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THE POWER AND THE LIMITS OF THE ABACUS

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Abstract: The abacus is a well-known calculating tool with a limited number of placeholders for digits of operands and results. Given a number of rods n of the abacus, a chosen basis of the number system and the first operand a, this paper deals with the possible values of the other operand b in the four basic arithmetic operations performed with integers on the abacus.

1. Introduction

The abacus is an ideal tool to demonstrate the basic arithmetic operations in various number systems. This, probably first and definitely the most well-known calculating tool, is still widely used and has survived through centuries due to its simplicity, adaptability and creativity which it provides. Certain historical facts can be found in [3] and in an interesting manuscript [2].

We take a moment to recall that every abacus consists of a frame

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with several rods, with equal number of beads on each rod (actually, there is one rod with few beads less on the Russian abacus, but that will not be of much importance for us). Calculations are made by moving beads, representing digits in that way, while beads on a single rod represent a single digit. Every basic arithmetic operation can be carried out on the abacus and is performed very similarly to the standard pencil-and-paper algorithm. In fact, every pencil-and-paper algorithm can be transferred to abacus, after possible slight modifications. Despite known ways of making such modifications, users may harmonize the calculating procedure according to their requirements and wishes (this property also develops a higher level of creativity).

The number systems representable on an abacus depend on the number of beads on a rod. For instance, the basis of a number system representable on a traditional Chinese abacus is less than or equal to 16, while such basis on a traditional Japanese abacus is not greater than 10. From now on, when working with the numbers in base B, we assume that there are at least $B-1$ beads on each rod.

It is worth pointing out that the abacus has numerous pedagogical applications and therefore it is widely used as a teaching tool, especially in learning arithmetic, mental calculation and manipulating with various number systems. Also, some special sorts of abacus are still used by the blind individuals as a decisive help in learning arithmetic. Some of the above mentioned subjects are described in [5] and [6]. For a deeper discussion we also refer the reader to [1].

Besides many advantages, a major drawback while calculating on abacus is the insufficiency of space needed for implementation of some calculating procedures. A problem that naturally arises is to find explicit upper bounds for a given arithmetic operation, on an abacus with arbitrary (but fixed) number of rods and the fixed first operand. Since abacus allows computation in various number systems, this problem can be generalized by considering operands in any basis (assuming that they are representable on the rods of the abacus). Surprisingly, this was not done before, except by the second-named author et al. in [4] for basis 10, where division was considered only in the special case of a traditional Chinese 13-rod abacus. Since on each abacus a single rod can be used to represent at most one digit, we will not specify the type of the abacus that we use. Also, our inquiry could be viewed as a problem of determining a minimal space required for some integer computations, which seems to be of independent interest, especially in some aspects of the number theory and computer science.

For the sake of completeness, before deriving the constraints on some basic arithmetic operation, we shortly describe its usage on the abacus. We shall describe all the considered operations under the natural assumption that the person making the calculations is not memorizing any of the data.

Given the number of rods on an abacus and the first operand a, for each basic arithmetic operation we have considered the maximal values of the second operand b such that it is performable on our abacus. While the maximal values for b are quite obvious in the case of addition and subtraction, they are not so obvious (although relatively simple to prove) for multiplication. The case of division is by far the most complicated case, with many special cases and answers that are neither obvious nor simple to formulate. The most of the paper is devoted to that problem, in which appear some extremely non-trivial cases of numbers with so-called critical number of digits, which have to be studied separately. We have completely solved the problem of finding the maximal value of the divisor b for a with at most N_n digits, N_n being the smallest number of digits for a such that a cannot be divided by all smaller divisors on the n-rod abacus. Although there are still several open questions to be solved, the most usual 13-rod case, which has turned up several times as a specially complicated, has been completely solved. It is expected that the remaining situations can be solved in pretty much the same way. This needs longer case-by-case examination, and we plan to write it elsewhere.

We now describe the content of the paper in more detail. In the second section we recall some basic notations that will be used through the paper. In the third section we briefly sketch addition and subtraction on the abacus and, as a motivation, obtain explicit bounds for this operations. Sec. 4 is devoted to the study of the limitations of multiplication using abacus. Sec. 5 provides a detailed exposition of the limitations which arise during the division.

2. Notation

The base of the number system we are working in shall be denoted by B ($B \in \mathbb{N}$, $B > 1$); accordingly, the digits of a number represented in the base B number system are $0, 1, \ldots, B-1$. The number of digits of a number a in base B shall be denoted by $\delta_B(a)$. If a number a is represented in the base B number system, the digits shall be denoted by a_i, $i = 0, 1, \ldots, m = \delta_B(a) - 1$ and we shall write

$$a = (a_m a_{m-1} \ldots a_2 a_1 a_0)_B.$$

If a number is represented by its digits without reference to base it is understood, as usual, that the chosen base is 10. We shall frequently make use of the fact that for a number a that has $\delta_B(a)$ digits with respect to base B the following inequalities hold:

$$B^{\delta_B(a)-1} \leq a \leq B^{\delta_B(a)} - 1.$$

The number of the abacus rods shall be denoted by n ($n \in \mathbb{N}$). Note that a number a is representable in base B on an abacus with n rods if (and only if) $\delta_B(a) \leq n$. Throughout the text we use the phrases "a can be multiplied with b" and "a can be divided by b" in the sense that the corresponding arithmetical operation is feasible on the abacus for a given number n of rods (i.e. under "a can be divided by b" we don't mean that a is a multiple of b).

3. Addition and subtraction

Let a and b be two positive integers. In the case of the calculation of $a - b$ we suppose that $b \leq a$. For adding $a + b$ or subtracting $a - b$ one starts with the number a registered on the abacus; we suppose that it is registered in such a way that the last digit of a is represented on the last (rightmost) rod of our abacus. Starting from right, one adds or subtracts the corresponding digit of b by moving the corresponding number of beads on the rod. Say we are adding the i-th digits of a and b i.e. performing addition $a_{i-1} + b_{i-1}$ on the i-th rod from the right. If the sum $a_{i-1} + b_{i-1}$ is less than B, we just move b_{i-1} beads to the a_{i-1} beads and then move to the next digit to the left. Otherwise, as soon as we have B beads moved to the bottom of the rod, we remove them and then on the same rod we represent $b_i - a_i$ and add 1 to the digit represented on the next rod to the left. Addition is illustrated in Fig. 1.

When subtracting b_i from a_i, if $a_i \geq b_i$ the operation is obvious. Otherwise one has to remove one bead from the next left rod and on the i-th rod the digit $(a_i + B) - b_i$ is represented.

The proof of the following proposition is elementary.

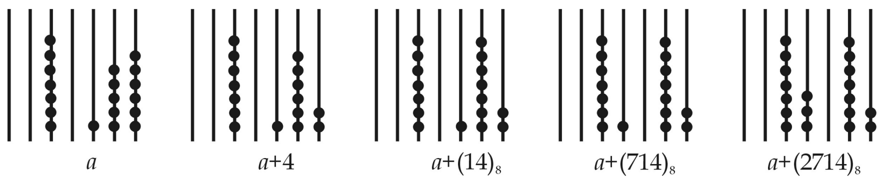

Figure 1. Addition on the abacus (base $B = 8$, $a = (70156)_8$, $b = (2714)_8$, $a + b = (73072)_8$).

