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# EXISTENCE, UNIQUENESS AND GLOBAL STABILITY OF COURNOT EQUILIBRIUM IN OLIGOPOLY WITH COST EXTERNALITIES

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**Abstract:** Cournot oligopoly without product differentiation and with cost externalities is formulated and analyzed from static and dynamic points of view. After characterization of the firms' best responses as functions of their rest of the industry output, a proof is given for the existence of a unique (Nash–) Cournot equilibrium on the basis of an alternative best response functions depending on the industry total output. Finally, the global stability condition for the equilibrium is derived using the firm's gradient output adjustment system.

## 1. Introduction

The existence and uniqueness of the equilibrium in Cournot oligopoly was first the attention of the researchers based on the pioneering work of Cournot [1]. The works of Debreu [2], Frank and Quandt [3]

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are among the most important early contributions. The existence of the equilibrium was based on fixed point theorems which do not provide methods to compute the equilibrium. The uniqueness of the equilibrium was usually proved by using the Gale–Nikaido theorem (Nikaido [5]), or the uniqueness result of Rosen [9] based on strictly diagonally concave games. From the mid-60s increasing attention has been given to dynamic extensions. Theocharis [10] examined linear oligopolies and proved that the equilibrium is asymptotically stable if only two firms form the industry. The paper of Hahn [4] has to be mentioned as a pioneering work in this subject. A comprehensive summary of the earlier results can be found in Okuguchi [6]. The extensions of the classical results for different variants of Cournot oligopoly are summarized in Okuguchi and Szidarovszky [7], which contains also detailed literature review. In most studies the interdependence of the firms in oligopolistic industry was taken into account through the inverse market demand function as a function of the industry total output. The firms, however, compete also in imperfectly competitive factor markets for acquiring factors of production such as labor, capital, etc., and the factor prices will be determined by the industry activity level shown by the industry total output.

In this paper we will formulate oligopoly without product differentiation but with cost externalities among the firms in the sense that the firm's cost is a function of its output as well as the industry output. As a preliminary for the existence proof of the unique equilibrium in our oligopoly model with cost externalities, we will characterize the firms' best response functions depending on their rest of the industry output. This is the task of Sec. 2. In Sec. 3, we will consider alternative best response functions for the firms as functions of the total industry output, on the basis of which we will prove the existence of a unique (Nash–) Cournot equilibrium which is identical to the fixed point of a function having the industry output as its only independent variable. The proof is based on the existence of a unique solution of a single-variable monotonic equation, which also provides an efficient method to compute the equilibrium. In Sec. 4, we will derive a sufficient condition for the global stability of the equilibrium for the firms' gradient output adjustment system, where we will apply a global stability theorem for the equilibrium in aggregate games due to Okuguchi and Yamazaki [8].

## 2. Firms best response function

Consider an $n$-firm oligopoly without product differentiation. Let $x_k$ be the output of firm $k$ and $X = \sum_{k=1}^{n} x_k$ be the industry total output. It is assumed that each firm has a finite capacity limit $L_k$, so $0 \le x_k \le L_k$. Furthermore, let $f(X)$ and $C_k(x_k, X)$ be the inverse market demand function and firm $k$'s cost function, respectively. Then firm $k$'s profit function can be given as

$$\pi_k = x_k f(x_k + X_k) - C_k(x_k, x_k + X_k), \tag{1}$$

where $X_k = \sum_{l \neq k} x_l$ is firm $k$'s rest of the industry output.

Assume that functions $f$ and $C_k$ $(k = 1, 2, \ldots, n)$ are twice continuously differentiable, furthermore that

(A) $f' < \min\{0; C''_{k11} + C''_{k12}\}$,

(B) $f' + x_k f'' \le \min\{0; C''_{k12} + C''_{k22}\}$

for all $k$ and feasible outputs, where

$$C''_{k11} = \frac{\partial^2 C_k}{\partial x_k^2}, \quad C''_{k12} = \frac{\partial^2 C_k}{\partial x_k \partial X}, \quad C''_{k22} = \frac{\partial^2 C_k}{\partial X^2}.$$

In the case of decreasing return to scale in the cost functions all second order partial derivatives $C''_{k11}$, $C''_{k12}$ and $C''_{k22}$ are positive, so Assumption (A) requires that $f' < 0$, and Assumption (B) means that $f' + x_k f'' = (x_k f')' < 0$. That is, both $f(X)$ and $x_k f'(x_k + X_k)$ are strictly decreasing in $X$ and $x_k$, respectively.

