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SOME NOTES ON THE ASYMPTOTICAL STABILITY OF DYNAMIC ECONOMIC SYSTEMS

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Abstract: The local and global asymptotical stability of equilibria of dynamic economic systems is examined. After a survey of the most important conditions is presented we introduce a new stability condition for discrete systems, when the iteration map is only piece-wise differentiable. Some examples of duopoly illustrate the theoretical findings.

1. Introduction

The long-term behavior of dynamic economic systems is one of the most important problem areas in mathematical economics. In the case of continuous time scales the asymptotic properties of trajectories of ordinary differential equations are examined, while in the case of discrete time scales the solutions of difference equations are studied. There are

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many different methods known from the literature in investigating the asymptotic properties of dynamic systems. In the case of time-invariant linear systems the locations of the eigenvalues of the coefficient matrix determine the stability properties of the system. If the linear system is time variant, then the system is marginally stable if the fundamental matrix is bounded, and if in addition the fundamental matrix converges to zero as $t \to \infty$, then the stability is asymptotical. In the case of linear systems local and global asymptotic stability are equivalent, however in the case of nonlinear systems we have to distinguish between local and global asymptotical stability. The most important results on the stability of linear systems can be found in all textbooks of linear systems theory (for example, Szidarovszky and Bahill, 1998).

The literature on nonlinear systems is less extensive. For continuous systems the stability issues are discussed in many books on ordinary differential equations (for example, Brauer and Nohel, 1969), for discrete systems the most relevant results are discussed, for example, in Gandolfo (1971), and La Salle (1976).

The asymptotical stability of nonlinear systems can be examined by several methods. The most common methodology is based on the different applications of the Lyapunov method. This approach is very useful in many cases however finding an appropriate Lyapunov function is usually a difficult problem, and the failure of finding a Lyapunov function does not prove the instability of the system. Local asymptotic stability can be shown by locating the eigenvalues of the Jacobian or by bounding the norm of the Jacobian at the equilibrium. There was an intensive research on extending the local asymptotical stability conditions into global stability and relaxing the sufficient conditions as much as possible. Parthasarathy (1983) gives an excellent background of this problem area in the continuous case, and Cima et al. (1999) discuss its discrete time scales counterpart.

In the classical mathematical literature there are several alternative stability conditions which guarantee global asymptotical stability of dynamic systems. More recent publications introduce more simple and more general stability conditions which could be very useful in economic studies.

In this paper we will give a brief overview of the current state of this research field. We will also present an extension of stability conditions to discrete system with piece-wise differentiable maps.

2. Stability conditions using norms

Consider first the one-dimensional discrete system

$$x(t+1) = f(x(t)) \tag{1}$$

where $f : \mathbb{D} \to \mathbb{R}$ is a continuously differentiable function and $\mathbb{D}$ is a (finite or infinite) interval. Let $\bar{x} \in \mathbb{D}$ be a fixed point of map f, that is, $\bar{x} = f(\bar{x})$. The following results are well known from numerical analysis:

Fact 1. Assume that $|f'(\bar{x})| < 1$, then $x(t) \to \bar{x}$ as $t \to \infty$ if $x(0)$ is selected sufficiently close to $\bar{x}$.

Fact 2. Assume that for all $x \in \mathbb{D}$, $|f'(x)| \leq q < 1$ with some constant q, then $x(t) \to \bar{x}$ as $t \to \infty$ with arbitrary $x(0) \in \mathbb{D}$.

Notice that in the case of Fact 1 only local convergence to the equilibrium is guaranteed based on only local information on the derivative, while Fact 2 guarantees global convergence based on global information on this derivative. The convergence to $\bar{x}$ also implies that in the case of Fact 1 there is a neighborhood of $\bar{x}$ such that $\bar{x}$ is the only fixed point there, and in the case of Fact 2, $\bar{x}$ is the only fixed point in the entire domain $\mathbb{D}$.

In multidimensional case the derivative $f'(x)$ is replaced by the Jacobian matrix, and naturally the absolute value of the derivative is replaced by the norm of the Jacobian. Consider therefore the n-dimensional system

$$\boldsymbol{x}(t+1) = \boldsymbol{f}(\boldsymbol{x}(t)) \tag{2}$$

where $\boldsymbol{f} : \mathbb{D} \mapsto \mathbb{R}^n$ is a continuously differentiable function and $\mathbb{D}$ is a convex subset of $\mathbb{R}^n$. Let $\bar{\boldsymbol{x}} \in \mathbb{D}$ be a fixed point of map $\boldsymbol{f}$, and let $\boldsymbol{J}(\boldsymbol{x})$ denote the Jacobian of $\boldsymbol{f}$ at $\boldsymbol{x}$. The following results are well known from systems theory:

Fact 3. Assume that with some matrix norm $||\boldsymbol{J}(\bar{\boldsymbol{x}})|| < 1$. Then $\boldsymbol{x}(t) \to \bar{\boldsymbol{x}}$ as $t \to \infty$ if $\boldsymbol{x}(0)$ is selected sufficiently close to $\bar{\boldsymbol{x}}$.