Proposition 1. *Given the number of rods n and a number $a \in \mathbb{N}$ representable on the abacus, then the largest number $b \in \mathbb{N}$ for which it is possible to calculate $a + b$ is*

$$b_{\max} = B^n - 1 - a.$$

The largest number $b \in \mathbb{N}$ for which it is possible to calculate $a - b$ is $b_{\max} = a$.

4. Multiplication

The multiplication procedure on the abacus is similar to the algorithm that is used in pencil-and-paper calculations. The main difference in the algorithm is that the first factor a is sequentially multiplied with digits of the second factor b starting with the *rightmost* digit and the results are added. This ensures we don't have to plan ahead how many columns we need for the product $a \cdot b$. In order to keep the calculation neat we shall reserve two columns of the abacus to separate the factors and the second factor from the result (we can imagine these two empty columns represent the symbols $\cdot$ and $=$). Secondly, we suppose that we need to represent both factors a and b on the abacus so the user doesn't need to memorize any of them.

We shall suppose that we write a and b with the empty columns on the left side of the abacus.

Before we continue with the limitations of the multiplication using the abacus we have to describe the algorithm in detail. The first factor a is first multiplied by the last digit of the second factor b. This is a multiplication of a by a single-digit number and is performed right to left and the result is represented on the abacus starting from the rightmost column to the left; whenever we have to carry a digit because the multiplication of digits resulted in a number greater or equal to the base B we perform the corresponding addition. After the multiplication of a with the last digit of b is completed, we continue sequentially with the other digits of b (right to left), every time positioning the last digit of this partial result one place more to the left and adding it to the previously obtained partial product. The algorithm is illustrated in Fig. 2.

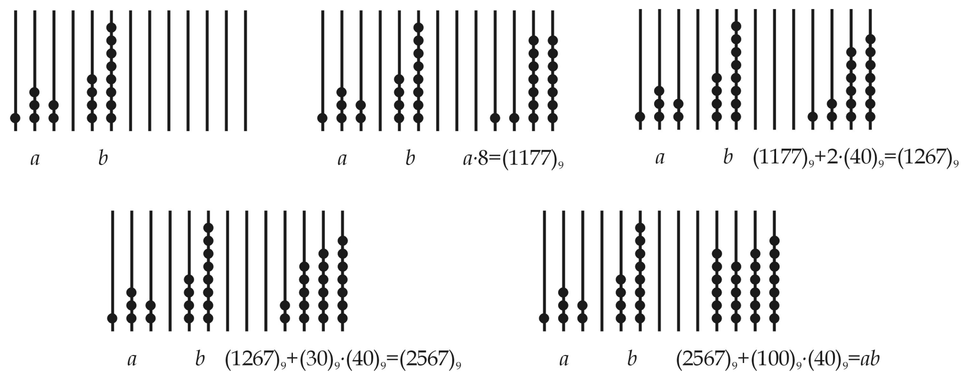

Figure 2. Multiplication on the abacus (base $B = 9$, $a = (132)_9$, $b = (48)_9$, $ab = (6567)_9$).

Considering the algorithm, it is obvious that in order to be able to perform the multiplication ab one needs an abacus with (at least) $2 + \delta_B(a) + \delta_B(b) + \delta_B(ab)$ rods. Given a number a, observe that if it cannot be multiplied by 1, then it cannot be multiplied by any larger number b. So, a can be multiplied on the abacus if it can be multiplied at least by 1. To multiply a by 1 we need $\delta_B(a)$ columns for a, one for 1, two additional separating columns and $\delta_B(a)$ columns for the result $a \cdot 1 = a$. Altogether, we need $2\delta_B(a) + 3$ columns. Consequently, we have proven

Proposition 2. *A number a can be multiplied on an abacus with n rods if*

$$n \geq 2\delta_B(a) + 3,$$

i.e. if a has at most $\left\lfloor \dfrac{n-3}{2} \right\rfloor$ digits.

In particular, it is not possible to multiply any numbers on an abacus with less than 5 rods. On the usual 13-rod abacus one cannot multiply a number having 6 or more digits with any other number.

Now suppose that a has at most $\left\lfloor \dfrac{n-3}{2} \right\rfloor$ digits. What is the largest number b that a can be multiplied with? To answer this question first we shall determine the maximal number k of digits that b can have. If a can be multiplied with any k-digit number, then it can be multiplied by the smallest k-digit number B^{k-1}. Note that $\delta_B(aB^{k-1}) = k-1+\delta_B(a)$, so $2k \leq n-1-2\delta_B(a)$. Thus, $k = \lfloor \frac{n-1}{2} \rfloor - \delta_B(a)$ is the maximal number of digits the second factor can have. The following theorem describes the constraints which arise during the multiplication.

Theorem 1. *Let n be the number of rods on the abacus and a be a number such that $\delta_B(a) \leq \left\lfloor \dfrac{n-3}{2} \right\rfloor$. Then:*

(a) *if n is odd, a can be multiplied on the abacus with all numbers not greater than $\left\lfloor \dfrac{B^{\frac{n-3}{2}}-1}{a} \right\rfloor$;*

(b) *if n is even, a can be multiplied on the abacus with numbers not greater than $B^{\frac{n}{2}-1-\delta_B(a)}-1$.*

Proof. If n is odd, then $k = \frac{n-1}{2} - \delta_B(a)$, so multiplication of a with a k-digit number b requires n or $n+1$ columns (2 separating columns, $\delta_B(a)$ columns for a, k columns for b and $\delta_B(a)+k = \frac{n-1}{2}$ or $\delta_B(a)+k-1 = \frac{n-3}{2}$ columns for ab). Accordingly, the upper limit for b is the largest number such that the product ab has $\delta_B(a)+k-1 = \frac{n-3}{2}$ digits. That means that the largest feasible product is $B^{\frac{n-3}{2}}-1$, i.e., $b \leq \frac{B^{\frac{n-3}{2}}-1}{2}$. Thus we have proven case (a). For case (b), the situation is simpler: we need 2 columns for separation purposes, $\delta_B(a)$ columns for a, $k = \frac{n-2}{2} - \delta_B(a) = \frac{n}{2}-1-\delta_B(a)$ columns for b and $\frac{n}{2}-1$ or $\frac{n}{2}-2$ columns for ab. That makes altogether n or $n-1$ columns, so we can calculate the product of a with any k-digit number b. It follows that the upper limit for b is the largest k-digit number $B^k - 1 = B^{\frac{n}{2}-1-\delta_B(a)}-1$. $\Diamond$

Note that we can significantly increase the maximal feasible number of digits of the second factor if we allow its digits to be sequentially deleted during the computation. Namely, since we are multiplying the first factor a with the second factor's digits right to left, as soon as we have completed the multiplication of a with a digit of the second factor, we can delete this digit and use the column of the deleted digit as the separation column between b and ab.

In the same way as before, we obtain the following theorem:

Theorem 2. *Let n be the number of rods on the abacus and a be a number such that $\delta_B(a) \leq \left\lfloor \dfrac{n-3}{2} \right\rfloor$. Let us suppose the modified multiplication* *procedure is to be used, i.e. in each calculation step after multiplying a with a digit of b the column of b containing that digit is cleared and used as the separation column between b and the result. Then a can be multiplied on the abacus with numbers up to* $\left\lfloor \frac{B^{n-3-\delta_B(a)}-1}{a} \right\rfloor$.

