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HOPF BIFURCATION IN A DELAYED HUMAN MIGRATION MODEL

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Abstract: In this paper, a mathematical model will be developed that describes the spread of information in a human population. This model, consisting of a two dimensional system of differential equations, has been motivated by attempts to explain the behaviour of individuals (students) who may choose between two alternatives with little, respectively more risk.

We shall show that by an additional assumption, namely, the growth rate of the population in the option with more risk does not depend only on the present density of the other population in the option with little risk but also on past densities, how the increase of this delay influences the behaviour of the system: as the delay is increased the originally asymptotic stable interior equilibrium loses its stability and at a certain critical value a Hopf bifurcation takes place: small amplitude periodic solutions arise. The stability condition for bifurcating periodic solutions is derived by using the method of Poore. A numerical simulation for supporting the theoretical analysis is also given.

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1. Introduction

In [9] the first author of this paper has considered the following system (proposed by Scheurle and Seydel (cf. [13])) that describes the spread of information among individuals

$$\text{(1)} \qquad \begin{cases} \dot{S}_1 = \lambda - aS_1S_2 + \beta S_2 - \delta_1 S_1, \\ \dot{S}_2 = aS_1S_2 - \beta S_2 - \delta_2 S_2 \end{cases}$$

where the dot means differentiation with respect to time t; $S_1(t) \geq 0$ and $S_2(t) \geq 0$ are the numbers or densities of individuals. By these individuals students at universities were meant, who, during their studies, face the necessity to choose between a popular option with little risk (a subject having reputation of comparably easy examinations with favourable grades) on the one side, and a demanding and more risky option on the other (a subject having reputation of being more difficult). $\lambda > 0$, $a > 0$, $\beta > 0$ and $\delta_i > 0$ $(i \in \{1, 2\})$ are the input of new participants, the contact rate (the measure of the effectiveness of the communication between the two groups), the backflow rate of individuals who are disappointed in the second option and the rates of successful final examinations (which correspond to mortalities in biological models), respectively. It was also shown that the following condition is of more interest

$$\text{(2)} \qquad a\lambda > \delta_1 (\beta + \delta_2).$$

(2) is needed to have a boundary and an interior equilibrium point in the plane $[S_1, S_2]$: $E_1 := (\lambda/\delta_1, 0)$ which is unstable and one with positive coordinates: $E_2 := (1/a) \cdot (\beta + \delta_2, (a\lambda - \delta_1(\beta + \delta_2))/\delta_2)$ which is globally asymptotically stable. Starting from the point of view that in the course of interaction among individuals immediate change does not always occur, it was reasonable to assume that the migration of the individuals from the option with little risk into the one with more risk is subject to delay because of the scaring effect of the second one. This delay was a constant and it was assumed that the delaying effect for each individual is the same, i.e. (1) was replaced by

$$\text{(3)} \qquad \begin{cases} \dot{S}_1(t) = \lambda - aS_1(t-\tau)S_2(t) + \beta S_2(t) - \delta_1 S_1(t) \\ \dot{S}_2(t) = aS_1(t-\tau)S_2(t) - \beta S_2(t) - \delta_2 S_2(t) \end{cases}$$

equipped by the initial conditions $S_1(\theta)=\varphi(\theta)$ $(\theta\in[-\tau,0])$, $S_2(0) = S_2^0 >$ > 0, where $\varphi : [-\tau, 0] \to \mathbb{R}$ is a non-negative continuous function with $\varphi(0) > 0$. It was shown that the originally asymptotically stable interior equilibrium loses its stability and at a certain critical value of delay τ a Hopf bifurcation takes place: a small amplitude periodic solution arises (cf. Fig. 1). Surprisingly enough it explains the unusual phenomenon, that repetitively one of the options prevails over the other one and vice versa.

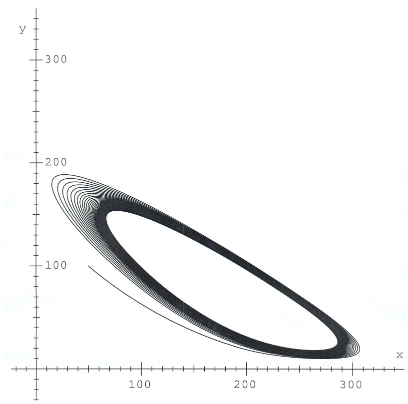

Figure 1: Limit cycle of system (3) with $x = S_1$ and $y = S_2$ (DifEqu®)

Now, we can have a more realistic model assuming that the response lag for the given individual is always different. In treating this problem the method applied in [7], [8] (c.f. [5], [6]) and in [10] is followed. Therefore a continuous density function ρ will be introduced whose role is to weight moments of the past and an infinitely distributed delay into the second equation of the system (1) for the S_1 density, i.e. we replace S_1 in the second equation by

$$R(t) := \int_{-\infty}^{t} S_1(\tau)\rho(t-\tau)d\tau \quad (t \in [0, +\infty)) \tag{4}$$

where the density function satisfies the requirements

$$\rho \in C^1[0, +\infty),\ \rho \geq 0, \qquad \int_0^{+\infty} \rho(s)ds = 1. \tag{5}$$

Thus, system (1) will be replaced by the integro-differential equation

$$(6)\qquad \begin{cases} \dot{S}_1 &= \lambda - aS_1S_2 + \beta S_2 - \delta_1 S_1 \\ \dot{S}_2 &= aRS_2 - \beta S_2 - \delta_2 S_2 \end{cases}$$

where R is given by (4).