Fact 4. Assume that $||\boldsymbol{J}(\boldsymbol{x})|| \leq q < 1$ for all $\boldsymbol{x} \in \mathbb{D}$, where q is a constant. Then $\boldsymbol{x}(t) \to \bar{\boldsymbol{x}}$ as $t \to \infty$ with arbitrary $\boldsymbol{x}(0) \in \mathbb{D}$.

Similarly to the single dimensional case, there is a neighborhood of $\bar{\boldsymbol{x}}$ such that $\bar{\boldsymbol{x}}$ is the only fixed point in it under the conditions of Fact 3, and in the case of Fact 4, $\bar{\boldsymbol{x}}$ is the only fixed point in the entire domain $\mathbb{D}$. These results are the most frequently used sufficient stability conditions to prove local or global asymptotic stability of the equilibrium. If a given matrix norm is greater than one, it does not prove instability, since there is the possibility that by selecting another norm stability still can be proved. The most commonly used matrix norms are as follows. Let a_{ij} denote the (i, j) element of an $n \times n$ matrix $\boldsymbol{A}$. Then

$$||\boldsymbol{A}||_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \quad \text{(column norm)} \tag{3}$$

$$||\mathbf{A}||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \quad \text{(row norm)} \tag{4}$$

$$||\mathbf{A}||_2 = \max_k \sqrt{\lambda^{(k)}_{\boldsymbol{A}^T\boldsymbol{A}}} \quad \text{(Euclidean norm)} \tag{5}$$

where the eigenvalues of $\boldsymbol{A}^T\mathbf{A}$ are $\lambda^{(k)}_{\boldsymbol{A}^T\boldsymbol{A}}$ $(k = 1, 2, \ldots, n)$.

This norm requires the computation of eigenvalues of the symmetric matrix. In many cases the Euclidean norm is replaced by the Frobenius norm:

$$||\boldsymbol{A}||_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2} \; (\geq ||\mathbf{A}||_2) \tag{6}$$

Example 1. As a simple example consider matrices

$$\boldsymbol{A}_1 = \begin{pmatrix} 0.8 & 0.8 \\ 0 & 0 \end{pmatrix}, \boldsymbol{A}_2 = \begin{pmatrix} 0.8 & 0 \\ 0.8 & 0 \end{pmatrix}, \text{ and } \boldsymbol{A}_3 = \begin{pmatrix} 0.58 & 0.58 \\ 0.58 & 0 \end{pmatrix}.$$

Then clearly

$$||\boldsymbol{A}_1||_1 = 0.8 \quad ||\boldsymbol{A}_1||_\infty = 1.6 \quad ||\boldsymbol{A}_1||_2 = ||\boldsymbol{A}_1||_F = 0.8 \cdot \sqrt{2} \approx 1.131$$
$$||\boldsymbol{A}_2||_1 = 1.6 \quad ||\boldsymbol{A}_2||_\infty = 0.8 \quad ||\boldsymbol{A}_2||_2 = ||\boldsymbol{A}_2||_F = 0.8 \cdot \sqrt{2} \approx 1.131$$

and

$$||\boldsymbol{A}_3||_1 = 1.16 \quad ||\boldsymbol{A}_3||_\infty = 1.16|| \quad \boldsymbol{A}_3||_2 = 0.58 \cdot \sqrt{\frac{3+\sqrt{5}}{2}} \approx 0.938$$
$$\text{and } ||\boldsymbol{A}_3||_F = 1.009.$$

Notice that in each case only one norm is below 1, the other three norms are greater than one. $\triangledown$

In many economic models function $\boldsymbol{f}$ is only piece-wise differentiable on $\mathbb{D}$ because of nonnegativity conditions on production levels and prices as well as presence of capacity limits. The above mentioned Facts 3 and 4 cannot be applied in such cases without further consideration. If $\boldsymbol{f}$ is continuously differentiable in a neighborhood of $\bar{\boldsymbol{x}}$, that is, $\bar{\boldsymbol{x}}$ is in the interior of a subregion of $\mathbb{D}$ in which $\boldsymbol{f}$ is continuously differentiable, then Fact 3 can be used to prove local asymptotical stability. However if $\bar{\boldsymbol{x}}$ is on the boundary between two subregions, then the Jacobian usually does not exist at $\bar{\boldsymbol{x}}$, so this result cannot be applied. Fact 4 assumes the existence of $\boldsymbol{J}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}$, which is not the case if $\boldsymbol{f}$ is only piecewise differentiable. With some additional considerations we can however extend these results.

