t}_{2n}$ in the sequence (2.11a) be called *conjugate tangent lengths* concerning the same triple (R_1, r_1, d_1).

The following conjecture is strongly suggested.

Conjecture 2. *Let $t_1, t_3, \ldots, t_{2n-1}$ and $t_2, t_4, \ldots t_{2n}$ be conjugate tangent lengths concerning triple (R_1, r_1, d_1) and let also $u_1, u_3, \ldots, u_{2n-1}$ and $u_2, u_4, \ldots, u_{2n}$ be conjugate tangent lengths concerning triple (R_1, r_1, d_1). Then*

$$\left(\sum_{i=1}^{n} t_{2i-1}\right)\left(\sum_{i=1}^{n} t_{2i}\right) = \left(\sum_{i=1}^{n} u_{2i-1}\right)\left(\sum_{i=1}^{n} u_{2i}\right). \tag{2.19}$$

We have found that this conjecture is a true one for many numerical examples and that using computer algebra it is not difficult to prove it generally for $2n = 6$ and $2n = 8$. In the case when $2n = 6$ we have found that

(2.20a)

$$\left(\sum_{i=1}^{3} t_{2i-1}\right)\left(\sum_{i=1}^{3} t_{2i}\right) = $$
$$= 5t_M t_m + 2t_M\sqrt{2R_0r_0 + 2r_0d_0 - r_0^2} + 2t_m\sqrt{2R_0r_0 - 2r_0d_0 - r_0^2}$$

where $t_M^2 = (R_0 + d_0)^2 - r_0^2, \quad t_m^2 = (R_0 - d_0)^2 - r_0^2$.

In the case when $2n = 8$ it holds

$$\left(\sum_{i=1}^{4} t_{2i-1}\right)\left(\sum_{i=1}^{4} t_{2i}\right) = 4\sqrt{(R_0^2 - d_0^2)(3R_0^2 - d_0^2 + 2r_0^2)}. \tag{2.20b}$$

Conjugate tangent lengths here defined are very connected with conjugate bicentric n-gons whose definition is based on Cor. 2.2. Namely, if $A_1 \ldots A_n$ is a given bicentric n-gon from the class $C_n(R_0, r_0, d_0)$ and $t_1, t_3, \ldots, t_{2n-1}$ are its tangent lengths,then conjugate bicentric n-gon to the n-gon $A_1 \ldots A_n$ can be found by calculation of tangent lengths $t_2, t_4, \ldots, t_{2n}$ such that t_2 be calculated using t_1 and the triple (R_1, r_1, d_1), then using triple (R_0, r_0, d_0) can be calculated tangent lengths $t_4, \ldots, t_{2n}$.

So for every integer $n \geq 3$ for which Conj. 2 is true the following is valid. Any two conjugate bicentric n-gons from the class $C_n(R_0, r_0, d_0)$ have the same product of theirs perimeters. Thus, they also have the same product of theirs areas. From this it is clear that bicentric n-gon with maximal perimeter is conjugate to the bicentric n-gon with minimal perimeter. Of course, both of them must be from the same class $C_n(R_0, r_0, d_0)$.

The following conjecture is also strongly suggested.

Conjecture 3. *Let $A_1 \dots A_n$ and $B_1 \dots B_n$ be polygons from the class $C_n(R_0, r_0, d_0)$ such that one has maximal perimeter and the other has minimal perimeter. Then each of these polygons is axially symmetric in relation to the axis OI where O is center of circumcircle and I is center of incircle. That one has maximal perimeter which has maximal tangent length t_M.*