5. Division

The algorithm of division $a : b$ on the abacus is also performed similarly to the pencil-and-paper algorithm. Obviously it is not interesting to consider the division $a : b$ in case $a < b$, so in all of the following we suppose that $a \geq b$. We start by representing a and b on the leftmost rods of the abacus. In each step we determine one digit of the integer quotient. After the digit is determined we have to calculate the partial product, i.e. the product of this digit with b, and represent it on the rightmost columns of our abacus. Then we subtract this partial product from a (this can be done in the columns where a was represented in the beginning). Accordingly, in each step of the division we need $\delta_B(a)$ columns on the left to represent a (or the current remainder obtained by sequentially subtracting the previous partial products), $\delta_B(b)$ columns to represent b, and columns for the currently known digits of the integer quotient $q = \lfloor a : b \rfloor$ and columns for the present partial product. To keep the calculations neat, we shall suppose that three columns shall be used for separation purposes (one separating a from b and thus representing the symbol :, one separating b from the quotient and thus representing the symbol = and one separating the quotient from the partial product). The algorithm is illustrated in Fig. 3.

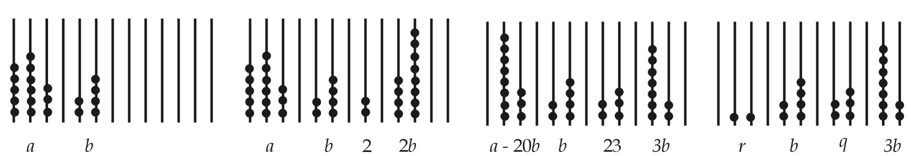

Figure 3. Division on the abacus (base $B = 10$, $a = 563$, $b = 24$, $q = \lfloor a : b \rfloor = 23$ and the remainder $r = 11$).

The following lemma contains well-known results; both statements are simple consequences of the equation stated in the introductory section about notation.

Lemma 1. (a) *The integer quotient* $q = \lfloor a : b \rfloor$ *has either* $\delta_B(a) - \delta_B(b)$ *or* $\delta_B(a) - \delta_B(b) + 1$ *digits.*

(b) *The product of a number* b *with a one digit number and thus every nonzero partial product in the calculation of* $a : b$ *has* $\delta_B(b)$ *or* $\delta_B(b) + 1$ *digits.*

A simple, but important, observation is contained in the following lemma:

Lemma 2. *If the abacus allows enough space to represent the last nonzero partial product, then it is also possible to represent all the previous partial products.*

Proof. The last nonzero partial product is the product of the last nonzero digit from q with b. If this product is representable on the abacus, the previous was also representable since it has the same number of digits or one more, and there was one less digit of q determined. Inductively we conclude that all the previous steps could be performed. $\Diamond$

First we shall, for a given a, determine when it is not possible to divide it by any b and vice versa, for a given b when it cannot be a divisor for any dividend a. In all of the following we suppose that we need one column for a zero partial product.

Proposition 3. *A number* a *cannot be a dividend on an abacus with* n *rods if*

$$n < 4 + 2\delta_B(a).$$

A number b *cannot be a divisor on an abacus with* n *rods if*

$$n < 4 + 3\delta_B(b).$$

Proof. For a given a the most favorable situation we obtain when b is such that the quotient has $\delta_B(a) - \delta_B(b)$ digits and all but the first partial products are zero (i.e. all but the first digit of the quotient are zero) and thus we need only one column to represent these partial products. In such a case we need

$$N_{a,b,\min} = 3 + \delta_B(a) + \delta_B(b) + \delta_B(a) - \delta_B(b) + 1 = 2\delta_B(a) + 4$$

columns. Consequently, a cannot be divided by any b if $n < 2\delta_B(a) + 4$.

For a given b the most favorable situation we obtain when $a = b$ since and in this case we need only one step that uses $3 + \delta_B(a) + \delta_B(b) + +1 + \delta_B(b) = 3\delta_B(b) + 4$ columns. This proves the second statement. $\Diamond$

It follows that on abaci with less than seven rods one cannot divide any numbers, so in all of the following we shall suppose $n \geq 7$.

In the rest of the paper we shall deal with the following question: given a number a, what is the largest number b such that it is possible to calculate $a : b$ on a n-rod abacus? Such a b shall be denoted by $b_{\max}(a)$. Although the question is simple, the answer shall be quite nontrivial to prove. Because of the previous proposition we shall consider only a with at most $\left\lfloor \frac{n-4}{2} \right\rfloor$ digits. For example, on a 13-rod abacus one cannot divide a number with five or more digits (in any base) with any other number.

The maximal number of rods needed for a division $a : b$ is

$$\begin{aligned} N_{a,b,\max} = 3 + \delta_B(a) + \delta_B(b) + \delta_B(a) - \delta_B(b) + \\ + 1 + \delta_B(b) + 1 = 2\delta_B(a) + \delta_B(b) + 5. \end{aligned}$$

Since $\delta_B(b) \leq \delta_B(a)$, we conclude that $N_{a,b,\max}$ is at most $3\delta_B(a) + 5$, and this can happen only if b has the same number of digits as a. But, if this is the case the quotient has to have only one digit and the one and only partial product cannot be greater than a, thus it has the same number of digits as a and b and we need only $3\delta_B(a) + 4$ columns. On the other side, if $\delta_B(b) < \delta_B(a)$ then $N_{a,b,\max} < 3\delta_B(a) + 5$, so in all cases we need at most $3\delta_B(a) + 4$ columns for the division of a given a by any b. Consequently, if $3\delta_B(a) + 4 \leq n$, then a can be divided by all $b \leq a$. We summarize this as

Proposition 4. *If a has at most $\left\lfloor \frac{n-4}{3} \right\rfloor$ digits, then it is possible to calculate $a : b$ on a n-rod abacus for all $b \leq a$ and $b_{\max}(a) = a$.*

For example, on a 13-rod abacus every number with three digits can be divided by all numbers not greater than it and it is the smallest number of rods which allows a three-digit dividend to be divided by any smaller divisor. On a 7-rod abacus all quotients of two one-digit numbers can be calculated.

The previous two propositions suggest that there is a possibility (and unfortunately, it occurs) that there are specific numbers of digits k, depending on n, of a dividend such that some of k-digit dividends cannot be divided by all smaller divisors. Such numbers of digits k are called critical. Critical numbers of digits are those greater than $\left\lfloor \frac{n-4}{3} \right\rfloor$ and not greater than $\left\lfloor \frac{n-4}{2} \right\rfloor$, i.e. there are $c(n)$ such critical numbers of digits, where $c : \mathbb{N} \setminus \{1, 2, 3, 4, 5\} \to \mathbb{N}_0$ is the function defined by

$$c(n) = \left\lfloor \frac{n-4}{2} \right\rfloor - \left\lfloor \frac{n-4}{3} \right\rfloor .$$

Checking for all possible remainders of the division of n by 6 it is easy to show that

$$c(n+1) = \begin{cases} c(n)-1, & n \equiv 0 (\operatorname{mod} 6), \\ c(n)+1, & n \equiv 5 (\operatorname{mod} 6), \\ c(n), & \text{otherwise.} \end{cases}$$

The largest value n for which $c(n) = 1$ (i.e. the largest number of rods for which there is only one critical number of digits for a dividend) is 13. As n increases the situation becomes more complicated. Let us for each n denote the smallest critical number of digits of a dividend a by N_n, i.e.

$$N_n = 1 + \left\lfloor \frac{n-4}{3} \right\rfloor .$$

Note that:

$$N_n = \begin{cases} m-1, & n = 3m, \\ m, & n = 3m+1 \text{ or } n = 3m+2. \end{cases} \tag{1}$$

Since there are no critical numbers of digits for $n = 7$, we shall now restrict ourself to abaci with $n > 7$ rods.