Similarly to [2], [11], [17], [5], [6] and [1] we assume $\rho(t) \equiv \equiv \alpha \cdot \exp(-\alpha t)$ $(\alpha \in (0, +\infty))$ and

$$\alpha \int_{-\infty}^{t} \exp(-\alpha(t-\tau))\, d\tau = \alpha \int_{0}^{+\infty} \exp(-\alpha s) ds = 1$$

holds. The smaller the α the longer is the time interval in the past in which the values of S_1 are taken into account, i.e. $1/\alpha$ is the "measure of the influence of the past".

Now we have

$$\dot{R}(t) \;= \alpha(S_1(t) - R(t)) \quad (t \in [0, +\infty)),$$

therefore (6) is equivalent in its qualitative dynamical behaviour to the three-dimensional system of ordinary differential equations

$$(7)\qquad \begin{aligned} \dot{S}_1 &= F_1(S_1, S_2, R; \alpha) := \lambda - aS_1S_2 + \beta S_2 - \delta_1 S_1 \\ \dot{S}_2 &= F_2(S_1, S_2, R; \alpha) := aRS_2 - \beta S_2 - \delta_2 S_2 \\ \dot{R} &= F_3(S_1, S_2, R; \alpha) := \alpha(S_1 - R) \end{aligned}$$

on $[0, \infty)$ in the following sense (cf. [4]). If $(S_1, S_2) : [0, +\infty) \to \mathbb{R}^2$ is the solution of (6) corresponding to the continuous and bounded initial function $\tilde{S}_1 : (-\infty, 0] \to \mathbb{R}$ and the initial value $S_2^0 := S_2(0)$ (i.e. $S_1(t) := \tilde{S}_1(t)$ $(t < 0)$), then $(S_1, S_2, R) : [0, +\infty) \to \mathbb{R}^3$ is the solution of (7) satisfying the initial values $S_1(0) = \tilde{S}_1(0)$, $S_2(0) = S_2^0$ and $R(0) = = R^0 := \alpha \int_{-\infty}^{0} \tilde{S}_1(\tau) \exp(\alpha\tau) d\tau$ and vice versa. (Clearly, if the initial values $S_1(0)$, S_2^0 and R^0 related to system (7) are prescribed then the function $\tilde{S}_1$ is not uniquely determined.)

2. Local stability analysis and Hopf bifurcation

If (2) holds then the system (7) has the following equilibria: the boundary equilibrium $E_1^d := (\lambda/\delta_1, 0, \lambda/\delta_1)$ and the interior equilibrium

$$E_2^d := \left(\frac{\beta + \delta_2}{a}, \frac{a\lambda - \delta_1(\beta + \delta_2)}{a\delta_2}, \frac{\beta + \delta_2}{a} \right) .$$

In order to check the stability of the last two equilibria we linearize the system (7) at these points. The coefficient matrix is

$$J(S_1, S_2, R) := \begin{bmatrix} -aS_2 - \delta_1 & -aS_1 + \beta & 0 \\ 0 & aR - \beta - \delta_2 & aS_2 \\ \alpha & 0 & -\alpha \end{bmatrix},$$

specially

$$J\left(E_1^d\right) = \begin{bmatrix} -\delta_1 & \beta - \dfrac{a\lambda}{\delta_1} & 0 \\ 0 & \dfrac{a\lambda - \delta_1(\beta + \delta_2)}{\delta_1} & 0 \\ \alpha & 0 & -\alpha \end{bmatrix}$$

and

$$J\left(E_2^d\right) = \begin{bmatrix} \dfrac{\delta_1\beta - a\lambda}{\delta_2} & -\delta_2 & 0 \\ 0 & 0 & \dfrac{a\lambda - \delta_1(\beta + \delta_2)}{\delta_2} \\ \alpha & 0 & -\alpha \end{bmatrix}.$$

The characteristic polynomials of these matrices assume the form

$$p_1(x) :\equiv (\delta_1 + x) \cdot \left(\frac{a\lambda - \delta_1(\beta + \delta_2)}{\delta_1} - x \right) \cdot (\alpha + x) \tag{8}$$

and

$$p_2(x) :\equiv x^3 + \frac{\alpha\delta_2 + a\lambda - \delta_1\beta}{\delta_2} \cdot x^2 + \frac{\alpha}{\delta_2}(a\lambda - \delta_1\beta) \cdot x + \alpha\left(a\lambda - \delta_1\left(\beta + \delta_2\right)\right). \tag{9}$$

