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ON UMBILIC AXES OF CIRCLES OF THE NON-CYCLIC QUADRANGLE IN THE ISOTROPIC PLANE

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Abstract: A power of the point and a power of the line with respect to the circle in the isotropic plane are studied in this paper. To begin with, the radical axis and the radical center of circles in the isotropic plane are presented. As the principle of duality is preserved in the isotropic plane, notions dual to the radical axis and to the radical center, the umbilic point and the umbilic axis are introduced. Non-cyclic quadrangle is called standard if a special hyperbola with the equation $xy = 1$ is circumscribed to it. The properties of the standard quadrangle related to the umbilic point and the umbilic axis are given.

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1. Isotropic plane

The isotropic plane is a real affine plane where the metric is introduced by an absolute figure (ω, Ω) where ω is a line at infinity and Ω is a point incident with it. If $T = (x, y, z)$ denotes any point in the plane presented in homogeneous coordinates, then usually an projective coordinate system where $\Omega = (0, 1, 0)$ and the line ω with the equation $z = 0$ is chosen. The line ω is said to be *the absolute line* and the point Ω *the absolute point*. All projective transformations that preserve the absolute figure form a 5-parametric group, *the group of similarities* of the isotropic plane, usually denoted by G_5 ([7, p. 10]).

All the notions related to the geometry of the isotropic plane can be found for example in [7] and [8].

For two non-parallel points $T_1 = (x_1, y_1)$ and $T_2 = (x_2, y_2)$ *a distance* between these two points is defined as $d(T_1, T_2) := x_2 - x_1$. In the case of parallel points, e.g. $T_1 = (x, y_1)$ and $T_2 = (x, y_2)$, $s(T_1, T_2) := y_2 - y_1$ defines *a span* between points T_1, T_2.

For two non-isotropic lines $y = k_i x + l_i,\ i = 1, 2$ *an angle* between two of them is defined by $\varphi = \angle(p_1, p_2) := k_2 - k_1$. The distance, the span and the angle are directed quantities.

All aforementioned quantities are kept invariant under a subgroup G_3 of G_5 being of the form

$$\begin{aligned} \overline{x} &= a + x, \\ \overline{y} &= b + cx + y, \end{aligned} \qquad a,\, b,\, c \in \mathbb{R}.$$

G_3 is called *the motion group* of the isotropic plane.

Later on we will need the notion of a bisector of two lines. It is easy to show that the bisector of the lines $y = k_i x + l_i,\ i = 1, 2$ has the slope $k = \frac{k_1 + k_2}{2}$.

2. A power of the point with respect to the circle

In this section we are concerned with a power of the point with respect to the circle. Although studied in several books and articles (e.g. [6], [7]) we give a different approach to the topic, suitable for the research conducted in the sections that follow. It should be pointed out that all proofs are novel as are the obtained forms for the power of a point with respect to a circle and the radical axis of two circles.

Every circle in the isotropic plane can be represented in the form $y = ux^2 + vx + w,\ \ u, v, w \in \mathbb{R},\ u \neq 0$ ([7, p. 23]). The first part of the theorem that we start with has been stated and proved in [7, p. 38], in a different way. We have achieved the new form (1) of the power of point with respect to the circle, suitable for further investigation.

Theorem 1. *Let $\mathcal{K}$ be a circle with the equation $y = ux^2+vx+w$ and let $T = (x', y')$. If points T_1, T_2 denote the points of intersection of any line passing through the point T and of the circle $\mathcal{K}$ then the product $TT_1 \cdot TT_2$ is constant and equal to*

$$P_{T,\mathcal{K}} = x'^2 + \frac{vx'}{u} - \frac{y'}{u} + \frac{w}{u}. \tag{1}$$

$P_{T,\mathcal{K}}$ is called the power of the point T with respect to the circle $\mathcal{K}$.