Notice first that

$$\frac{\partial \pi_k}{\partial x_k} = f(x_k + X_k) + x_k f'(x_k + X_k) - C'_{k1}(x_k, x_k + X_k) - C'_{k2}(x_k, x_k + X_k) \tag{2}$$

where $C'_{k1} = \frac{\partial C_k}{\partial x_k}$, $C'_{k2} = \frac{\partial C_k}{\partial X}$. Furthermore

$$\frac{\partial^2 \pi_k}{\partial x_k^2} = 2f' + x_k f'' - C''_{k11} - 2C''_{k12} - C''_{k22} < 0 \tag{3}$$

by assumptions (A) and (B). Since with fixed values of $X_k$, $\pi_k$ is strictly concave in $x_k$, the best response of firm $k$ exists and is unique, and can be obtained as

$$
(4) \quad R_k(X_k) = \begin{cases} 0 & \text{if } f(X_k) - C'_{k1}(0, X_k) - C'_{k2}(0, X_k) \le 0, \\ L_k & \text{if } f(L_k + X_k) + L_k f'(L_k + X_k) - \\ & \qquad -C'_{k1}(L_k, L_k + X_k) - C'_{k2}(L_k, L_k + X_k) \ge 0, \\ x_k^* & \text{otherwise,} \end{cases}
$$

where $x_k^*$ is the unique solution of the equation

$$
(5) \quad f(x_k + X_k) + x_k f'(x_k + X_k) - C'_{k1}(x_k, x_k + X_k) - C'_{k2}(x_k, x_k + X_k) = 0
$$

in the interval $(0, L_k)$. Notice that the left-hand side strictly decreases in $x_k$, positive at $x_k = 0$ and negative at $x_k = L_k$, so there is a unique solution which is continuously differentiable in $X_k$. By implicitly differentiating equation (5) we have

$$
\begin{aligned} &(1 + R'_k) f' + R'_k f' + (1 + R'_k) x_k f'' - C''_{k11} R'_k - \\ &\qquad - (1 + R'_k) C''_{k12} - C''_{k21} R'_k - (1 + R'_k) C''_{k22} = 0. \end{aligned}
$$

Hence

$$
(6) \qquad R'_k = -\frac{f' + x_k f'' - C''_{k12} - C''_{k22}}{2f' + x_k f'' - C''_{k11} - 2C''_{k12} - C''_{k22}}.
$$

Conditions (A) and (B) imply that

$$
(7) \qquad -1 < R'_k \le 0.
$$

Notice that this relation holds in the first two cases of (4) as well, so it is valid at all points, except at the boundary between the three cases. It is also clear that $R_k(X_k)$ is continuous in its entire domain.

The existence of unique best responses of the firms does not imply the existence of an equilibrium in general. However in this case based on the continuity of the best response functions and the compact strategy sets the Brouwer fixed point theorem can be applied to prove existence, however the uniqueness of the equilibrium still remains unresolved. In the next section we will introduce a constructive proof for the existence and uniqueness of the equilibrium, which also provides an efficient method to compute the equilibrium.