Clearly, p_1 is an unstable polynomial because of (2), and when introducing a delay the instability of the boundary equilibrium does not change. Condition (2) implies that all coefficients of p_2 are positive for $\alpha > 0$. Thus, applying the Routh–Hurwitz criterion p_2 is a stable polynomial if and only if the following inequality holds:

$$P(\alpha) :\equiv \alpha \left(\frac{(a\lambda - \delta_1\beta)\left(\alpha\delta_2 + a\lambda - \delta_1\beta\right)}{\delta_2^2} - \left[a\lambda - \delta_1\left(\beta + \delta_2\right)\right] \right) > 0 \tag{10}$$

i.e.

$$P^*(\alpha) :\equiv \delta_2(a\lambda - \delta_1\beta)\alpha + (a\lambda - \delta_1\beta)^2 - \delta_2^2\left[a\lambda - \delta_1(\beta + \delta_2)\right] > 0 \tag{11}$$

If

$$(a\lambda - \delta_1\beta)^2 - \delta_2^2\left[a\lambda - \delta_1(\beta + \delta_2)\right] \geq 0 \tag{12}$$

holds then (11) follows from (2) for all $\alpha > 0$ and E_2^d remains asymptotically stable.

In order to have Hopf bifurcation, one has to show that there is a smooth family of eigenvalues $x_{\pm}(\alpha) = \rho(\alpha) \pm \omega(\alpha) \cdot \imath$ of $J(E_2^d)$ such that the following conditions are fulfilled:

- $\rho(\alpha_H) = 0$, i.e. for the critical parameter value α_H there is a pair $\pm\omega(\alpha_H)\imath$ of purely imaginary eigenvalues of the matrix $J\left(E_2^d\right)$;
- there are no other eigenvalues of $J\left(E_2^d\right)$ on the imaginary axis for $\alpha = \alpha_H$;
- the transversality condition $\rho'(\alpha_H) \neq 0$ holds.

The left-hand side of the inequality (12) can be written as a polynomial of β:

$(a\lambda - \delta_1\beta)^2 - \delta_2^2\left[a\lambda - \delta_1(\beta + \delta_2)\right] = \delta_1^2\beta^2 - \delta_1(2a\lambda - \delta_2^2)\beta + a^2\lambda^2 + \delta_2^2(\delta_1\delta_2 - a\lambda)$

whose discriminant is

$$\begin{aligned} &\delta_1^2\left(2a\lambda - \delta_2^2\right)^2 - 4\delta_1^2\delta_2^2\left(\delta_1\delta_2 - a\lambda\right)^2 - 4\delta_1^2 a^2\lambda^2 = \\ &= \delta_1^2\left[4a^2\lambda^2 - 4a\lambda\delta_2^2 + \delta_2^4 - 4\delta_2^3\delta_1 + 4\delta_2^2 a\lambda - 4a^2\lambda^2\right] = \\ &= \delta_1^2\delta_2^3(\delta_2 - 4\delta_1). \end{aligned}$$

The leading coefficient of this quadratic polynomial is positive, therefore if

$$\delta_2 \leq 4\delta_1 \tag{13}$$

then (12) holds for all β and E_2^d is asymptotically stable. Thus, instability can occur only if (13) does not hold i.e.

$$\delta_2 > 4\delta_1 \ . \tag{14}$$

If (12) does not hold then, since the linear term in (11) is positive, $P(\alpha)$ has a unique positive root:

$$\alpha_H := \frac{\delta_2^2\left[a\lambda - \delta_1(\beta + \delta_2)\right] - (a\lambda - \delta_1\beta)^2}{\delta_2(a\lambda - \delta_1\beta)} \ . \tag{15}$$

In this case the equilibrium E_2^d is asymptotically stable for large values of α, i.e. for small delays.

Using α as a bifurcation parameter, we show that this equilibrium is losing its stability by a Hopf bifurcation when α is decreased below α_H (the delay is increased), i.e. we prove the following

Theorem 2.1. *Suppose that* (2) *and* (14) *holds then as the bifurcation parameter* α *is decreased at* α_H *the equilibrium* E_2^d *undergoes a Poincaré–Andronov–Hopf bifurcation, i.e. system* (7) *has a branch of periodic solutions bifurcating from* E_2^d *near* $\alpha = \alpha_H$.