Lemma 3. *Let $n \geq 8$.*

If n is odd, every number a with a critical number of digits can be divided by at least one number not greater than a.

If n is even, every number a with less than the maximal critical number of digits, i.e. with $\delta_B(a) < \left\lfloor \frac{n-4}{2} \right\rfloor$, can be divided by at least one number not greater than a.

Proof. Note first that for every a the division of a by 1 requires $5+2\delta_B(a)$ columns.

If $n = 2l+1$ is odd and $N_n < \delta_B(a) \leq \left\lfloor \frac{n-4}{2} \right\rfloor$, then $5 + 2\delta_B(a) \leq$ $\leq 5+2\left\lfloor \frac{2l+1-4}{2} \right\rfloor = 5+2(l-2) = 2l+1 = n$ columns, i.e. a can be divided at least by 1.

Consider now even $n = 2l$, $n \geq 8$. Now, $5+2\delta_B(a) \leq 5+2\left\lfloor \frac{2l-4}{2} \right\rfloor =$ $= 2l+1 = n+1$. Thus, if $\delta_B(a) < \left\lfloor \frac{n-4}{2} \right\rfloor$ the division of a by 1 is possible. Consequently, of all numbers a with a critical number of digits only those with $\left\lfloor \frac{n-4}{2} \right\rfloor$ digits cannot be divided by 1, and this happens only for even n. $\diamondsuit$

Let now n be even and $\delta_B(a) = \left\lfloor \frac{n-4}{2} \right\rfloor$. Since the least possible number of columns for a division of a by some b is $N_{a,\min} = 4+2\delta_B(a) = n$, if there is no b such that the division needs $N_{a,\min}$ columns, then a cannot be divided by any b on an abacus with n rods. Since the first step of the division $a : b$ by any b needs at least $4 + \delta_B(a) + 2\delta_B(b)$ columns and this must not exceed $n = 4 + 2\delta_B(a)$, we get the following condition on the possible number of digits for the divisor b: $\delta_B(b) \leq \left\lfloor \frac{\delta_B(a)}{2} \right\rfloor$. It is now easy to find examples of even n, bases B and numbers a with $\left\lfloor \frac{n-4}{2} \right\rfloor$ digits that cannot be divided by any number $b \leq a$. If $n = 8$, then considering numbers a with $\delta_B(a) = 2$ digits the only possible divisors resulting in a division using only 8 columns are one-digit numbers. Taking e.g. $B = 10$ it is easy to see that the division of $a = 21$ by all one-digit numbers requires at least 9 columns. If $n = 10$, then considering numbers a with $\delta_B(a) = 3$ digits the only possible divisors resulting in a division using only 10 columns are also one-digit numbers. Taking e.g. $B = 10$ it is easy to see that the division of $a = 297$ by all one-digit numbers requires at least 11 columns. Since for $n = 8$ and $n = 10$ the only critical number of digits is $\left\lfloor \frac{n-4}{2} \right\rfloor$ we conclude that there exist numbers a with a critical number of digits for which the calculation $a : b$ cannot be performed for any b. Another example for even n is the case $n = 12$, $B = 2$, $a = (1001)_2$. In this case the only possible divisors b with $\delta_B(b) \leq \left\lfloor \frac{\delta_B(a)}{2} \right\rfloor$ are 1, $(10)_2$ and $(11)_2$ and it is easy to check that the division of a by any of these divisors requires 13 columns.

Because of these problems we shall now restrict ourselves to abaci with 9 or at least 11 rods and to the identification of maximal divisors for numbers a with the minimal critical number N_n of digits.

Lemma 4. *To divide a number with k digits by a number with the same number of digits the abacus has to have at least $3k+4$ rods. In particular, a number with a critical number of digits cannot be divided by any number with the same number of digits.*

Proof. The division of a number by another one with the same number of digits necessary yields a one-digit quotient (this is a consequence of Lemma 1). Since the only partial product in this case has at least the value of the divisor and at most the value of the dividend, we need the same number k of columns for representing it. This means that for the calculation one needs exactly $4 + 3k$ columns.

If $k \geq N_n$, we conclude that for dividing a k-digit number by a k-digit number one needs at least $4 + 3N_n = 7 + 3\left\lfloor \frac{n-4}{3} \right\rfloor$ columns. Using equation (1) it is easy to see that this number is always greater than n. $\Diamond$

Thus the largest number that a N_n-digit number can be divided by has at most $N_n - 1$ digits. In many, but not in all, cases it is possible to perform the division of a N_n-digit number by a (N_n-1)-digit divisor. Let a be a number with N_n digits and b a number with one digit less. Because of Lemma 1 we know that the integer quotient of a by b has 1 or 2 digits. In the first step of the division we obtain the first (and possibly the last) digit of the quotient and the partial product is necessarily nonzero. For this step we need $3N_n+2$ or $3N_n+3$ columns, depending on if the partial product has the same number of digits as b has or one more. Because of equation (1), we conclude that for determining the first digit of the quotient we need at most $3m = n$ columns if $n = 3m$, at least $3m+2 > n$ columns if $n = 3m + 1$ and either $3m + 2 = n$ or one more columns if $n = 3m + 2$. So we conclude:

Proposition 5. *If* $n \equiv 1 \pmod 3$ *then it is not possible to divide a number with* N_n *digits with any number with* $N_n - 1$ *digits. In particular, it is not possible to divide any four-digit number with any three-digit number on the classical* 13*-rod abacus.*

We shall continue to consider separately the three possible cases in three separate subsections. In all of the three subsections we assume $\delta_B(a) = N_n$.

5.1. Maximal divisors for N_n-digit numbers on abaci with $n \equiv 0 (\mathrm{mod}\, 3)$ rods

As we have seen, on abaci with $n = 3m \geq 9$ rods it is always possible to obtain the first digit of the result $a : b$ for $\delta_B(b) = N_n - 1$. Since we are searching for the maximal divisor, we are primarily interested in quotients that have only one digit and so we check the division of such numbers a by the maximal $(N_n - 1)$-digit number

$$b^* = B^{N_n-1} - 1.$$

The division $a : b^*$ can be performed if the integer quotient has one digit. The quotient of a by b^* has two digits only if $a \geq Bb^* = B^{N_n} - B$. But, there are only B such numbers a. Since $b^* > B - 1$ ($n \geq 9$ implies that $\delta_B(b)$ is at least 2) the division theorem implies that for $a \geq Bb^*$ we have $\lfloor a : b^* \rfloor = B = (10)_B$. Consequently the second partial product must be zero and for these divisions we need $3 + N_n + N_n - 1 + 2 + 1 = 2N_n + 5$ columns. As $n = 3m \geq 9$, equation (1) implies that we need $2m+3 \leq 3m$

columns. For all $m \geq 3$ this condition is fulfilled and the division can be performed. Thus we have proven

Theorem 3. *If $n \equiv 0 (\mathrm{mod}\, 3)$ then the largest divisor b of an N_n-digit number a such that $a : b$ can be computed on a n-rod abacus is the largest $(N_n - 1)$-digit number $b^* = B^{N_n - 1} - 1$.*

Note that this does not imply that all a with N_n digits can be divided by all b with $N_n - 1$ digits (but a can be divided with all b with at most $N_n - 2$ digits: if b has p digits, the division requires at most $2N_n + 5 + p$ digits, and this is at most $2N_n + 5 + N_n - 2 = 3m = n$). For example, if $n = 15$, i.e. $N_n = 4$, the number 9989 can be divided by 999 (quotient 9 and partial product 8991, altogether this makes 15 columns), but the division of 9989 by 112 yields 89 as the quotient and the last partial product is $9 \cdot 112 = 1008$, so the calculation would need 16 columns.