Assume now that $\mathbb{D}$ is convex in $\mathbb{R}^n$ and it is the union of sets $\mathbb{D}^{(1)}, \mathbb{D}^{(2)}, \ldots$ with mutually exclusive interiors. Assume furthermore that $\boldsymbol{f}$ is continuous on $\mathbb{D}$ and its restriction $\boldsymbol{f}^{(k)}$ on $\mathbb{D}^{(k)}$ is continuously differentiable in the interior of $\mathbb{D}^{(k)}$ $(k = 1, 2, \ldots)$. Assume in addition that one of the following two conditions hold:

(A) For all k, there is an open set containing $\mathbb{D}^{(k)}$ such that $\boldsymbol{f}^{(k)}$ can be extended to it, and $\boldsymbol{f}^{(k)}$ remains continuously differentiable there. Assume also that for the linear segment connecting $\bar{\boldsymbol{x}}$ and any $\boldsymbol{x} \in \mathbb{D}$ there are finitely many values $0 = t_0 < t_1 < \cdots < t_{K(\boldsymbol{x})} = 1$ such that all points of the linear segment connecting $\bar{\boldsymbol{x}} + t_l(\boldsymbol{x} - \bar{\boldsymbol{x}})$ and $\bar{\boldsymbol{x}} + t_{l+1}(\boldsymbol{x} - \bar{\boldsymbol{x}})$ belongs to the same $\mathbb{D}^{(k_l)}$ $(l = 0, 1, \ldots, K(\boldsymbol{x}) - 1)$.

(B) For all linear subsegments connecting $\bar{\boldsymbol{x}}+t_l(\boldsymbol{x}-\bar{\boldsymbol{x}})$ and $\bar{\boldsymbol{x}}+t_{l+1}(\boldsymbol{x}-\bar{\boldsymbol{x}})$ defined in the previous assumption there are sequences $\{\boldsymbol{u}_i\} \to \bar{\boldsymbol{x}}+$ $+t_l(\boldsymbol{x} - \bar{\boldsymbol{x}})$ and $\{\boldsymbol{v}_i\} \to \bar{\boldsymbol{x}} + t_{l+1}(\boldsymbol{x} - \bar{\boldsymbol{x}})$ as $i \to \infty$ such that the entire linear segment connecting $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ is in the interior of $\mathbb{D}^{(k_l)}$ for all i.

Assume finally that there is a constant q, such that for all k, and for all $\boldsymbol{x} \in \mathbb{D}^{(k)}$ (under assumption (A)) or $\boldsymbol{x} \in \mathrm{int}\mathbb{D}^{(k)}$ (under assumption (B)), $||\boldsymbol{J}^{(k)}(\boldsymbol{x})|| \leq q < 1$, where $\boldsymbol{J}^{(k)}$ is the Jacobian of $\boldsymbol{f}^{(k)}$.

Theorem 1. *Under the above conditions $\bar{\boldsymbol{x}}$ is the only equilibrium and it is globally asymptotically stable.*

Proof. For all $\boldsymbol{x} \in \mathbb{D}$,

$$\begin{aligned}
||\boldsymbol{f}(\boldsymbol{x}) - \bar{\boldsymbol{x}}|| &= ||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\bar{\boldsymbol{x}})|| = \\
&= \left\| \sum_{l=0}^{K(\boldsymbol{x})-1} \left(\boldsymbol{f}(\bar{\boldsymbol{x}} + t_{l+1}(\boldsymbol{x} - \bar{\boldsymbol{x}})) - \boldsymbol{f}(\bar{\boldsymbol{x}} + t_l(\boldsymbol{x} - \bar{\boldsymbol{x}}))\right) \right\| \le \\
&\le \sum_{l=0}^{K(\boldsymbol{x})-1} \left\| \int_{t_l}^{t_{l+1}} \boldsymbol{J}^{(k_l)}\left(\bar{\boldsymbol{x}} + t(\boldsymbol{x} - \bar{\boldsymbol{x}})\right)(\boldsymbol{x} - \bar{\boldsymbol{x}}) dt \right\| \le \\
&\le \sum_{l=0}^{K(\boldsymbol{x})-1} \int_{t_l}^{t_{l+1}} \left\| \boldsymbol{J}^{(k_l)}\left(\bar{\boldsymbol{x}} + t(\boldsymbol{x} - \bar{\boldsymbol{x}})\right) \right\| \cdot \|\boldsymbol{x} - \bar{\boldsymbol{x}}\| \, dt \le \\
&\le q \cdot ||\boldsymbol{x} - \bar{\boldsymbol{x}}|| \sum_{l=0}^{K(\boldsymbol{x})-1} (t_{l+1} - t_l) = q \cdot ||\boldsymbol{x} - \bar{\boldsymbol{x}}||.
\end{aligned}$$

Therefore for $t \ge 1$,

$$||\boldsymbol{x}(t) - \bar{\boldsymbol{x}}|| \le q \cdot ||\boldsymbol{x}(t-1) - \bar{\boldsymbol{x}}|| \le \cdots \le q^t ||\boldsymbol{x}(0) - \bar{\boldsymbol{x}}||,$$

which converges to zero as $t \to \infty$. The uniqueness of the equilibrium is a simple consequence of this convergence. $\Diamond$

We will show some applications of this theorem in Sec. 4.