Proof. Introducing the notation $K := a\lambda - \delta_1\beta > 0$ the value (15) is expressed by

$$\alpha_H = \frac{\delta_2^2(K - \delta_1\delta_2) - K^2}{K\delta_2} .$$

At α_H the characteristic polynomial p_2 has the form

$$\begin{aligned} p_2(x) \equiv x^3 + \frac{1}{\delta_2}\left(\frac{\delta_2^2(K-\delta_1\delta_2)-K^2}{K} + K\right)\cdot x^2 + \frac{\delta_2^2(K-\delta_1\delta_2)-K^2}{\delta_2^2}\cdot x + \\ + \frac{\delta_2^2(K - \delta_1\delta_2) - K^2}{K\delta_2}(K - \delta_1\delta_2) \equiv \\ \equiv \left[x^2 + \frac{\delta_2^2(K - \delta_1\delta_2) - K^2}{\delta_2^2}\right] \times \left[x + \frac{\delta_2(K - \delta_1\delta_2)}{K}\right], \end{aligned}$$

whose roots are

$$x_H(\alpha_H) = \frac{\delta_2(\delta_1\delta_2 - K)}{K} = \frac{\delta_2(\delta_1\delta_2 - a\lambda + \delta_1\beta)}{K} = \frac{\delta_2\left[\delta_1\left(\delta_2 + \beta\right) - a\lambda\right]}{K}$$

which is negative and $x_\pm(\alpha_H) = \pm\imath\omega$, where

$$\omega := \frac{\sqrt{\delta_2^2(K - \delta_1\delta_2) - K^2}}{\delta_2} > 0 . \tag{16}$$

Thus, we have to determine the derivative with respect to "α" of the real part of the smooth extension of the root $x_+(\alpha_H)$. Let us denote by $x_+(\alpha)$ the root of p_2 that assumes the value $\imath\omega$ at α_H and by

$$\mathcal{F}(x,\alpha) \;:\equiv\; x^3 + \frac{\alpha\delta_2 + K}{\delta_2}\cdot x^2 + \frac{\alpha K}{\delta_2}\cdot x + \alpha\left(K - \delta_1\delta_2\right)$$

the characteristic polynomial in (9) as a function of the parameter "α". Since

$$\mathcal{F}\left(x_+(\alpha_H), \alpha_H\right) = \mathcal{F}\left(\imath\omega, \alpha_H\right) = 0$$

and $\imath\omega$ is a simple root of the polynomial $\mathcal{F}\left(x, \alpha_H\right)$, the smooth function x_+ is uniquely determined by $\mathcal{F}\left(x_+(\alpha), \alpha\right) \equiv 0$, $x_+(\alpha_H) = \imath\omega$. We are going to determine the derivative of the implicit function x_+ at α_H:

$$x'_{+}(\alpha_H) = -\frac{\partial_\alpha \mathcal{F}(\imath\omega, \alpha_H)}{\partial_x \mathcal{F}(\imath\omega, \alpha_H)} = -\frac{\delta_2 x^2 + Kx + \delta_2(K - \delta_1\delta_2)}{3\delta_2 x^2 + 2(\alpha\delta_2 + K)x + \alpha K}\Bigg|_{\substack{x=\imath\omega \\ \alpha=\alpha_H}} =$$

$$= \frac{\delta_2\omega^2 - \delta_2(K - \delta_1\delta_2) - K\imath\omega}{\alpha_H K - 3\delta_2\omega^2 + 2(\alpha_H\delta_2 + K)\imath\omega}.$$

Hence, we have

$$\rho'(\alpha_H) = \frac{d\Re(x_+(\alpha))}{d\alpha}\bigg|_{\alpha=\alpha_H} = \Re\left(\frac{dx_+(\alpha)}{d\alpha}\bigg|_{\alpha=\alpha_H}\right) =$$

$$= \left\{\alpha_H\delta_2\left[\delta_1(\delta_2+\beta) - a\lambda\right] - 3\delta_2^2\omega^4 - \right.$$

$$\left. - \left[3\delta_1\delta_2^3 + K\left((\alpha_H - 3\delta_2)\,\delta_2 + 2K\right)\right]\omega^2\right\} \times$$

$$\times \left\{9\omega^4\delta_2^2 + 2\omega^2\delta_2\alpha_H K + \alpha_H^2 K^2 + 4\omega^2\alpha_H^2\delta_2^2 + 4K^2\omega^2\right\}^{-1} < 0$$

and this completes the proof of the theorem. ◊

3. Stability of the bifurcating periodic solution

In this section, we shall deduce the condition for the supercriticality resp. subcriticality of the bifurcation. Under supercritical bifurcation we mean the case when the equilibrium E_2^d has lost its stability with occurrence of periodic solutions which are orbitally asymptotically stable (i.e. for values of the bifurcation parameter α less than α_H), while in the subcritical case the periodic solutions are unstable and exist for such αs when the equilibrium E_2^d is still asymptotically stable (i.e. for values of the α greater than α_H).