The proof for $n = 3$ is given in [4]. The proof for $n = 4$ can be as follows. Let $A_1\,A_2\,A_3\,A_4$ be a bicentric quadrilateral from the class $C_4(R_0, r_0, d_0)$. Denote by $P(t_1)$ its perimeter. Thus, $P(t_1) = t_1 + t_2 + t_3 + t_4$, that is,

$$P(t_1) = t_1 + \frac{(R_0^2 - d_0^2)t_1 + r_0\sqrt{D_1}}{r_0^2 + t_1^2} + \frac{r^2}{t_1} + \frac{(R_0^2 - d_0^2)t_1 - r_0\sqrt{D_1}}{r_0^2 + t_1^2}$$

or

$$P(t_1) = t_1 + \frac{2(R_0^2 - d_0^2)t_1}{r_0^2 + t_1^2} + \frac{r_0^2}{t_1}.$$

From $\frac{\mathrm{d}}{\mathrm{d}t_1}P(t_1) = 0$ we obtain the equation which can be written as

$$\left(t_1^2 - r_0^2\right)\left(t_1^4 - 2\left(R_0^2 - r_0^2 - d_0^2\right)t_1^2 + r_0^4\right) = 0.$$

Its positive roots are given by

$$(t_1)_1 = r_0, \quad (t_1)_{2,3} = R_0^2 - d_0^2 - r_0^2 \pm \sqrt{(R_0^2 - r_0^2 - d_0^2)^2 - r_0^4}.$$

It can be found that $\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}P(t_1) < 0$ for $(t_1)_1$ and $\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}P(t_1) > 0$ for both of $(t_1)_2$ and $(t_1)_3$.

Here let us remark that the first quadrilateral has tangent lengths t_M, r_0, t_m, r_0 and the second has tangent lengths $(t_1)_2$, $(t_1)_3$, $(t_1)_2$, $(t_1)_3$.

It seems that Conj. 3 can be without difficulties proved for some other small n, say, for $n = 6$ and $n = 8$.

Now we prove the following theorem as one of the main results in the article.

Theorem 3. *Let R_0, r_0, d_0 be any given positive numbers such that $R_0 > r_0 + d_0$ and let R_1, r_1, d_1 be positive numbers given by*

(2.21a)

$$R_1^2 = R_0\left(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \quad r_1^2 = (R_0 + r_0)^2 - d_0^2,$$

(2.21b)
$$d_1^2 = R_0\left(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2}\right).$$

Let $\hat{R}_1$, $\hat{r}_1$, $\hat{d}_1$ *be given by*

$$\hat{R}_1 = \frac{1}{c}R_1, \quad \hat{r}_1 = \frac{1}{c}r_1, \quad \hat{d}_1 = \frac{1}{c}d_1, \tag{2.22}$$

where

$$c = \sqrt{\frac{R_0 + r_0 + \sqrt{(R_0+r_0)^2 - d_0^2}}{R_0}}, \tag{2.23}$$

that is, $c = \frac{R_1}{R_0}$ *and* $\frac{1}{c} = \frac{R_0}{R_1}$. *(Thus* $\hat{R}_1 = R_0$*). Further, let* C_1, C_2 *and* $\hat{K}_1$, $\hat{K}_2$ *be circles in the same plane such that*

R_0*: radius of* C_1, r_0*: radius of* C_2,
d_0*: distance between centers of* C_1 *and* C_2,
$\hat{R}_1$*: radius of* $\hat{K}_1$, $\hat{r}_1$*: radius of* $\hat{K}_2$,
$\hat{d}_1$*: distance between centers of* $\hat{K}_1$ *and* $\hat{K}_2$,

where $\hat{K}_1$ *is concentric to* C_1 *and equal to* C_1 *(since* $\hat{R}_1 = R_0$*).*

Finally, let $P(u,v)$ *be any given point on the circle* C_1 *and let by* $|PT_1|$ *and* $|PT_2|$ *be denoted, respectively, lengths of the tangents drawn from* P *to* $\hat{K}_2$ *and from* P *to* C_2*. Then*

$$c\,|PT_1| = |PT_2|\,. \tag{2.24}$$

Proof. The proof easily follows from Th. A and Th. B. ◊

Here is an example.