As is obvious from the considerations above, the calculation of $a : b$ (for $\delta_B(a) = N_n$ and $\delta_B(b) = N_n - 1$) is not feasible if the integer quotient $\lfloor a : b \rfloor$ has two digits and the last partial product has more digits than b i.e. when $\lfloor a : b \rfloor \geq B$ and $q_0 b \geq B^{N_n - 1}$, where q_0 denotes the last digit of $\lfloor a : b \rfloor$. This cannot happen if $q_0 = 0$ or $q_0 = 1$ (in particular, if $B = 2$ the division is always possible!). Otherwise we have the (necessary and sufficient) condition: $b < \frac{B^{N_n - 1}}{q_0}$. Since $q_0 \leq B - 1$ we conclude

Proposition 6. *If $n \equiv 0 (\mathrm{mod}\, 3)$ then a N_n-digit number a can be divided by all $b < \frac{B^{N_n - 1}}{B - 1}$. In particular, for binary calculations such an a can be divided by all $(N_n - 1)$-digit numbers b.*

As conditions similar to the one above the previous proposition shall appear in the subsequent subsections, we formulate the following lemma:

Lemma 5. *If q^* is a one-digit number, i.e. $q^* \in \{1, 2, \ldots, B - 1\}$, and b has $\delta_B(b)$ digits, then the product $q^* \cdot b$ has $\delta_B(b)$ digits if and only if*

$$b < \frac{B^{\delta_B(b)}}{q^*}.$$

Otherwise the product has $\delta_B(b) + 1$ digits.

5.2. Maximal divisors for N_n-digit numbers on abaci with $n \equiv 1 (\mathrm{mod}\, 3)$ rods

As we have seen, the divisors in this case can have at most $N_n - 2$ digits. For a given a, with $\delta_B(a) = N_n = m$ (if $n = 3m + 1$), we want to determine[1] $b_{\max}(a)$. Note also that $m \geq 4$ since we restricted ourselves to abaci with 9 or at least 11 rods, so the smallest $n \equiv 1 (\mathrm{mod}\, 3)$ we are considering in this section is $n = 13$. We consider only (unless otherwise stated) divisors b with $\delta_B(b) = N_n - 2 = m - 2$. The maximal such number b shall be denoted b^*:

$$b^* = B^{N_n - 2} - 1 = B^{m-2} - 1.$$

By Lemma 1, the quotients $q = \lfloor a : b \rfloor$ will have two or three digits. The first calculation step needs at most $3 + N_n + N_n - 2 + 1 + N_n - 1 = n$ columns, so we can always determine the first digit of the result. The second step needs either $2N_n + 4 < n$ (if the second digit of the quotient is zero), $3N_n + 1 = n$ or $n+1$ columns; the last, problematic, case appears if the product of b with the second digit of the quotient has more digits than b, i.e. if the partial product is at least $B^{N_n - 2}$. This cannot happen if the second digit of the quotient is 0 or 1, in particular it cannot happen in binary calculations. If the second digit is 2 or larger, then b has to meet the condition from Lemma 5 (where q^* is the second digit of the quotient and $\delta_B(b) = N_n - 2$). If needed, the third step requires one column more for the quotient and thus either $2N_n + 5 < n$ columns (if the third digit of the quotient is 0), or at least $3N_n + 2 > n$ columns. So, if the quotient has three digits, then it can be calculated if and only if the last digit is zero. We summarize this as

Proposition 7. *If $n = 3m + 1$ for some $m \in \mathbb{N}$ ($m \geq 4$), $\delta_B(a) = m$ and $\delta_B(b) = m - 2$, then the calculation of the integer quotient q of a by b on a n-rod abacus is not feasible if the product of b with the second digit of q is at least B^{m-2} or if q has three digits and the last one is not zero. If $B = 2$ the calculation is always possible, except for the cases when the quotient has three digits the last of which is* 1.

If $B = 2$ the only case when $b_{\max}(a) \neq b^*$ happens if the quotient is a three-digit binary number such that the last digit is 1. It can easily be seen by inspection of the possible cases for $m > 4$ and separately for $m = 4$ that the only case when $\lfloor a : b^* \rfloor$ cannot be calculated is for

[1] The number $b_{\max}(a)$ depends not only on a, but also on B.

$a = (1111)_2$. By checking the possible divisors b for this a it is easily seen that $b_{\max}((1111)_2) = 1$.

Proposition 8. *For binary calculations on an abacus with* $n = 3m+1 \geq$ ≥ 13 *rods for all* a *with* m *digits the maximal number* b *such that the integer quotient of* a *by* b *can be calculated is* b^*, *except for* $m = 4$ *and* $a = (1111)_2$, *in which case the maximal divisor is* 1.

In the following we shall suppose that $B > 2$. As we want to determine the maximal b, we would prefer smaller quotients. Let us first consider the case when there exists a b such that the integer quotient $q = \lfloor a : b \rfloor$ has two digits. Since the smallest quotients of a given a we get for the largest b, and that is b^*, if we are sure that no quotient of a by b^* can have 0 or 1 as the last (second) digit, then a cannot be divided on a n-rod abacus by any b that results in a two-digit quotient. Now, the quotient q has to have the last digit greater than 1 if q is greater than or equal to $(B-1\ 2)_B = (B-1)B + 2 = B^2 - B + 2$. It follows that for

$$a \geq (B^2 - B + 2) \cdot b^* = B^m - B^{m-1} + 2B^{m-2} - B^2 + B - 2 = \overline{a}$$

it is not possible to divide a by any of the b with $m-2$ digits that yield a quotient with two digits (the smallest such quotient would be $B^2 - B + 2$, i.e. all possible two-digit quotients have the last digit greater than 1). We summarize this as

Lemma 6. *If* $a \geq \overline{a}$, *then none of the integer quotients of* a *by a* $(m-2)$-*digit divisor* b *has two digits.*

This means that for $a \geq \overline{a}$ we have to search for the maximal divisor $b_{\max}(a)$ among those yielding three-digit quotients. Note also that for $m = 4$ (i.e. $n = 13$) $\overline{a} = (B-1)B^{m-1} + B^{m-2} + (B-2) = (B-1\ 1\ 0\ B-2)_B$. For larger n,

$$\begin{aligned} \overline{a} &= (B-1)B^{m-1} + B^{m-2} + B^{m-2} - B^2 + (B-2) = \\ &= (B-1\ 1\ B-1\ B-1\ \ldots\ B-1\ 0\ B-2)_B. \end{aligned}$$

For example, if $B = 10$ then for $n = 13, 16, 19, 22, 25, \ldots$ we get the following values of $\overline{a}$: 9108, 91908, 919908, 9199908, 91999908, ...

Let us determine $b_{\max}(a)$ for a given a, $a \geq \overline{a}$. Since all quotients have three digits, if there exists a $b < B^{m-2}$ such that $\lfloor a : b \rfloor = B^2$ (the smallest three-digit number: $B^2 = (100)_B$), then $b_{\max}(a) = b$.