To examine the supercriticality resp. subcriticality of the bifurcating solution, we have to compute the sign of the first Lyapunov coefficient l_1 which can be calculated by

$$l_1 = \frac{1}{2\omega} \cdot \Re\Big[\langle \mathbf{p}, \mathbf{C}(\mathbf{q}, \mathbf{q}, \overline{\mathbf{q}})\rangle - 2\left\langle \mathbf{p}, \mathbf{B}\left(\mathbf{q}, \mathfrak{A}^{-1}\mathbf{B}(\mathbf{q}, \overline{\mathbf{q}})\right)\right\rangle + + \left\langle \mathbf{p}, \mathbf{B}\left(\overline{\mathbf{q}}, (2\imath\omega\mathfrak{I}_3 - \mathfrak{A})^{-1}\mathbf{B}(\mathbf{q}, \mathbf{q})\right)\right\rangle\Big] \tag{17}$$

where $\mathfrak{I}_3$ denotes the 3×3 identity matrix, the bilinear function $\mathbf{B} : \mathbb{R}^3 \times \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$B_i(\mathbf{x}, \mathbf{y}) := \sum_{j,k=1}^{3} \frac{\partial^2 F_i(\boldsymbol{\xi}, \alpha_H)}{\partial\xi_j\partial\xi_k}\bigg|_{\boldsymbol{\xi}=E_2^d} x_j y_k \quad (i \in \{1,2,3\})$$

while the function $\mathbf{C} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$C_i(\mathbf{x},\mathbf{y},\mathbf{z}) := \sum_{j,k,l=1}^{3} \frac{\partial^3 F_i(\boldsymbol{\xi},\alpha_H)}{\partial\xi_j\partial\xi_k\partial\xi_l}\bigg|_{\boldsymbol{\xi}=E_2^d} x_j y_k z_l \quad (i\in\{1,2,3\}),$$

furthermore $\mathbf{p},\mathbf{q}\in\mathbb{C}^3$ are left and right eigenvectors of

$$\mathfrak{A} := \partial_{(S_1,S_2,R)}\mathbf{F}(E_2^d\,;\alpha_H)$$

corresponding to the eigenvalues $\imath\omega$ and $-\imath\omega$, respectively, i.e. satisfying

$$\begin{cases} \mathfrak{A}\mathbf{q} = \imath\omega\mathbf{q} \\ \mathfrak{A}^T\mathbf{p} = -\imath\omega\mathbf{p} \end{cases} \tag{18}$$

and are normalized by setting

$$\langle \mathbf{p},\mathbf{q}\rangle = 1 \tag{19}$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{C}^3$, antilinear in the first argument (c.f. [12], resp. [10]). In case of $l_1<0$ (resp. $l_1>0$) we have supercritical (resp. subcritical) bifurcation.

Introducing $\zeta := K/\delta_2$ it is easy to calculate that

$$\mathfrak{A} = \begin{bmatrix} -\zeta & -K/\zeta & 0 \\ 0 & 0 & \zeta-\delta_1 \\ \omega^2/\zeta & 0 & -\omega^2/\zeta \end{bmatrix}$$

and the vectors

$$\mathbf{q} \sim \begin{bmatrix} 1+\imath\zeta/\omega \\ \imath(\delta_1-\zeta)/\omega \\ 1 \end{bmatrix}, \quad \mathbf{p} \sim \begin{bmatrix} \imath\omega^2\delta_2 \\ \omega\delta_2^2 \\ K\,(\omega+\imath\zeta) \end{bmatrix}$$

are eigenvectors of $\mathfrak{A}$, resp. $\mathfrak{A}^T$ corresponding to the eigenvalues $\imath\omega$, resp. $-\imath\omega$ and in order to achieve the normalization (19), we should scale these vectors:

$$\mathbf{q} = \begin{bmatrix} 1+\imath\zeta/\omega \\ \imath(\delta_1-\zeta)/\omega \\ 1 \end{bmatrix}, \qquad \mathbf{p} = \frac{1}{2\left\{\omega+\imath K(\delta_1/\zeta-1)/\zeta\right\}} \begin{bmatrix} \imath\omega^2/\zeta \\ \omega K/\zeta^2 \\ \omega+\imath\zeta \end{bmatrix}.$$

The linear part of the analysis is now complete.

There are only two nonlinear terms in (7). Therefore, the bilinear function $\mathbf{B}$ can be expressed as

$$\mathbf{B}(\mathbf{x},\mathbf{y}) = a\begin{bmatrix} -x_1y_2-x_2y_1 \\ x_2y_3+x_3y_2 \\ 0 \end{bmatrix} \qquad \big((\mathbf{x},\mathbf{y})\in\mathbb{R}^3\times\mathbb{R}^3\big)$$

while $\mathbf{C}(\mathbf{x},\mathbf{y},\mathbf{z}) \equiv \mathbf{0}$. Thus, we have

$$\mathbf{B}(\mathbf{q},\mathbf{q}) = 2aq_2 \begin{bmatrix} -q_1 \\ q_3 \\ 0 \end{bmatrix} = \frac{2a\imath(\delta_1-\zeta)}{\omega} \begin{bmatrix} -1-\imath\zeta/\omega \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{B}(\mathbf{q},\overline{\mathbf{q}}) = a \begin{bmatrix} -q_1\overline{q}_2 - q_2\overline{q}_1 \\ q_2\overline{q}_3 + q_3\overline{q}_2 \\ 0 \end{bmatrix} = a \begin{bmatrix} 2\zeta(\zeta-\delta_1)/\omega^2 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{B}(\overline{\mathbf{q}},\mathbf{r}) = a \begin{bmatrix} -\overline{q}_1 r_2 - \overline{q}_2 r_1 \\ \overline{q}_2 r_3 + \overline{q}_3 r_2 \\ 0 \end{bmatrix}.$$

