Mathematica Pannonica
22/1 (2011), 19–23

ON GENERALIZED ϕ-RECURRENT SASAKIAN MANIFOLDS

Mahuya **Bandyopadhyay**

53 Cragg Street, Condell Park, NSW 2200, Australia (Formerly Lecturer at S. A. Jaipuria College, Kolkata, India)

Received: January 2010

MSC 2000: 53 C 25

Keywords: Sasakian manifold, generalized recurrent manifold, generalized ϕ-recurrent manifold, Einstein manifold.

Abstract: The objective of this paper is to study generalized ϕ-recurrent Sasakian manifolds.

1. Introduction

A Riemannian manifold (M^n, g) is called generalized recurrent [2] if its curvature tensor R satisfies the condition

$$(1.1)\quad (\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)\big[g(Z,W)Y - g(Y,W)Z\big]$$

where α and β are two 1-forms, β is non-zero and these are defined by

$$(1.2)\qquad g(X,A) = \alpha(X), \quad g(X,B) = \beta(X)$$

where A and B are vector fields associated with 1-forms α and β respectively.

In [4], Q. Khan studied generalized recurrent Sasakian manifolds. In 2003 [3], U. C. De, A. A. Shaikh and S. Biswas considered ϕ-recurrent Sasakian manifolds. In this study we consider generalized ϕ-recurrent

E-mail address: mahuya7@gmail.com

Sasakian manifolds and obtain some interesting results. Here it is shown that a generalized ϕ-recurrent Sasakian manifold is an Einstein manifold. Also the relationship between the linear forms α, β and the corresponding vector fields A, B in a generalized ϕ-recurrent Sasakian manifold is found. Later, it is proved that any generalized ϕ-recurrent Sasakian manifold is a space of constant curvature.

2. Preliminaries

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian manifold where ϕ is a skew-symmetric tensor field of type (1, 1), ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is known that the structure (ϕ, ξ, η, g) satisfy the following relations [1]

$$\phi(\xi) = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$\phi^2 X = -X + \eta(X)\xi, \; g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$R(\xi, X)Y = (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

$$\nabla_X \xi = -\phi X, \; (\nabla_X \eta)(Y) = g(X, \phi Y), \tag{2.5}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.7}$$

$$R(X, \xi)\xi = X - \eta(X)\xi, \tag{2.8}$$

$$\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z), \tag{2.9}$$

$$S(X, \xi) = 2n\eta(X), \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y) \tag{2.11}$$

for all vector fields X, Y, Z, where ∇ denotes the operator of covariant differentiation with respect to the metric g, S is the Ricci tensor of type (0, 2) and R is the Riemann curvature tensor of the manifold.

3. Generalized ϕ-recurrent Sasakian manifolds

Definition 1. A Sasakian manifold (M^{2n+1}, g) is said to be a generalized ϕ-recurrent if its curvature tensor R satisfies the condition

$$\phi^2\big((\nabla_W R)(X,Y)Z\big) = \alpha(W)R(X,Y)Z + \beta(W)\big[g(Y,Z)X - g(X,Z)Y\big] \tag{3.1}$$

where α and β are associated 1-forms as defined in (1.2).

From (3.1), using (2.2) we have

$$-(\nabla_W R)(X,Y)Z + \eta\big((\nabla_W R)(X,Y)Z\big)\xi = \alpha(W)R(X,Y)Z + \beta(W)\big[g(Y,Z)X - g(X,Z)Y\big] \tag{3.2}$$

from which it follows that

$$-g\big((\nabla_W R)(X,Y)Z,U\big) + \eta\big((\nabla_W R)(X,Y)Z\big)\eta(U) = \alpha(W)g(R(X,Y)Z,U) + \beta(W)\big[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\big]. \tag{3.3}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, 2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.3) and taking summation over i, $1 \leqslant i \leqslant 2n+1$, we get

$$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta\big((\nabla_W R)(e_i,Y)Z\big)\eta(e_i) = \alpha(W)S(Y,Z) + 2n\beta(W)g(Y,Z). \tag{3.4}$$

Replacing $Z = \xi$ in (3.4) and using (2.2), (2.4), (2.5) and (2.10) we have

$$-(\nabla_W S)(Y,\xi) = 2n\alpha(W)\eta(Y) + 2n\beta(W)\eta(Y). \tag{3.5}$$

Now we have $(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi)$.