Lemma 7. *For* $n \equiv 1 \pmod 3$ *if* $a \geq \overline{a}$ *and* $\lfloor a : B^2 \rfloor > a - B^2 \lfloor a : B^2 \rfloor$, *then* $b_{\max}(a) = \lfloor a : B^2 \rfloor$.

Proof. If $\lfloor a : b \rfloor = B^2$, the division theorem implies that $a = bB^2 + r$ with $r < \lfloor a : B^2 \rfloor$. Thus $a - bB^2 < \lfloor a : B^2 \rfloor$. $\Diamond$

Which $a \geq \overline{a}$ fulfil the condition from the lemma? Except in the case $n = 13$, all do! Namely, if $a = (a_k a_{k-1} \dots a_2 a_1 a_0)_B$, then $\lfloor a : B^2 \rfloor = (a_k a_{k-1} \dots a_2)_B$ and $B^2 \lfloor a : B^2 \rfloor = (a_k a_{k-1} \dots a_2 00)_B$, and thus

$$a - B^2 \lfloor a : B^2 \rfloor = (a_1 a_0)_B < (a_k a_{k-1} \dots a_2)_B$$

if $k = 1 + N_n \geq 4$, i.e. if $n \geq 16$, $n \equiv 1 (\mathrm{mod}\, 3)$. Thus the only complicated case is $n = 13$.

If $n = 13$, all of the numbers $a \geq \overline{a} = (B-1\ 1\ 0\ B-2)_B$ fulfil the condition from the previous lemma except for the numbers $a = (B-1\ a_2 a_1 a_0)_B$ with the property $(B-1\ a_2)_B \leq (a_1 a_0)_B$. These are exactly the numbers a with the property that the remainder of the division $a : B^2$ is not smaller than the quotient. For these numbers there is no two-digit divisor b ($m - 2 = 2$ in this case) such that $\lfloor a : b \rfloor = B^2$. In the paper [4, pp. 83–84] the case $B = 10$ was solved.

We shall now show that there is a similar way to determine $b_{\max}(a)$ for bases $B > 2$. According to the previous discussion, we have to determine $b_{\max}(a)$ for numbers a that are of the form $(B-1\ 0\ B-1\ 0)_B + 101i + j$ for $1 \leq i \leq B-3$ and $0 \leq j \leq B-(i+1)$, and for numbers a in the three-element set $S = \{(B-1\ B-2\ B-1\ B-2)_B, (B-1\ B-2\ B-1\ B-1)_B, (B-1\ B-1\ B-1\ B-1)_B\}$.

Proposition 9. *If* $a = (B-1\ 0\ B-1\ 0)_B + 101i + j$, $1 \leq i \leq B-3$, $0 \leq j \leq B-(i+1)$, *then* $b_{\max}(a) = (B-2) \cdot B + i + 2 = (B-2\ i+2)_B$.

Proof. It suffices to prove that $\lfloor a : (B-2\ i+2)_B \rfloor = (110)_B$, i.e.

$$B^2 + B \leq \frac{B^4 - B^3 + (i+1) \cdot B^2 - B + i + j}{(B-2) \cdot B + i + 2} < B^2 + B + 1$$

for all $1 \leq i \leq B-3$ and $0 \leq j \leq B-(i+1)$. This is easy to check by direct calculation. $\Diamond$

Proposition 10. *For* $a \in S$ *the number* $b_{\max}(a)$ *is the largest number* b *such that* $B \leq b < B^2$ *and* $\lfloor a : b \rfloor = (i\ j\ 0)_B$, *where* $0 < j \leq i \leq B-1$, *if such a number* b *exists. Otherwise* $b_{\max}(a)$ *is equal to the largest number* b', $b' \leq B-1$, *such that* $b' \cdot q_0 < B$, *where* q_0 *denotes the last digit of the number* $\lfloor a : b' \rfloor$.

Proof. It can be shown similarly as before that there is no number b with the property that $\lfloor a : b \rfloor$ is of the form $(i\ 0\ 0)_B$. If there exist a number b as in the first statement of the proposition, it is obvious that a

can be divided by b on a 13-rod abacus (because both partial products have at most two digits).

To finish the proof of the first part of the proposition we have to check that a cannot be divided by b if $\lfloor a : b \rfloor = (i\ j\ 0)_B$, $j > i$. Namely, if we suppose that this division is feasible, the product $b \cdot j$ would be smaller than B^2. This is not possible if $b \cdot i \geq (B-1\ B-2)_B$. The inequality $b \cdot i < (B-1\ B-2)_B < b \cdot (i+1)$ implies $b \cdot j = (B-1\ B-1)_B$ and $j = i+1$. Since in the first partial division we have obtained the integer quotient i, the remainder in this division is $b-1$. Since $(B-1\ B-1)_B$ is divisible by b, we get $B \leq b-1$. The second digit of the quotient (i.e., j) we obtain by the division of $(b-1\ B-1)_B$ by b. We get $j = B-1$ because $(b-1\ B-1)_B + 1 = (b\ 0)_B$. In the same way we get that the third digit of the quotient cannot be zero (it is in fact equal to $B-1$) and we have obtained a contradiction with our presumption.

The second part of the proposition is now obviously valid, noting that such a number b' always exists and is in the worst case equal to 1. $\Diamond$

Table 1 shows a few interesting examples.

a	B	$b_{\max}(a)$
6565, 6566	7	5
6666	7	6
7676, 7677	8	1
7777	8	42
8787, 8788	9	1
8888	9	28
9898, 9899	10	47
9999	10	9

Table 1. Values of $b_{\max}(a)$ for some cases of $a \in S$

Now let us turn to numbers $a < \overline{a}$ (for $B > 2$). For such numbers there is at least one b with $\delta_B(b) = N_n - 2$ such that $\lfloor a : b \rfloor$ has two digits. The necessary and sufficient condition that the calculation is performable is contained in Lemma 5 (with q^* being the second digit of the quotient $\lfloor a : b \rfloor$). There are two cases when the maximal b such that the division is possible is equal to b^*; these two cases are contained in the following two lemmas. We first introduce some additional notation: $\underline{a} = (B+2)b^*$ and $\widetilde{a} = (B^2 - B)b^*$.

Lemma 8. *Let n be the number of rods on the abacus, $n = 3m+1$, $B \geq 3$ and a be a number smaller than $\overline{a}$. If $a < \underline{a}$ or $a \geq \widetilde{a}$, then a*

can be divided by b^*, i.e. b^* is the maximal divisor b such that a can be divided by b on the abacus.

Proof. If $a < \overline{a}$, the integer quotient $\lfloor a : b^* \rfloor$ has two digits. In fact, the largest possible integer quotient of such a number a by b^* is $\lfloor(\overline{a}-1) : b^*\rfloor = \lfloor((B^2-B+2)b^*-1) : b^*\rfloor = B^2-B+1 = B(B-1)+1 = (B-1\ 1)_B$.

Now, for $a < \underline{a}$, because $\underline{a}$ is a multiple of b^* we obtain the following sharp inequality: $\lfloor a : b^* \rfloor < \lfloor \underline{a} : b^* \rfloor = B+2$. Thus $\lfloor a : b^* \rfloor = B$ or $\lfloor a : b^* \rfloor = B+1$, so the second (and the last) digit of the quotient of a by b^* is 0 or 1 and a can be divided by b^*.