Inverting the matrix $\mathfrak{A}$ yields

$$\mathbf{s} := \mathfrak{A}^{-1}\mathbf{B}(\mathbf{q},\overline{\mathbf{q}}) = \begin{bmatrix} 0 & \dfrac{1}{\zeta-\delta_1} & \dfrac{\zeta}{\omega^2} \\ \dfrac{-\zeta}{K} & \dfrac{-\zeta}{\delta_2(\zeta-\delta_1)} & \dfrac{-\zeta^3}{\omega^2 K} \\ 0 & \dfrac{1}{\zeta-\delta_1} & 0 \end{bmatrix} \cdot \mathbf{B}(\mathbf{q},\overline{\mathbf{q}}) = \begin{bmatrix} 0 \\ \dfrac{2a\zeta^2(\delta_1-\zeta)}{\omega^2 K} \\ 0 \end{bmatrix}$$

resp.

$$\mathbf{B}\left(\mathbf{q},\mathfrak{A}^{-1}\mathbf{B}(\mathbf{q},\overline{\mathbf{q}})\right) = \frac{2a^2\zeta^2(\delta_1-\zeta)}{\omega^2 K} \begin{bmatrix} -1-\dfrac{\imath\zeta}{\omega} \\ 1 \\ 0 \end{bmatrix}.$$

Hence the second term in l_1 has the form

$$\begin{aligned} \langle \mathbf{p}, \mathbf{B}\left(\mathbf{q},\mathfrak{A}^{-1}\mathbf{B}(\mathbf{q},\overline{\mathbf{q}})\right)\rangle &= \frac{2a^2\zeta^2(\delta_1-\zeta)}{\omega^2 K} \cdot \frac{1}{2\left\{\omega - K(\delta_1/\zeta - 1)\imath/\zeta\right\}} \cdot \\ &\cdot \left[-\frac{\imath\omega^2}{\zeta}\cdot\left(-1-\frac{\imath\zeta}{\omega}\right) + \frac{\omega K}{\zeta^2}\right] = \\ &= \frac{a^2(\delta_1-\zeta)\zeta^2}{K\omega} \cdot \frac{-\omega\zeta + \imath(K-\zeta^2)}{K(\delta_1-\zeta)+\imath\omega\zeta^2} \end{aligned}$$

with real part

$$\begin{aligned} \Re\left(\langle \mathbf{p}, \mathbf{B}\left(\mathbf{q},\mathfrak{A}^{-1}\mathbf{B}(\mathbf{q},\overline{\mathbf{q}})\right)\rangle\right) &= \frac{a^2(\delta_1-\zeta)\zeta^2}{K} \cdot \frac{-\zeta K(\delta_1-\zeta) + (K-\zeta^2)\zeta^2}{K^2(\delta_1-\zeta)^2 + \omega^2\zeta^4} = \\ &= -\frac{a^2\zeta^3(\delta_1-\zeta)\left[K(\delta_1-2\zeta)+\zeta^3\right]}{K^3(\delta_1-\zeta)^2 + K\omega^2\zeta^4}. \end{aligned}$$

Denoting

$$\mathbf{r} := \left(2\imath\omega\mathfrak{I}_3 - \mathfrak{A}\right)^{-1}\mathbf{B}(\mathbf{q},\mathbf{q})$$

and bringing the matrix $2\imath\omega\mathfrak{I}_3 - \mathfrak{A}$ into upper Hessenberg form by interchanging the first and second rows and columns, we get the following system for the coordinates of $\mathbf{r}$:

$$(20)\qquad \begin{bmatrix} 2\imath\omega & 0 & \delta_1 - \zeta \\ K/\zeta & 2\imath\omega + \zeta & 0 \\ 0 & -\omega^2/\zeta & 2\imath\omega + \omega^2/\zeta \end{bmatrix} \cdot \begin{bmatrix} r_2 \\ r_1 \\ r_3 \end{bmatrix} = 2aq_2 \begin{bmatrix} 1 \\ -q_1 \\ 0 \end{bmatrix}.$$

Thus, one can compute them recursively:

$$r_3 = \frac{aq_2\left(\frac{\imath}{\omega} - 2\frac{\zeta q_1}{K}\right)}{\frac{\zeta}{K}(2\imath\omega + \zeta)\left(1 + \frac{2\zeta\imath}{\omega}\right) + \frac{q_2}{2}},$$

$$r_2 = -\frac{q_2 a}{\omega}\imath + \frac{q_2 r_3}{2} = \frac{q_2}{2\omega}(\omega r_3 - 2a\imath),$$

$$r_1 = \frac{\frac{-Kr_2}{\zeta} - 2aq_1q_2}{2\imath\omega + \zeta},$$

resp. explicitly:

$$r_1 = \frac{2aq_2(\omega + 2\zeta\imath)(K + 2\omega q_1\zeta\imath)}{\omega\left[d_1K + \zeta\left(4(\omega^2 + \zeta^2) - K + 6\imath\omega\zeta\right)\right]},$$

$$r_2 = \frac{2aq_2\zeta\left[\omega\left(\zeta(3 + q_1) - \delta_1 q_1\right) - 2(\omega^2 + \zeta^2)\imath\right]}{\omega\left[d_1K + \zeta\left(4(\omega^2 + \zeta^2) - K + 6\imath\omega\zeta\right)\right]},$$

$$r_3 = \frac{2aq_2(K + 2\imath\omega q_1\zeta)}{d_1K + \zeta\left(4(\omega^2 + \zeta^2) - K + 6\imath\omega\zeta\right)}.$$