Example 5. Let $R_0 = 6$, $r_0 = 4$, $d_0 = 1$ and let $P(-2, 5.65685425\ldots)$. Then

$\hat{R}_1 = 6$, $\hat{r}_1 = 5.456612823\ldots$, $\hat{d}_1 = 0.300753772\ldots$, $c = 1.823452514\ldots$

$$x_1 = 0.733417674\ldots, \qquad y_1 = 5.439432451\ldots,$$
$$x_2 = 2.58870939\ldots, \qquad y_2 = 3.670967512\ldots,$$
$$|PT_1| = 2.742051134\ldots, \quad |PT_2| = 5, \quad c\,|PT_1| = |PT_2|\,.$$

Now we state some corollaries of Th. 3.

Theorem 4. *From Th. 3 and Th. A and Th. B it follows Conj. 1.*

Proof. We can without loss of generality consider the case when, say, $2n = 8$, since there exists a complete analogy. So we start from a triple (R_0, r_0, d_0) which is a (positive) solution of Fuss' relation $F_4(R, r, d) = 0$. In this case, using Th. B stated in Introduction, we have the situation shown in Fig. 6, where

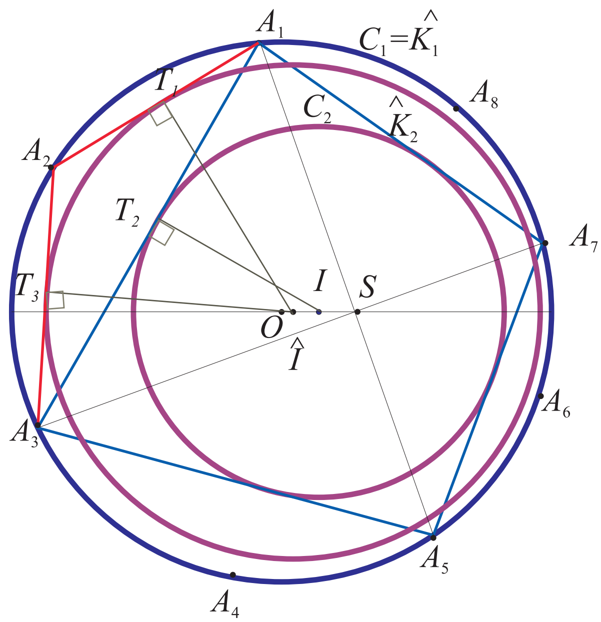

Figure 6

$$R_0 = 7, \quad r_0 = 4.8, \quad d_0 = 1, \tag{2.25}$$

(2.26)
$$R_1 = 12.841450672\ldots, \quad r_1 = 11.757550765\ldots, \quad d_1 = 0.545109755\ldots,$$

$$\hat{R}_1 = R_0, \quad \hat{r}_1 = 6.546477218\ldots, \quad \hat{d}_1 = 0.303511221\ldots, \tag{2.27}$$

$$c = 1.796011867\ldots, s = 1.96. \tag{2.28}$$

The following notation is used.

O = center of C_1, I = center of C_2, $\hat{I}$ = center of $\hat{K}_2$, $S(s, 0)$ is the characteristic point of the circles C_1 and C_2 and also of the circles $\hat{K}_1$ and $\hat{K}_2$. The quadrilateral $A_1A_3A_5A_7$ is from the class $C_4(R_0, r_0, d_0)$ where $R_0 = 7$, $r_0 = 4.8$, $d_0 = 1$. (For convenience its vertices are denoted by A_1, A_3, A_5, A_7 instead of A_1, A_2, A_3, A_4.)

We have to prove that straight lines A_1T_1 and A_3T_3 intersect in a point of the circle C_1 between vertices A_1 and A_3. To prove this we have to prove that the situation shown in Fig. 7 is impossible. The proof is as follows.