On the other side, if $a \geq \widetilde{a}$, then $\lfloor a : b^* \rfloor \geq B^2 - B = (B-1\ 0)_B$. Since the largest possible quotient is $(B-1\,1)_B$, for such numbers a the only possible quotients with b^* are $(B-1\,0)_B$ and $(B-1\,1)_B$, so we can again conclude that the division is possible. $\diamondsuit$

The numbers a such that $\underline{a} \leq a < \widetilde{a}$ can be divided in two classes:

$$I = \{a : iBb^* \leq a < (iB+2)b^*,\ i = 2, 3, \ldots, B-2\}$$

and

$$II = \{a : (iB+2)b^* \leq a < (i+1)Bb^*,\ i = 1, 2, \ldots, B-2\}.$$

For all numbers a of the form described as class I the largest divisor is b^*:

Lemma 9. *Let n be the number of rods on the abacus, $n = 3m+1$, $B \geq 3$ and a be a number such that $\underline{a} \leq a < \widetilde{a}$. If there exist integers i and j such that $a = iBb^* + j$, $i \in \{2, 3, \ldots, B-2\}$ and $j \in \{0, 1, \ldots, 2b^* - 1\}$, then the maximal divisor b such that a can be divided by b on the abacus with n rods is b^*.*

Proof. If a can be represented in the form $a = iBb^* + j$ with i and j as described in the statement of the lemma, then either $a = iBb^* + j$ with $j < b^*$ or $a = iBb^* + b^* + j' = (iB+1)b^* + j'$ with $j' < b^*$. In the first case we have $\lfloor a : b^* \rfloor = iB = (i\ 0)$ and in the second $\lfloor a : b^* \rfloor = iB+1 = (i\ 1)$, so in both cases the last digit of the quotient is 0 or 1 and the division of a by b^* is feasible. $\diamondsuit$

Before we continue, note that for numbers $a \in II$ the maximal divisor cannot be b^*. Namely, for $a \geq \underline{a}$ and $b \leq b^*$ we have $\lfloor a : b \rfloor \geq \lfloor \underline{a} : b^* \rfloor = B+2 = (1\ 2)_B$. Thus for $a \in II$ we have

$$\begin{aligned}\lfloor a : b^* \rfloor \in \big\{B+2, B+3, \ldots, B^2-B-1\big\}\setminus \\ \setminus\big\{iB, iB+1 : i = 2, 3, \ldots, B-2\big\} = \\ = \big\{(1\ 2)_B, (1\ 3)_B, \ldots, (B-2\ B-1)_B\big\}\setminus \\ \setminus\big\{(2\ 0)_B, (2\ 1)_B, \ldots, (B-2\ 0)_B, (B-2\ 1)_B\big\},\end{aligned}$$

i.e. the second digit of the quotient is larger than 1 and so the last partial product of the division by b^* must be larger than b^*. In class II we shall show that the largest divisor is the number

$$b^\natural = \left\lfloor \frac{a}{\left\lceil \frac{\lfloor a:b^* \rfloor}{B} \right\rceil B} \right\rfloor . \tag{2}$$

Although the formula seems complicated, the idea behind it is quite simple: We check the number $\lfloor a : b^* \rfloor$ and now we decrease the divisor downwards starting with $b^* - 1$ until we find the first b (that is our $b^\natural$) such that the integer quotient of a by b is a two-digit number with the second digit equal to zero, i.e. $\lfloor a : b \rfloor$ is a multiple of B and it is the smallest multiple of B less than $\lfloor a : b^* \rfloor$.

To prove that $b_{\max}(a) = b^\natural$ for $a \in II$ we have to prove that a can be divided by $b^\natural$ and that there is no larger b ($b^\natural < b \leq b^* - 1$) such that a can be divided by b on an abacus with $n = 3m + 1$ rods. Denote by $k = \left\lceil \frac{\lfloor a:b^* \rfloor}{B} \right\rceil$ the smallest integer k such that $\lfloor a : b^* \rfloor < kB = (k\ 0)_B$. Since $a \in II$ there exists some $i \in \{1, 2, \ldots, B-2\}$ such that $(iB+2)b^* \leq$ $\leq a < (i+1)Bb^*$. By the definition of k we have $i + 1 = k$ and

$$(k-1)B < \frac{a}{b^*} < kB.$$

Lemma 10. *Let $a \in II$ and suppose $(iB + 2)b^* \leq a < (i + 1)Bb^*$, for some $i \in \{1, 2, \ldots, B-2\}$. Then $a \in II$ can be divided by $b^\natural = \left\lfloor \frac{a}{(i+1)B} \right\rfloor$. The corresponding quotient $\lfloor a : b^\natural \rfloor$ is equal to $(i + 1)B = (k\ 0)_B$ and thus $b^\natural$ is the largest b such that $\lfloor a : b \rfloor = (i + 1)B$.*

Proof. The quotient theorem implies that a can be represented in the form $a = p(i+1)B+r$, with $0 \leq r < (i+1)B$. Obviously $p = \lfloor \frac{a}{(i+1)B} \rfloor$. If we show that $p > r$, the same theorem implies that $\lfloor a : b^\natural \rfloor = (i+1)B =$ $= (k\ 0)_B$. It is sufficient to show that $a \geq (i+1)^2B^2$, because this implies $p \geq (i + 1)B$.

As a is at least $(iB+2)b^*$, it is sufficient to check that $(i+1)^2B^2 <$ $< (iB+2)(B^2-1)$. Note that $(iB+2)(B^2-1) > iB^3$ and $(i+1)^2B^2 < iB^3$

for $i < B-2$. In the case $i = B-2$, the required inequality is obtained by direct calculation. $\Diamond$

Now let us suppose that there is a $b > b^\natural$ such that a can be divided by b on our abacus. Since we have shown that $b^\natural$ is the largest divisor b such that $\lfloor a : b \rfloor = kB$ we conclude that for $b > b^\natural$ we have $\lfloor a : b \rfloor \leq (k-1\ B-1)_B$. On the other side, $b < b^*$ and our choice of k implies that $(k-1)B + 2 < a : b^* < a : b$. Thus $\lfloor a : b \rfloor$ has to be one of the $B-1$ numbers $(k-1\ 2)_B$, $(k-1\ 3)_B$, $\ldots$, $(k-1\ B-1)_B$. Now, it is enough to show that $2b^\natural \geq b^*$, because this would imply $2b > b^*$ so all of the possible quotients result in a too large second partial product and then we can conclude that a cannot be divided by any $b > b^\natural$.

Lemma 11.

$$2b^\natural \geq b^*.$$

Proof. We have $((k-1)B+2)b^* \leq a < kBb^*$ and thus

$$b^\natural = \left\lfloor \frac{a}{kB} \right\rfloor \geq \left\lfloor \frac{(k-1)B+2}{kB} \cdot b^* \right\rfloor .$$

It is easy to see by elementary calculus that the function $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \frac{(x-1)B+2}{xB} = 1 - \frac{B-2}{xB}$ is increasing and has the line $y = 1$ as a horizontal asymptote. Consequently, $1 > f(k) \geq f(2) = \frac{2+B}{2B} > \frac{1}{2}$ for all k.

Now, if B is odd, i.e. b^* is even, we have $b^\natural \geq \left\lfloor \dfrac{b^*}{2} \right\rfloor = \dfrac{b^*}{2}$.