The first row of the system (20) can be rearranged to discover the second component of $\mathbf{B}(\overline{\mathbf{q}}, \mathbf{r})$ as follows

$$2\imath\omega r_2 - (\zeta - \delta_1)r_3 = 2aq_2, \quad \text{i.e.} \quad 2\imath\omega r_2 - \imath\omega q_2 r_3 = 2aq_2.$$

After dividing by $\imath\omega$ we have

$$\frac{2aq_2}{\imath\omega} = 2r_2 - q_2r_3 = r_2 + \overline{q}_3 r_2 + \overline{q}_2 r_3 = r_2 + \frac{B(\overline{\mathbf{q}}, \mathbf{r})_2}{a}$$

from which the second component is expressible as

$$B(\overline{\mathbf{q}}, \mathbf{r})_2 = \frac{2a^2q_2}{\imath\omega} - ar_2 = a\left[-r_2 - \frac{2aq_2\imath}{\omega}\right].$$

For the whole vector $\mathbf{B}(\overline{\mathbf{q}}, \mathbf{r})$ we have the form

$$\mathbf{B}(\overline{\mathbf{q}},\mathbf{r})=\frac{2a^2(\delta_1-\zeta)}{\omega^3\left\{K(\delta_1-\zeta)+4\zeta(\omega^2+\zeta^2)+6\imath\omega\zeta^2\right\}}\begin{bmatrix}A+B\imath\\ C+D\imath\\ 0\end{bmatrix}$$

where

$$A:=-\omega\left[\zeta\left(11\zeta^2-K+2\omega^2\right)+\delta_1(K-6\zeta^2)\right]$$

and

$$B:=-\zeta\left[3\zeta^3-2K\zeta+\delta_1\left(2K+\omega^2-5\zeta^2\right)\right]$$

resp.

$$C:=\omega\left[\zeta\left(3\zeta^2-K+2\omega^2\right)+\delta_1(K-\zeta^2)\right]\quad\text{and}\quad D:=\omega^2\zeta(\delta_1+2\zeta).$$

A tedious calculation shows that the third term in l_1 assumes the form

$$\left\langle\mathbf{p},\mathbf{B}\left(\overline{\mathbf{q}},(2\imath\omega\mathfrak{I}_3-\mathfrak{A})^{-1}\mathbf{B}(\mathbf{q},\mathbf{q})\right)\right\rangle=$$
$$=a^2(\delta_1-\zeta)\left\{\delta_1K^2+\zeta\left[2\omega(\omega+\delta_1\imath)-K\right]+\left[-\delta_1(3K+\omega^2)+\imath\omega(K+2\omega^2)\right]\zeta^2+\right.$$
$$\left.+(5K-6\imath\omega\delta_1)\zeta^3+(5\delta_1+11\imath\omega)\zeta^4-3\zeta^5\right\}/$$
$$/\left\{\omega\left[\omega\zeta^2+K(\zeta-\delta_1)\imath\right]\left[\delta_1K+\zeta\left(4(\omega^2+\zeta^2)-K+6\imath\omega\zeta\right)\right]\right\}$$

with real part (resubstituting the values of ω and ζ)

$$\Re\left(\left\langle\mathbf{p},\mathbf{B}\left(\overline{\mathbf{q}},(2\imath\omega\mathfrak{I}_3-\mathfrak{A})^{-1}\mathbf{B}(\mathbf{q},\mathbf{q})\right)\right\rangle\right)=$$
$$=-a^2\delta_2K(\delta_1\delta_2-K)\left\{\delta_1^2(5\delta_1-2\delta_2)\delta_2^7+4\delta_1\delta_2^6\cdot K(\delta_2-4\delta_1)+\right.$$
$$+\delta_2^4K^2(-18\delta_1^2+23\delta_1\delta_2-2\delta_2^2)+3\delta_2^3K^3(13\delta_1-4\delta_2)-$$
$$\left.-3\delta_2K^4(4\delta_1+5\delta_2)+18K^5\right\}/3\left\{\delta_1^2\delta_2^6-2\delta_1\delta_2^3K(\delta_2^2+2K)+\right.$$
$$\left.+K^2\left(\delta_2^4+4\delta_2^2K-4K^2\right)\right\}\left\{\delta_1^2\delta_2^6-\delta_1\delta_2^3K\cdot\left(2\delta_2^2+K\right)+\right.$$
$$\left.+K^2\left(\delta_2^2(\delta_2^2+K)-K^2\right)\right\}.$$