Local asymptotic stability is guaranteed, if the conditions of the theorem are true with $\mathbb{D}$ replaced by a neighborhood of the equilibrium that contains the equilibrium in its interior.

3. Stability conditions using eigenvalues

Consider first the discrete system (2). It is well known that if all eigenvalues of an $n \times n$ matrix $\boldsymbol{A}$ are inside the unit circle, then there is a matrix norm such that $||\boldsymbol{A}|| < 1$ (see for example, Ortega and Rheinboldt, 1970). This observation and Fact 3 imply the following result:

Fact 5. Assume $\boldsymbol{f}$ is continuously differentiable in a neighborhood of $\bar{\boldsymbol{x}}$ and all eigenvalues of the Jacobian $\boldsymbol{J}(\bar{\boldsymbol{x}})$ are inside the unit circle, then $\bar{\boldsymbol{x}}$ is locally asymptotically stable.

In the case of time invariant linear systems the Jacobian is the constant coefficient matrix, and in this case the condition of the theorem is sufficient and necessary. In the case of nonlinear systems this result gives only sufficient conditions, since if some eigenvalues have unit absolute values and all others are inside the unit circle, then the system can be unstable, or marginally stable, or even globally asymptotically stable as it is shown in the following examples:

Example 2. Consider system

$$\boldsymbol{x}(t+1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \boldsymbol{x}(t)$$

with both eigenvalues being equal to one. There are infinitely many equilibria: $\bar{x}_1 =$ arbitrary and $\bar{x}_2 = 0$. It is easy to see that

$$\boldsymbol{x}(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^t \boldsymbol{x}(0) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \boldsymbol{x}(0)$$

which show that if $x_2(0) \neq 0$, then $x_1(t)$ converges to $+\infty$ or $-\infty$ so the system is unstable.

Consider next the one-dimensional system

$$x(t+1) = -x(t)$$

with eigenvalue -1. Clearly, $x(t) = (-1)^t x(0)$, so the zero equilibrium is only marginally stable.

And finally consider again a one-dimensional system

$$x(t+1) = x(t)\mathrm{e}^{-x(t)^2}.$$

Notice that $x(0) > 0$ implies that for all t, $0 < x(t+1) < x(t)$ and if $x(0) < 0$, then $x(t) < x(t+1) < 0$. Hence $x(t)$ is monotonic and bounded, so convergent. Letting $t \to \infty$ in the systems equation implies that the limit equals zero, which is the unique equilibrium of the system. Hence the system is globally asymptotical stable. $\triangledown$

The extension of Fact 4 in terms of the eigenvalues of the Jacobian is not true in general. If the eigenvalues of $\boldsymbol{J}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}$ are inside the unit circle, then for all $\boldsymbol{x} \in \mathbb{D}$ there is a matrix norm such that the norm of $\boldsymbol{J}(\boldsymbol{x})$ is below one, however this norm is usually different for different values of $\boldsymbol{x}$. The following example gives a general n-dimensional discrete system in which the eigenvalues of $\boldsymbol{J}(\boldsymbol{x})$ are inside the unit circle even there is a constant $q \in [0, 1)$ such that the absolute values of the eigenvalues of $\boldsymbol{J}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ are less than q, and the system is not globally asymptotically stable (for more details see Cima et al., 1999).

Example 3. Consider the n-dimensional system (1) with

$$\boldsymbol{f}(\boldsymbol{x}) = \left(-\frac{kx_2^3}{1+x_1^2+x_2^2}, \frac{kx_1^3}{1+x_1^2+x_2^2}, \frac{1}{2}x_3, \ldots, \frac{1}{2}x_n \right)^T$$

where $k \in \left(1, \frac{2}{\sqrt{3}}\right)$. This function is continuously differentiable in $\mathbb{R}^n$, $\bar{\boldsymbol{x}} = 0$ is a fixed point and it has the following properties:

(a) The eigenvalues of the Jacobian are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \cdots = \lambda_n = \frac{1}{2}$ if $x_1 x_2 = 0$, otherwise λ_1 and λ_2 are complex with absolute values less than $\frac{\sqrt{3}}{2}k < 1$ and $\lambda_3 = \cdots = \lambda_n = \frac{1}{2}$;

(b) $\boldsymbol{f}^4\left(\frac{1}{\sqrt{k-1}}, 0, \ldots, 0\right) = \left(\frac{1}{\sqrt{k-1}}, 0, \ldots, 0\right)$. $\triangledown$

In important special cases however such counter example cannot be found, as the following results state (see Cima et al., 1999).

Fact 6. Assume that $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a continuously differentiable triangular map, that is,

$$\boldsymbol{f}(\boldsymbol{x}) = \left(f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, x_2, \ldots, x_n)\right).$$

Let $\bar{\boldsymbol{x}}$ be an equilibrium of system (2) and assume that the absolute values of the eigenvalues of $\boldsymbol{J}(\boldsymbol{x})$ are less than one for all $\boldsymbol{x} \in \mathbb{R}^n$. Then $\bar{\boldsymbol{x}}$ is globally asymptotically stable.