On the other side, if B is even and thus b^* is odd,

$$b^\natural \geq \left\lfloor \frac{B+2}{2B} \cdot b^* \right\rfloor = \left\lfloor \frac{B^{m-2}}{2} + B^{m-3} - \frac{B+2}{2B} \right\rfloor =$$
$$= \frac{B^{m-2}}{2} + B^{m-3} - 1 > \frac{b^*}{2}. \qquad \Diamond$$

The previous two lemmas imply that the formula (2) is valid, i.e. we have proven

Proposition 11. *Let $n \geq 13$ be the number of rods on the abacus, $n = 3m+1$, and $B \geq 3$. For $a \in II$*

$$b_{\max}(a) = b^\natural = \lfloor a : (kB) \rfloor,$$

where k is the largest of the numbers $2, 3, \ldots, B-1$ such that $\lfloor a : b^* \rfloor \leq$ $\leq kB$.

We summarize the previous results as

Theorem 4. *Let $n \geq 13$ be the number of rods on the abacus, $n = 3m+1$ and $B \geq 3$. Let a be a number such that $\delta_B(a) = N_n$ and $b^* = B^{N_n - 2} - 1$.*

(a) *If* $a \geq (B^2 - B + 2) \cdot b^*$ *then* $b_{\max}(a) = \lfloor a : B^2 \rfloor$, *except if* $n = 13$ *and* a *is of the form* $a = (B-1\ a_2 a_1 a_0)_B$ *where* $(B-1\ a_2)_B \leq (a_1 a_0)_B$. *If this is the case, the value of* $b_{\max}(a)$ *is determined either by Prop. 9 or Prop. 10.*

(b) *If* $a < (B+2)b^*$ *or* $(B^2 - B)b^* \leq a < (B^2 - B + 2)b^*$ *or if there exists some* $i \in \{2, 3, \ldots, B-2\}$ *such that* $a \in \big\{(i-1\ B-1\ B-1\ \ldots\ B-1\ B-i\ 0),\ (i-1\ B-1\ B-1\ \ldots\ B-1\ B-i\ 1),\ (i-1\ B-1\ B-1\ \ldots\ B-1\ B-i\ 2), \ldots, (i\ 1\ B-1\ \ldots\ B-1\ B-(i+1)\ \ B-3)\big\}$, *then* $b_{\max}(a) = b^*$.

(c) *In all other cases* $b_{\max}(a) = b^\natural = \lfloor a : (\lceil \lfloor a : b^* \rfloor : B \rceil\ B) \rfloor$.

5.3. Maximal divisors for N_n-digit numbers on abaci with $n \equiv 2 (\operatorname{mod} 3)$ rods

In this subsection we assume that $n = 3m+2$ with $m \geq 3$, $\delta_B(a) = N_n = m$ and (unless otherwise stated) $\delta_B(b) = N_n - 1 = m-1$. The maximal number $B^{m-1} - 1$ with $m-1$ digits shall be denoted by b^*. The integer quotient $q = \lfloor a : b \rfloor$ has either one or two digits. The first division step needs either $3 + N_n + N_n - 1 + 1 + N_n - 1 = n$ or $n+1$ columns, so we get the first condition for the division to be feasible: $b < \frac{B^{m-1}}{q_1}$, where q_1 denotes the first (and possibly the only) digit of q. If needed, the second step requires either $2N_n + 5 < n$ columns (if the second digit of q is zero) or at least $2N_n + 4 + N_n - 1 > n$ columns. Thus we have a second condition: if $q \geq B$, then the second digit of q has to be zero.

Lemma 12. *All integer quotients* $q = \lfloor a : b^* \rfloor$, *where* a *has* m *digits and* $b^* = B^{m-1} - 1$, *are smaller than* $(11)_B = B + 1$.

Proof. If $q \geq B+1$ then $qb^* \geq B^m + B^{m-1} - B - 1 > B^m$ for all $B \geq 2$ and $m \geq 3$. In particular it is impossible for an m-digit number to be of the form $qb^* + r$ with $q \geq B+1$ and $r \geq 0$. ◊

For binary calculations this amounts to: the smallest quotient for which the calculation cannot be performed is $q = (11)_2 = 3$. The previous lemma implies that this cannot happen, i.e. for $B = 2$ and $n = 3m+2$, for numbers a with m digits we have $b_{\max}(a) = b^*$.

Now we turn to bases $B > 2$. The conditions stated above for the division to be performable imply that the only cases when $b_{\max}(a) \neq b^*$ arise when $q_1 b^* \geq B^{m-1}$ or if there is a second digit in q that is not zero. Because of the previous lemma, the maximal possible two-digit quotient

$\lfloor a : b^* \rfloor$ is $(10)_B$, so in fact we have only one condition: the first partial product has to have $m-1$ digits. The possible quotients of a by b^* are $1, 2, \ldots, B$. If $q = 1$ or $q = B$, the division is feasible, i.e. for $a < 2b^*$ and for $a \geq Bb^*$ we have $b_{\max}(a) = b^*$. For $a \in \{2b^*, \ldots, Bb^* - 1\}$ the first partial product is at least $2b^* > B^m$, so $b_{\max}(a) \neq b^*$. Also, it is impossible to obtain $q = 1$ in the division of $a \in \{2b^*, \ldots, Bb^* - 1\}$ by a number $b < b^*$ ($a = 1 \cdot b + r$ would imply that $a < 2b^*$, contrary to our assumption). Since we want the largest b, i.e. the smallest possible q, we now check if there are performable divisions of $a \in \{2b^*, \ldots, Bb^* - 1\}$ by b such that q has one digit. In such a case b must meet the condition $qb < B^{m-1}$ and thus $2b^* \leq a = qb + r < (q+1)b < \frac{q+1}{q}B^{m-1}$. The last inequality implies that $B^{m-1} < \frac{2q}{q-1}$. Since $m, B \geq 3$ we have $B^{m-1} \geq$ $\geq 3^2 = 9$, so the only possibilities for a division of a by b to be performable would be in the cases when $9 < \frac{2q}{q-1}$, i.e. when $7q < 9$, which is impossible for $q \geq 2$. Thus there are no cases where $a \geq 2b^*$, q has only one digit and the division is performable. Because of the previous lemma, the only possibility for a division of a number $a \in \{2b^*, \ldots, Bb^* - 1\}$ by a number b with $m-1$ digits to be performable is the case when $q = (10)_B = B$.

Let now a be any m-digit number in the set $\{2b^*, \ldots, Bb^* - 1\}$. If there is a b with $m-1$ digits such that $a = Bb + r$, $0 \leq r < b$, then this b is the searched for maximal divisor. Set $b = \lfloor \frac{a}{B} \rfloor$. Obviously this b fulfils the equality $a = Bb + r$, so we only have to check that b has one digit less than a. But, dividing any number $a = (a_{m-1} \ldots a_2 a_1 a_0)_B$ by B results in the integer quotient $b = (a_{m-1} \ldots a_2 a_1)_B$ with $m-1$ digits. So we have proven

Theorem 5. *Let $n = 3m + 2 \geq 11$ be the number of rods on the abacus. Let a be a number with $N_n = m$ digits and let $b^* = B^{N_n - 1} - 1$ be the maximal $(N_n - 1)$-digit number. The maximal divisor b such that the division of a by b on an abacus with n rods can be performed is:*

(a) b^* *if* $a < 2b^*$ *or* $a \geq Bb^*$ *or*

(b) $\lfloor a : B \rfloor$ *otherwise.*

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