## 3. Existence of unique equilibrium

In this section we will show the existence of a unique (Nash–) Cournot equilibrium. The basic idea of the proof is a new form of the individual firm's best response as a function of the industry total output instead of its rest of the industry output. We will then show the strict monotonicity of the resulting single variable equation regarding the industry output. The main result of this section is the following

**Existence Theorem.** *Under assumptions* (A)–(B)*, there exists a unique (Nash–) Cournot equilibrium.*

**Proof.** We can rewrite the firms' best response functions in terms of the industry output as

(8)
$$\overline{R}_k(X_k) = \begin{cases} 0 & \text{if } \ f(X) - C'_{k1}(0, X) - C'_{k2}(0, X) \le 0, \\ L_k & \text{if } \ f(X) + L_k f'(X) - C'_{k1}(L_k, X) - C'_{k2}(L_k, X) \ge 0, \\ \overline{x_k} & \text{otherwise,} \end{cases}$$

where $\overline{x_k}$ is the solution of equation

$$f(X) + x_k f'(X) - C'_{k1}(x_k, X) - C'_{k2}(x_k, X) = 0 \tag{9}$$

in the interval $(0, L_k)$. This equation has a unique solution, since at $x_k = 0$ the left-hand side is positive, at $x_k = L_k$ it is negative, and it is strictly decreasing in $x_k$ by (A). By implicit differentiation of equation (9) with respect to $X$ we have

$$f' + \overline{R}'_k f' + x_k f'' - C''_{k11}\overline{R}'_k - C''_{k12} - C''_{k21}\overline{R}'_k - C''_{k22} = 0,$$

which implies that

$$\overline{R}'_k = -\frac{f' + x_k f'' - C''_{k12} - C''_{k22}}{f' - C''_{k11} - C''_{k12}} \le 0 \tag{10}$$

in light of (A) and (B). This is also true in the first two cases of (8) except at the boundaries between the cases. Hence $\overline{R}_k$ decreases in $X$, and is continuous.

Consider finally the following equation containing the industry output as a single variable:

$$\sum_{k=1}^{n} \overline{R}_k(x) - X = 0. \tag{11}$$

The left-hand side is clearly continuous and strictly decreasing. At $X = 0$ it is nonnegative, and at $X = \sum_{k=1}^{n} L_k$ it is nonpositive, so there is a unique solution $X^*$, which yields the firm's equilibrium output as $x_k^* = \overline{R}_k(X^*)$ for all $k$.

Some comments on our assumptions might be in order. Assumption (A) requires $f$ to be strictly decreasing. If $C''_{k11} + C''_{k12} < 0$, then the slope of the price function has to have sufficiently large absolute value. If $f$ is concave, then the first part of condition (B) holds, and if $C''_{k12} + C''_{k22} < 0$, then $f' + x_k f''$ must have sufficiently large absolute value meaning that function $x_k f'$ has to decrease sufficiently fast. In the case of linear cost functions these conditions reduce to the negativity of $f'$ and to the nonpositivity of $f' + x_k f''$. In the classical Cournot models, when $C_k$ depends only on $x_k$, conditions (A) and (B) reduce to the usual assumptions used in the literature.

## 4. Stability analysis

In this section we will derive sufficient conditions for the global asymptotic stability of the unique (Nash–) Cournot equilibrium whose existence has been established in Sec. 3. Assume now that the equilibrium is interior, that is, for all firms the third cases of (4) and (8) apply and consider the following gradient adjustment process for the firm's actual output:

$$(12)\qquad \frac{dx_i}{dt} = \alpha_i \frac{\partial \pi_i}{\partial x_i} = $$

$$= \alpha_i \left[ f\left(\sum_{l=1}^{n} x_l\right) + x_i f'\left(\sum_{l=1}^{n} x_l\right) - C'_{i1}\left(x_i, \sum_{l=1}^{n} x_l\right) - C'_{i2}\left(x_i, \sum_{l=1}^{n} x_l\right)\right]$$

for $i = 1, 2, \ldots, n$, where $t$ denotes continuous time and $\alpha_i$ is the constant speed of adjustment for firm $i$.