Let by M be denoted intersection of the straight line A_1T_1 and the

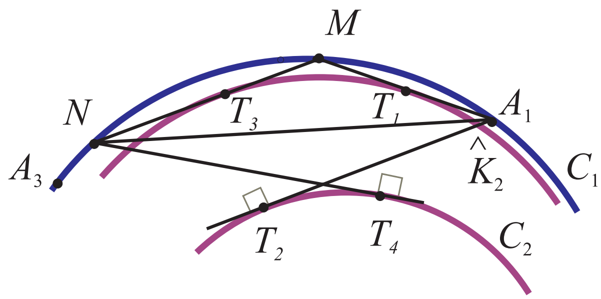

Figure 7

circle C_1 and let by N be denoted intersection of the straight line NT_3 and the circle C_1. Then by Th. A it is valid

$$c\,|A_1T_1| + c\,|NT_3| = |A_1N|\,,$$

and by Th. 3 we have

$$c\,|A_1T_1| = |A_1T_2|\,, \quad c\,|NT_3| = |NT_4|\,.$$

Obviously, $|A_1T_2| + |NT_4| > |A_1N|$.

In the same way can be concluded that N can not be between A_3 and A_5. Thus $[A_1A_2]$ and $[A_2A_3]$ are tangential segments to the circle $\hat{K}_2$.

Now we can proceed and easily conclude that between vertices A_3 and A_5 there exists a point A_4 such that $[A_3A_4]$ and $[A_4A_5]$ are tangential segment to the circle $\hat{K}_2$. And so on.

Thus starting from bicentric quadrilateral $A_1A_3A_5A_7$ we have obtained bicentric octagon $A_1 \ldots A_8$ which has the following properties.

a) Each of its main diagonals $[A_iA_{i+4}]$, $i = 1, 2, 3, 4$, contain the characteristic point $S(s, 0)$ since the circles C_1 and $\hat{K}_2$ determine the same characteristic point as the circles C_1 and C_2.

b) Let by $\bar{t}_1, \ldots, \bar{t}_8$ be denoted tangent lengths of the octagon $A_1 \ldots A_8$. Then by Th. B it is valid

$$\bar{t}_i\bar{t}_{i+4} = \frac{t_M}{c} \cdot \frac{t_m}{c}, \quad i = 1, 2, 3, 4,$$

where t_M and t_m are maximal and minimal tangent lengths of the class $C_4(R_0, r_0, d_0)$.

Since $c(\hat{R}_1, \hat{r}_1, \hat{d}_1) = (R_1, r_1, d_1)$ there exists a bicentric octagon $\hat{A}_1 ... \hat{A}_8$ from the class $C_8(R_1, r_1, d_1)$ such that its tangent lengths $\hat{t}_1, ..., \hat{t}_8$ have the property that

$$\hat{t}_i = c\bar{t}_i, \quad i = 1, \ldots, 8,$$

where $\bar{t}_1, \ldots, \bar{t}_8$ are tangent lengths of the octagon $A_1 \ldots A_8$. Thus

$$\hat{t}_i = t_i, \quad i = 1, 3, 5, 7 \quad \text{and} \quad \hat{t}_i = t_i, \quad i = 2, 4, 6, 8,$$

where t_1, t_3, t_5, t_7 are tangent lengths of the bicentric quadrilateral $A_1A_3A_5A_7$ and t_2, t_4, t_6, t_8 are tangent lengths of the bicentric quadrilateral $A_2A_4A_6A_8$ (which is not drawn in Fig. 6). The proof that $[A_{i+1}A_{i+3}]$, $i = 1, 3, 5, 7$ are tangential segments to the circle C_2 can be as follows. See Fig. 8. Since $[A_2A_3]$ and $[A_3A_4]$ are tangential segments to the circle $\hat{K}_2$ and by Th. A it is valid

$$|A_2A_4| = c(\bar{t}_2 + \bar{t}_4),$$

the situation shown in Fig. 8 is impossible. The segment $[A_2A_4]$ must be a tangential segment to the circle C_2 since by Th. 3 must be $c\bar{t}_2 = |A_2T_2|$, $c\bar{t}_4 = |A_4T_4|$. See relation (2.24).