Fact 7. Let $\boldsymbol{f} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a polynomial map with all eigenvalues of $\boldsymbol{J}(\boldsymbol{x})$ having absolute values less than one for all $\boldsymbol{x} \in \mathbb{R}^2$. Then system (2) has a unique equilibrium that is globally asymptotically stable.

The following example (see Cima et al., 1997) shows that Fact 7 does not hold for all higher dimensional systems.

Example 4. Assume now that

$$\boldsymbol{f}(\boldsymbol{x}) = \left(\frac{1}{2}x_1 + x_3 d(\boldsymbol{x})^2, \frac{1}{2}x_2 - d(\boldsymbol{x})^2, \frac{1}{2}x_3, \ldots, \frac{1}{2}x_n\right)^T$$

with $d(\boldsymbol{x}) = x_1 + x_2 x_3$. This is clearly a polynomial map with zero fixed point and it satisfies the following properties:

(a) For all $\boldsymbol{x} \in \mathbb{R}^n$, the eigenvalues of $\boldsymbol{J}(\boldsymbol{x})$ are equal to $\frac{1}{2}$;

(b) If $\boldsymbol{x}(0) = \left(\frac{147}{32}, -\frac{63}{32}, 1, 0, \ldots, 0\right)^T$, then for $t \geq 1$,

$$\boldsymbol{x}(t) = \left(\frac{147}{32} \cdot 2^t, -\frac{63}{32} \cdot 2^{2t}, \left(\frac{1}{2}\right)^t, 0, \ldots, 0\right)^T,$$

so the system is unstable. $\triangledown$

The following statement gives a sufficient condition for the instability of the equilibrium

Fact 8. Assume that at least one eigenvalue of $\boldsymbol{J}(\bar{\boldsymbol{x}})$ has absolute value larger than one. Then $\bar{\boldsymbol{x}}$ is unstable.

A simple elementary proof of this result is given in Li and Szidarovszky (1999).

We now turn our attention to the continuous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \tag{7}$$

where $\boldsymbol{f} : \mathbb{D} \mapsto \mathbb{R}^n$ with $\mathbb{D} \subset \mathbb{R}^n$. Assume that $\bar{\boldsymbol{x}}$ is an equilibrium of this system, which is in the interior of $\mathbb{D}$.

The continuous counterpart of Fact 5 is well known and can be given as follows.

Fact 9. Assume $\boldsymbol{f}$ is continuously differentiable in a neighborhood of $\bar{\boldsymbol{x}}$ and all eigenvalues of $\boldsymbol{J}(\bar{\boldsymbol{x}})$ have negative real parts. Then $\bar{\boldsymbol{x}}$ is locally asymptotically stable.

Similarly to the discrete case, this condition is sufficient and necessary for time invariant, linear systems. However in the case of nonlinear systems the above condition is only sufficient, since if some eigenvalues have zero real parts and all other eigenvalues have negative real parts, then the system can be unstable, or marginally stable, or even globally asymptotically stable, as it is illustrated in the following examples.

Example 5. Consider first the two-dimensional system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \boldsymbol{x}$$

where the equilibrium is $\bar{x}_1 =$ arbitrary and $\bar{x}_2 = 0$. Both eigenvalues of the Jacobian are equal to zero, and it is easy to see that

$$\boldsymbol{x}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \boldsymbol{x}(0),$$

so the system is unstable.

Consider next the one-dimensional system

$$\dot{x} = 0$$

with zero eigenvalue. Then all solutions are constant, the equilibrium is any real number and all are marginally stable.

Consider finally the system

$$\dot{x} = -x^3$$

where $\overline{x} = 0$ is the only equilibrium, and the eigenvalue is zero at $\overline{x} = 0$. This equation is separable, so it can be easily solved:

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx(0)^2}}.$$

Clearly with $x(0) \neq 0$, $x(t) \to 0$ as $t \to \infty$ showing the global asymptotical stability of the equilibrium. $\triangledown$

The extension if Fact 9 to global asymptotical stability is true only in the two-dimensional case (see Gutierrez, 1995; Fernandes et al., 2004).

Fact 10. Assume that $\boldsymbol{f} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is differentiable and $\bar{\boldsymbol{x}}$ is an equilibrium of system (7). Assume furthermore that for all $\boldsymbol{x} \in \mathbb{R}$, the eigenvalues if $\boldsymbol{J}(\boldsymbol{x})$ have negative real parts. Then $\bar{\boldsymbol{x}}$ is globally asymptotically stable.

This result however is not true in higher dimensions as the following example (Cima et al., 1997) shows.