In the analysis that follows we will apply a general stability theorem for aggregate games formulated and proved by Okuguchi and Yamazaki [8]. Let $\pi_i \equiv U_i(x_i, X)$ be firm $i$'s profit function and define

$$(13)\qquad h^i(x_i, X) = \frac{\partial}{\partial x_i} U_i(x_i, X) + \frac{\partial}{\partial X} U_i(x_i, X) = \frac{\partial}{\partial x_i} U_i(x_i, x_i + X_i),$$

and rewrite the adjustment process (12) as

$$\frac{dx_i}{dt} = \alpha_i h^i(x_i, X) \tag{14}$$

for $i = 1, 2, \ldots, n$. According to Okuguchi and Yamazaki [8], the equilibrium is globally asymptotically stable if

$$\alpha_j h_2^i + \alpha_i h_2^j \le 0 \tag{15}$$

and

$$2\alpha_i(h_1^i + h_2^i) < \sum_{j\neq i}(\alpha_j h_2^i + \alpha_i h_2^j) \tag{16}$$

for all $i$, $j$ and for all feasible outputs, where $h_1^i = \partial h^i/\partial x_i$ and $h_2^i = \partial h^i/\partial X$. Our main result is the following.

**Stability Theorem.** *Assume* (A)–(B) *and identical speeds of adjustment, $\alpha_i \equiv \alpha_1$. Then under condition* (19) *the interior equilibrium is globally stable with respect to the gradient process* (12).

**Proof.** In the case of our oligopoly model with cost externalities, we have

$$U_i(x_i, X) = x_i f(X) - C_i(x_i, X), \tag{17}$$

$$h^i(x_i, X) = f(X) - C'_{i1}(x_i, X) + x_i f'(X) - C'_{i2}(x_i, X). \tag{18}$$

Hence,

$$\begin{aligned} h_1^i &= f' - C''_{i11} - C''_{i21} < 0, \\ h_2^i &= f' + x_i f'' - C''_{i12} - C''_{i22} \le 0 \end{aligned}$$

by (A) and (B). Relation (15) holds in light of these relations. Notice that condition (16) has the form

$$\begin{aligned} &2\big(2f' + x_i f'' - C''_{i11} - 2C''_{i12} - C''_{i22}\big) < \\ &< \sum_{j\neq i}\big(2f' + (x_i + x_j)f'' - C''_{i12} - C''_{i22} - C''_{j12} - C''_{j22}\big) \end{aligned}$$

which can be rewritten as

(19)

$$2(f' - C''_{i11} - C''_{i12}) + (4-n)(f' + x_i f'' - C''_{i12} - C''_{i22}) < \sum_{j=1}^{n}(f' + x_j f'' - c''_{j12} - C''_{j22})$$

The right-hand side is nonpositive by Assumption (B). The first term in the left-hand side is negative by Assumption (A), and the multiplier of $(4-n)$ is also nonnegative. Therefore, condition (19) holds if the absolute value of the negative term $f' - C''_{i11} - C''_{i12}$ is sufficiently large.

To further clarify the economic implications of the stability condition (19), consider a Cournot oligopoly without cost externalities. In this case (19) reduces to

$$f' - C''_i < (n-2)f' + \frac{n-4}{2}x_i f'' + \frac{1}{2}Xf'' = (n-2)(f' + \overline{\overline{x_i}}f''), \tag{20}$$

where we define

$$\overline{\overline{x_i}} \equiv \frac{(n-4)x_i + X}{2(n-2)}.$$

Assumption (B) then implies that the right-hand side is nonpositive, and by (A) the left-hand side is negative. So (20) is a meaningful relation. If the inverse demand function is linear, (20) reduces to

$$(3-n)f' - C''_i < 0. \tag{21}$$

If $n = 2$, then (21) is true by (A), if $n = 3$, then it is true if $C_i$ is strictly convex, and if $n > 3$, it may still hold for sufficiently large values of $C''_i$. If the cost functions are also linear, then (21) holds for only $n = 2$, which is the well-known result of Theocharis [10].

## 5. Conclusions

In this paper we have formulated Cournot oligopoly without product differentiation and with cost externalities, which arise if, for example, firms compete for obtaining production factors in imperfectly competitive factor markets. We have proved the existence of a unique (Nash–) Cournot equilibrium in our model based on the best response functions of individual firms as functions of the industry output. The proof has provided a simple computer method to find the equilibrium. Finally, we have derived a sufficient condition for the global stability of the equilibrium using a general stability theorem for aggregate games and clarified its economic implications for some simple cases.

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