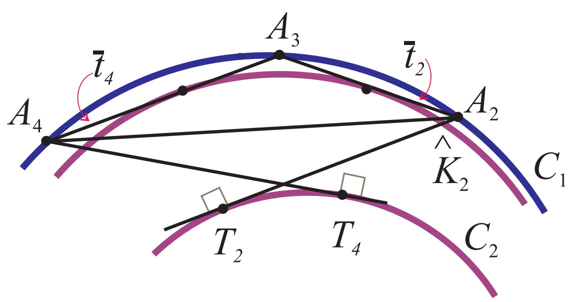

Figure 8

Of course, starting now from bicentric octagon $\hat{A}_1 \ldots \hat{A}_8$ we can in exactly the same way obtain a bicentric 16-gon from the class $C_{16}(R_{11}, r_{11}, d_{11})$ such that its tangent lengths $u_1, \ldots, u_{16}$ have the property that

$$u_{2i-1} = \hat{t}_i, \quad i = 1, \ldots, 8.$$

There is a complete analogy with the case when we start from bicentric quadrilateral.

So the proof that Conj. 1 is a true one follows from Th. 3 and Th. A and Th. B can be accepted. $\Diamond$

As an important corollary of Th. 4 we have the following:

Theorem 5. *Let $n \geq 3$ be an integer such that is known Fuss' relation $F_n(R, r, d) = 0$ for bicentric n-gons. Let (R_0, r_0, d_0) be any positive solution of the relation $F_n(R, r, d) = 0$ and let R_0, r_0, d_0 in $F_n(R_0, r_0, d_0) = 0$ be replaced by expressions given by* (2.15)*. Then obtained equality can be written as*

$$F_{2n}(R_1, r_1, d_1) = 0,$$

where $F_{2n}(R, r, d) = 0$ is Fuss' relation for bicentric $2n$-gons.

For example, if $n = 3$, then using relation (1.1d) can be easily (even by hand, without using computer algebra) obtained relation (1.1b).

Notice 2. In [6] one algorithm is given which states how can be obtained Fuss' relation $F_n(R, r, d) = 0$ for any odd $n \geq 3$. From this and Th. 5 it is clear one way how can be obtained Fuss' relation for any even $n \geq 4$. Many interesting and important facts and relations in this connection are established in the present article. These are a great source for many further investigations of bicentric n-gons and their practical use.

Acknowledgement. The author wishes to express thankfulness to the referee for many useful remarks and proposals.

References

[1] FUSS, N.: De quadrilateris quibus circulum tam inscribere quam cicumscribere licet, *NAAPS* 1792 (*Nova acta*) t. **X** (1797), 103–125 (14. VII. 1794).

[2] FUSS, N.: De poligonis simmetrice irregularibus calculo simul inscriptis et circumscriptis, *NAAPS* 1792 (*Nova acta*) t. **XIII** (1802), 168–189 (19. 04. 1798).

[3] PONCELET, J. V.: *Traité des propriétés projectives des figures*, Paris, 1865 (first ed. in 1822).

[4] RADIĆ, M.: Extreme Areas of Triangles in Poncelet's Closure Theorem, *Forum Geometricorum* **4** (2004), 23–26.

[5] RADIĆ, M. and KALIMAN, Z.: About one relation concerning two circles where one is inside of the other, *Math. Maced.* **3** (2005), 45–50.

[6] RADIĆ, M.: An improved method for establishing Fuss' relations for bicentric polygons, *Comptes Rendus mathèmatique* **348** 7–8 (2010), 415–417.

[7] RADIĆ, M.: About one characteristic point concerning two circles where one is inside of the other, *Journal of Geometry* (to appear).

[8] RADIĆ, M.: Certain relation concerning bicentric polygons and 2-parametric presentation of Fuss' relation, *Mathematica Pannonica* 20/2 (2009), 219–248.