Example 6. Let

$$\boldsymbol{f}(\boldsymbol{x}) = \left(-x_1 + x_3 d(\boldsymbol{x})^2, -x_2 - d(\boldsymbol{x})^2, -x_3, \ldots, -x_n\right)^T$$

with $d(\boldsymbol{x}) = x_1 + x_2 x_3$ as in Example 4. It is easy to see that the eigenvalues of $\boldsymbol{J}(\boldsymbol{x})$ are equal to -1 for all $\boldsymbol{x} \in \mathbb{R}^n$, and

$$x_1(t) = 18\mathrm{e}^t, x_2(t) = -12\mathrm{e}^{2t}, x_3(t) = \cdots = x_n(t) = 0$$

is a solutions of system (7) which shows that the zero equilibrium cannot be globally asymptotically stable. $\triangledown$

The continuous counterpart of Fact 6 remains valid for continuous systems (see Markus and Yamabe, 1960).

Fact 11. Assume that $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a continuously differentiable triangular map such that for all i and $\boldsymbol{x} \in \mathbb{R}^n$, $\frac{\partial f_i}{\partial x_i}(\boldsymbol{x}) < 0$. Then the equilibrium of system (7) is globally asymptotically stable.

Another useful sufficient condition is the following (see Markus and Yamabe, 1960).

Fact 12. Assume $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuously differentiable, and for all $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{x}) + \boldsymbol{J}(\boldsymbol{x})^T$ is negative definite. Assume furthermore that there are positive constants α and β such that

$$|\mathrm{Trace}\boldsymbol{M}(\boldsymbol{x})| < \alpha \text{ and } |\mathrm{Det}\boldsymbol{M}(\boldsymbol{x})| > \beta.$$

Then $\bar{\boldsymbol{x}}$ is globally asymptotically stable.

Similarly to the discrete case, a simple sufficient instability condition is given in the following result.

Fact 13. Assume $\boldsymbol{f}$ is continuously differentiable in a neighborhood of $\bar{\boldsymbol{x}}$, and at least one eigenvalue of $\boldsymbol{J}(\bar{\boldsymbol{x}})$ has positive real part. Then $\bar{\boldsymbol{x}}$ is unstable.

4. Stability of special duopolies

It was earlier mentioned that if the eigenvalues of a matrix are inside the unit circle, then there is a matrix norm such that the norm of the matrix is below unity. This matrix norm is based on the eigenvalues and the Jordan canonical form of the matrix. So it is hard to compute and it is not monotonic in the matrix elements, so it cannot be used if one or more matrix elements is changed to zero. In the 2-dimensional case for a large group of matrices a very special type of norm can be selected which can be easily computed and is monotonic. Assume that all eigenvalues of matrix

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are inside the unit circle. The characteristic polynomial of $\boldsymbol{A}$ is given as $\varphi(\lambda) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + (ad-bc)$, so the matrix elements satisfy relations (see for example, Bischi et al., 2010)

$$\pm(a+d) + (ad-bc) + 1 > 0, \tag{8}$$

$$ad - bc < 1. \tag{9}$$

Select a diagonal matrix

$$\boldsymbol{T} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

with $x > 0$, and consider the row norm of $\boldsymbol{A}$ generated by matrix $\boldsymbol{T}$:

$$||\boldsymbol{TAT}^{-1}||_\infty = \left\| \begin{pmatrix} a & bx \\ \frac{c}{x} & d \end{pmatrix} \right\|_\infty = \max\left\{ |a| + |b|x, \frac{|c|}{x} + |d| \right\}.$$

This is below unity, if $|a| < 1, |d| < 1$, furthermore

$$|a| + |b|x < 1 \text{ and } \frac{|c|}{x} + |d| < 1,$$

which occurs if and only if

$$\frac{|c|}{1-|d|} < x < \frac{1-|a|}{|b|}. \tag{10}$$

Suitable x exists, if

$$|bc| < (1-|a|)(1-|d|)$$

or

$$-(|a| + |d|) + |ad| - |bc| + 1 > 0 \tag{11}$$

which is consequence of condition (8) if $ad \geq 0$ and $bc \geq 0$. Hence we have the following result.

Fact 14. Assume $|a| < 1, |d| < 1, ad \geq 0$ and $bc \geq 0$. Assume furthermore that the eigenvalues of matrix $\boldsymbol{A}$ are inside the unit circle. Then there is a row norm generated by a diagonal matrix with positive diagonal, such that the norm of $\boldsymbol{A}$ and that of all other matrices which can be obtained from $\boldsymbol{A}$ by decreasing the absolute value of at least one of its elements, are all less than unity.