Finally, formula (17) gives the first Lyapunov coefficient

$$l_1=\frac{a^2K\delta_2\left(\delta_1\delta_2-K\right)}{6\sqrt{\delta_2^2(K-\delta_1\delta_2)-K^2}}\cdot$$
$$\cdot\left\{\delta_1^2\delta_2^9(\delta_1+2\delta_2)-4\delta_1\delta_2^8K(2\delta_1+\delta_2)+\delta_2^7K^2(7\delta_1+2\delta_2)+21\delta_1\delta_2^5K^3-\right.$$
$$\left.-9\delta_2^3(4\delta_1+3\delta_2)K^4+54\delta_2^2K^5-24K^6\right\}/\left\{\delta_1^2\delta_2^6-2\delta_1\delta_2^3K\left(\delta_2^2+2K\right)+\right.$$
$$\left.+K^2\left(\delta_2^4+4\delta_2^2K-4K^2\right)\right\}\left\{\delta_1^2\delta_2^6-\delta_1\delta_2^3K\left(2\delta_2^2+K\right)+\right.$$
$$\left.+K^2\left(\delta_2^4+\delta_2^2K-K^2\right)\right\}.$$

Thus, we have proved the following

Theorem 3.1. *If* (2) *and* (14) *hold then the bifurcation is supercritical resp. subcritical according as the number*

$$\rho:=\left\{\delta_1^2\delta_2^9(\delta_1+2\delta_2)-4\delta_1\delta_2^8K(2\delta_1+\delta_2)+\delta_2^7K^2(7\delta_1+2\delta_2)+21\delta_1\delta_2^5K^3-\right.$$
$$\left.-9\delta_2^3(4\delta_1+3\delta_2)K^4+54\delta_2^2K^5-24K^6\right\}/\left\{\delta_1^2\delta_2^6-2\delta_1\delta_2^3K\left(\delta_2^2+2K\right)+\right.$$
$$\left.+K^2\left(\delta_2^4+4\delta_2^2K-4K^2\right)\right\}\left\{\delta_1^2\delta_2^6-\delta_1\delta_2^3K\left(2\delta_2^2+K\right)+K^2\left(\delta_2^4+\delta_2^2K-K^2\right)\right\}$$

is positive, resp. negative.

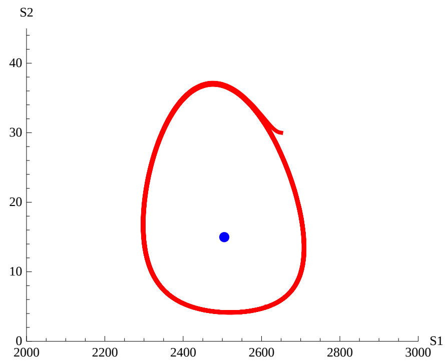

Figure 2: The limit cycle of system (7) (MATHEMATICA®)

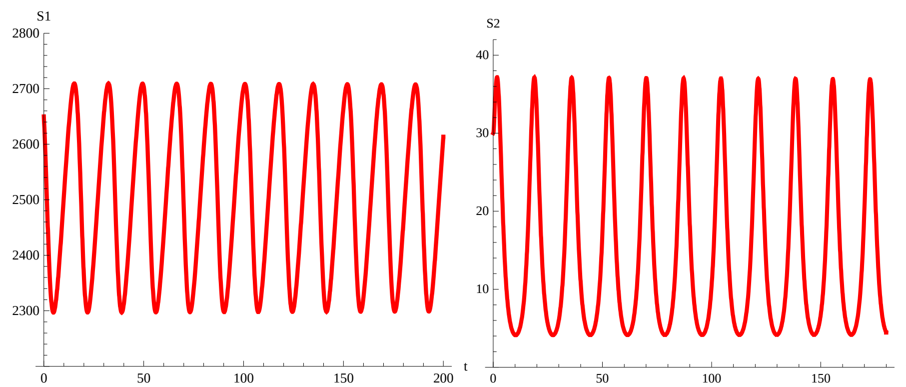

Figure 3: Time evolution of system (7) (MATHEMATICA®)

Example 3.1. Set $a = 0.0020$, $\lambda = 100.0000$, $\delta_1 = 0.0100$, $\delta_2 = 5.0000$, $\beta = 0.0100$. These values satisfy the conditions of Th. 3.1. We have $E_2^d = \text{col}\,(2505, 14.99, 2505)$, $\alpha_H = 3.7039$, $\omega = 0.3839$ and $\rho = 85.1701$. This means that a $0 < \varepsilon < \alpha_H$ exists such that for $(\alpha_H - \varepsilon, \alpha_H)$ system (7) has small amplitude orbitally asymptotically stable periodic solutions with approximate period $2\pi/\omega = 16.3167$ (cf. Figs. 2–3).

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