Using (2.5) and (2.10) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = -2ng(Y,\phi W) + S(Y,\phi W). \tag{3.6}$$

In view of (3.5) and (3.6) we obtain

$$-2ng(Y,\phi W) + S(Y,\phi W) = -2n\alpha(W)\eta(Y) - 2n\beta(W)\eta(Y). \tag{3.7}$$

Replacing Y by ξ in (3.7) and using (2.1), (2.2) and (2.10) we find

$$\alpha(W) + \beta(W) = 0. \tag{3.8}$$

Replacing Y by ϕY in (3.7) and then using (2.1), (2.3) and (2.11) we get

$$S(Y, W) = 2ng(Y, W) \quad \text{for all} \quad Y,\, W. \tag{3.9}$$

This leads to the following results:

Lemma 1. *A generalized ϕ-recurrent Sasakian manifold (M^{2n+1}, g) is an Einstein manifold.*

Lemma 2. *In a generalized ϕ-recurrent Sasakian manifold the linear forms α, β and the corresponding vector fields A, B satisfy the relations $\alpha + \beta = 0$, $A + B = 0$.*

As a consequence of Lemma 2 we can state the following alternative form:

Lemma 3. *There exists no generalized ϕ-recurrent Sasakian manifold (M^{2n+1}, g) if $\alpha + \beta$ is not everywhere zero.*

Now from (2.5) and (2.6), it can be easily seen that in a Sasakian manifold the following relations holds:

$$(\nabla_W R)(X, Y)\xi = g(W, \phi Y)X - g(W, \phi X)Y + R(X, Y)\phi W. \tag{3.10}$$

By virtue of (2.9) it follows from (3.10) that

$$\eta\big((\nabla_W R)(X, Y)\xi\big) = 0. \tag{3.11}$$

Again from Tanno [5] we have

$$\begin{aligned} R(X, Y)\phi Z = g(\phi X, Z)Y - g(Y, Z)\phi X - g(\phi Y, Z)X + \\ + g(X, Z)\phi Y + \phi R(X, Y)Z \end{aligned} \tag{3.12}$$

for any $X, Y, Z \in T_p M$. From (3.10) and (3.12), it follows that

$$(\nabla_W R)(X, Y)\xi = g(X, W)\phi Y - g(Y, W)\phi X + \phi R(X, Y)W. \tag{3.13}$$

In view of (3.13) and (3.11) we find from (3.2) that

$$\begin{aligned} g(X, W)\phi Y - g(Y, W)\phi X + \phi R(X, Y)W = \\ = -\big[\alpha(W) + \beta(W)\big]\big[\eta(Y)X - \eta(X)Y\big]. \end{aligned} \tag{3.14}$$

Now using (3.8) in (3.14), we have

$$g(X,W)\phi Y - g(Y,W)\phi X + \phi R(X,Y)W = 0. \tag{3.15}$$

Operating ϕ on both sides of (3.15) and using (2.2) we have

$$R(X,Y)W = g(Y,W)X - g(X,W)Y. \tag{3.16}$$

Hence we can state the following theorem:

Theorem 1. *A generalized ϕ-recurrent Sasakian manifold (M^{2n+1}, g) is a space of constant curvature* 1*.*

References

[1] BLAIR, D. E.: *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics No. 509, Springer, 1976.

[2] DE, U. C. and GUHA, N.: On generalized recurrent manifolds, *Proc. Math. Soc.* **7** (1991), 7–11.

[3] DE, U. C., SHAIKH, A. A. and BISWAS, S.: On ϕ-recurrent Sasakian manifolds, *Novi Sad J. Math.* **33** (2) (2003), 43–48.

[4] KHAN, Q.: On generalized recurrent Sasakian manifolds, *Kyungpook Math. J.* **44** (2004), 167–172.

[5] TANNO, S.: Isometric immersions of Sasakian manifold in spheres, *Kodai Math. Sem. Rep.* **21** (1969), 448–458.