(a) Consider first a duopoly with linear cost ($C_k(x_k) = c_k x_k + d_k$) and price ($p(x_1 + x_2) = A - B(x_1 + x_2)$) functions. Then the profit of firm k has the form

$$\varphi_k(x_1, x_2) = x_k(A - Bx_1 - Bx_2) - (c_k x_k + d_k)$$

where all coefficients are positive. Assume that both firms have finite capacity limits, so $0 \leq x_k \leq L_k$ for $k = 1, 2$. The best response of firm k is

$$R_k(x_l) = \begin{cases} 0 & \text{if } x_l \geq \dfrac{A - c_k}{B}, \\ L_k & \text{if } x_l \leq \dfrac{A - c_k - 2BL_k}{B}, \\ \dfrac{A - c_k - Bx_l}{2B} & \text{otherwise} \end{cases}$$

with $l \neq k$. By assuming partial adjustment to best responses, the dynamic system can be written as follows:

$$x_k(t+1) = x_k(t) + \alpha_k \cdot \left(R_k\left(x_l(t)\right) - x_k(t)\right) \quad (k = 1, 2). \tag{12}$$

Notice that the value of R_k' is either $-\frac{1}{2}$ or 0 in the different segments, so the four possible Jacobians are

$$\begin{pmatrix} 1 - \alpha_1 & -\frac{\alpha_1}{2} \\ -\frac{\alpha_2}{2} & 1 - \alpha_2 \end{pmatrix}, \begin{pmatrix} 1 - \alpha_1 & 0 \\ -\frac{\alpha_2}{2} & 1 - \alpha_2 \end{pmatrix}, \begin{pmatrix} 1 - \alpha_1 & -\frac{\alpha_1}{2} \\ 0 & 1 - \alpha_2 \end{pmatrix}$$
$$\text{and} \quad \begin{pmatrix} 1 - \alpha_1 & 0 \\ 0 & 1 - \alpha_2 \end{pmatrix}.$$

It is always assumed that $0 < \alpha_k \leq 1$, so the row norms of these matrices are

$$\max\left\{1 - \frac{\alpha_1}{2}; 1 - \frac{\alpha_2}{2}\right\}, \ \max\left\{1 - \alpha_1; 1 - \frac{\alpha_2}{2}\right\}, \ \max\left\{1 - \frac{\alpha_1}{2}; 1 - \alpha_2\right\}$$
$$\text{and} \quad \max\left\{1 - \alpha_1; 1 - \alpha_2\right\},$$

respectively. They are all less than or equal to

$$\max\left\{1 - \frac{\alpha_1}{2}; 1 - \frac{\alpha_2}{2}\right\} < 1,$$

consequently Th. 1 implies that global asymptotical stability of the equilibrium.

(b) Assume again linear price function but quadratic costs $\big(C_k(x_k) = c_k x_k + d_k x_k^2\big)$. The payoff of firm k is now as follows:

$$\varphi_k(x_1, x_2) = x_k(A - Bx_1 - Bx_2) - (c_k x_k + d_k x_k^2).$$

If $B + d_k > 0$ for $k = 1, 2$, then φ_k is strictly concave in x_k, so the best response of firm k is given as

$$R_k(x_l) = \begin{cases} 0 & \text{if } x_l \geq \dfrac{A - c_k}{B}, \\ L_k & \text{if } x_l \leq \dfrac{A - c_k - 2(B + d_k)L_k}{B}, \\ \dfrac{A - c_k - Bx_l}{2(B + d_k)} & \text{otherwise.} \end{cases}$$

By assuming again partial adjustment to best responses (dynamic equations (12)), the four possible Jacobians are

$$\begin{pmatrix} 1 - \alpha_1 & -\frac{\alpha_1 B}{2(B+d_1)} \\ -\frac{\alpha_2 B}{2(B+d_2)} & 1 - \alpha_2 \end{pmatrix}, \begin{pmatrix} 1 - \alpha_1 & 0 \\ -\frac{\alpha_2 B}{2(B+d_2)} & 1 - \alpha_2 \end{pmatrix}, \begin{pmatrix} 1 - \alpha_1 & -\frac{\alpha_1 B}{2(B+d_1)} \\ 0 & 1 - \alpha_2 \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} 1 - \alpha_1 & 0 \\ 0 & 1 - \alpha_2 \end{pmatrix}.$$

Since there is no guarantee that $B + d_1 > B$, the row norms of the Jacobians might be larger than unity. We will however apply Fact 14 to show the global asymptotical stability of the equilibrium. Notice first that the first matrix satisfies the conditions of Fact 14 concerning the matrix elements. The eigenvalues are inside the unit circle if and only if

$$(1 - \alpha_1)(1 - \alpha_2) - \frac{\alpha_1 \alpha_2 B^2}{4(B + d_1)(B + d_2)} < 1,$$

$$\pm(2 - \alpha_1 - \alpha_2) + (1 - \alpha_1)(1 - \alpha_2) - \frac{\alpha_1 \alpha_2 B^2}{4(B + d_1)(B + d_2)} + 1 > 0.$$

The first inequality is clearly satisfied, and the second holds if

$$-2 + \alpha_1 + \alpha_2 + 1 - \alpha_1 - \alpha_2 + \alpha_1 \alpha_2 \left(1 - \frac{B^2}{4(B + d_1)(B + d_2)}\right) + 1 > 0$$

which is valid if

$$B^2 < 4(B + d_1)(B + d_2).$$

Hence under this condition the equilibrium is globally asymptotically stable.

(c) Consider next the case of quadratic price function $p(x_1+x_2) = A - (x_1 + x_2)^2$ and linear costs $C_k(x_k) = c_k x_k + d_k$ with $A \geq c_k$. The profit of firm k is given as

$$\varphi_k(x_1, x_2) = x_k \left(A - (x_k + x_\ell)^2\right) - (c_k x_k + d_k),$$

where $\ell \neq k$ and we assume that $L_1 + L_2 \leq \sqrt{A}$, that is, price is always nonnegative. Notice that

$$\frac{\partial \varphi_k}{\partial x_k} = A - (x_k + x_\ell)^2 - 2x_k(x_k + x_\ell) - c_k$$

and

$$\frac{\partial^2 \varphi_k}{\partial x_k^2} = -4(x_k + x_\ell) - 2x_k < 0,$$

so φ_k is strictly concave in x_k, and the best response is unique. Let x_k^* be the solution of the first order condition
$A - (x_k + x_\ell)^2 - 2x_k(x_k + x_\ell) - c_k = -3x_k^2 - 4x_k x_\ell + (A - x_\ell^2) - c_k = 0,$
that is,

$$x_k^* = \frac{1}{3}\left(-2x_\ell + \sqrt{x_\ell^2 + 3(A - c_k)}\right).$$

Then the best response of firm k is given as

$$R_k(x_\ell) = \begin{cases} 0 & \text{if } x_k^* \leq 0, \\ L_k & \text{if } x_k^* \geq L_k, \\ x_k^* & \text{otherwise.} \end{cases}$$

There are three segments, in which the derivative of R_k is either 0 or

$$\frac{1}{3}\left(-2 + \frac{x_\ell}{\sqrt{x_\ell^2 + 3(A - c_k)}}\right) \in \left[-\frac{2}{3}, -\frac{1}{3}\right],$$

so

$$\left|R_k'\right| \leq \frac{2}{3}$$

except the boundaries between the segments. The Jacobian of the dynamic system (12) has now the forms

$$\begin{pmatrix} 1 - \alpha_1 & \alpha_1 R_1' \\ \\ \alpha_2 R_2' & 1 - \alpha_2 \end{pmatrix}$$

where $0 < \alpha_1 \leq 1$, $0 < \alpha_2 \leq 1$ and $|R_k'| \leq \frac{2}{3}$ $(k = 1, 2)$. Consequently in all segments the row norm of the Jacobian is below

$$\max\left\{1-\alpha_1+\frac{2\alpha_1}{3}, 1-\alpha_2+\frac{2\alpha_2}{3}\right\} = \max\left\{1-\frac{\alpha_1}{3}, 1-\frac{\alpha_2}{3}\right\}$$

which is less than unity implying the global asymptotical stability of the equilibrium.

(d) Let's turn our attention to general concave duopolies. Let p be the price function and C_k the cost function of firm k. It is assumed that these functions are twice continuously differentiable and

a) $p' < 0$;

b) $p' + x_k p'' \le 0$;

c) $p' - C_k'' < 0$

for all feasible values of the relevant variables. It is known from oligopoly theory (see, for example, Bischi et al., 2010) that in the case of finite capacity limits the best response of firm k is as follows:

$$R_k(x_l) = \begin{cases} 0 & \text{if } p(x_l) - C_k'(0) \le 0, \\ L_k & \text{if } p(x_l+L_k) + L_k p'(x_l+L_k) - C_k'(L_k) \ge 0, \\ x_k^* & \text{otherwise} \end{cases}$$

where x_k^* is the unique solution of the monotonic equation

$$p(x_l+x_k) + x_k p'(x_l+x_k) - C_k'(x_k) = 0$$

inside interval $(0, L_k)$. By implicit differentiation it is easy to see that in the case of interior best response

$$-1 < R_k'(x_l) \le 0$$

and in the first two cases $R_k'(x_l) \equiv 0$. By assuming dynamic equations (12), the Jacobian has again four possibilities:

$$\begin{pmatrix} 1-\alpha_1 & \alpha_1 R_1' \\ \alpha_2 R_2' & 1-\alpha_2 \end{pmatrix}, \begin{pmatrix} 1-\alpha_1 & 0 \\ \alpha_2 R_2' & 1-\alpha_2 \end{pmatrix}, \begin{pmatrix} 1-\alpha_1 & \alpha_1 R_1' \\ 0 & 1-\alpha_2 \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} 1-\alpha_1 & 0 \\ 0 & 1-\alpha_2 \end{pmatrix}.$$

The feasible region and all subregions are compact. Therefore there are constants $r_1,\ r_2$ such that

$$-1 < -r_1 \le R_1'(x_2) \le 0 \quad \text{and} \quad -1 < -r_2 \le R_2'(x_1) \le 0$$

everywhere in the feasible region except on the boundaries between the subregions. The row norms of the Jacobian matrices in all regions are less than or equal to

$$\max\{1-\alpha_1+r_1\alpha_1; 1-\alpha_2+r_2\alpha_2\} < 1$$

implying the global asymptotical stability of the equilibrium.

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