Proof. Let $\mathcal{K}$ be given with the equation

$$y = ux^2 + vx + w. \tag{2}$$

The line passing through the point $T(x', y')$ is of the form

$$y = kx + y' - kx' \tag{3}$$

where k stands for its slope. The abscissae of points of intersection of the line (3) and the circle (2) fulfil the equation

$$ux^2 + x(v - k) + w - y' + kx' = 0.$$

If these points are denoted by $T_1 = (x_1, y_1)$ and $T_2 = (x_2, y_2)$ then according to Viete's formulae their abscissae satisfy the equalities

$$\begin{aligned} x_1 + x_2 &= \frac{k - v}{u}, \\ x_1 \cdot x_2 &= \frac{w - y' + kx'}{u}. \end{aligned} \tag{4}$$

Thus, using (4)

$$TT_1 \cdot TT_2 = (x_1 - x')(x_2 - x') = x'^2 + \frac{vx'}{u} - \frac{y'}{u} + \frac{w}{u}. \qquad \Diamond$$

The radical axis of two circles is very well known notion in the Euclidean case. It is introduced in the isotropic plane as well, e.g. in ([7, p. 40]). Now on, we will obtain the equation of the radical axis in our terms.

Let $T = (x, y)$ be any point in the plane, and $\mathcal{K}_1$, $\mathcal{K}_2$ two circles with equations $y = u_i x^2 + v_i x + w_i$, $i = 1, 2$. A point T has the same power with respect to $\mathcal{K}_1, \mathcal{K}_2$ respectively if and only if $P_{T,\mathcal{K}_1} = P_{T,\mathcal{K}_2}$, i.e.

$$x^2 + \frac{v_1}{u_1}x - \frac{y}{u_1} + \frac{w_1}{u_1} = x^2 + \frac{v_2}{u_2}x - \frac{y}{u_2} + \frac{w_2}{u_2},$$

from where we get

$$y = \frac{v_1 u_2 - v_2 u_1}{u_2 - u_1}x + \frac{w_1 u_2 - w_2 u_1}{u_2 - u_1}.$$

In such a way we have reached the following theorem:

Theorem 2. *All the points having the same power with respect to two circles $\mathcal{K}_1$ and $\mathcal{K}_2$ given with $y = u_i x^2 + v_i x + w_i$, $i = 1, 2$ are incident with the same line that has the equation*

$$\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_2} \ldots y = \frac{v_1 u_2 - v_2 u_1}{u_2 - u_1}x + \frac{w_1 u_2 - w_2 u_1}{u_2 - u_1}. \tag{5}$$

$\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_2}$ is called the radical axis of the circles $\mathcal{K}_1$ and $\mathcal{K}_2$.

In [6, p. 204] has been proved that the radical axes $\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_2}$, $\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_3}$, $\mathcal{P}_{\mathcal{K}_2,\mathcal{K}_3}$ of three given circles are incident with one point. This point is called *the radical center* of the circles $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$, and it will be denoted by $P_{\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3}$.

3. The umbilic point and the umbilic axis

As the principle of duality is preserved in the isotropic plane (see, for example [7]), this is of great benefit for our further research on this topic. Hence, the notions of the power of the line with respect to the circle, the umbilic point and the umbilic axis will be introduced as the notions dual to the power of the point with respect to the circle, the radical axis and the radical center, respectively.

Theorem 3. *Let $\mathcal{K}$ be the circle with the equation $y = ux^2 + vx + w$ and let $\mathcal{P}$ be the line with the equation $y = kx + l$. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are tangents from any point of the line $\mathcal{P}$ to the circle $\mathcal{K}$, then the product $\angle(\mathcal{P}, \mathcal{T}_1) \cdot \angle(\mathcal{P}, \mathcal{T}_2)$ is constant and equal to*

$$P_{\mathcal{P},\mathcal{K}} = k^2 + 4ul - 2vk + v^2 - 4uw. \tag{6}$$

$P_{\mathcal{P},\mathcal{K}}$ is called the power of the line $\mathcal{P}$ with respect to the circle $\mathcal{K}$.

Proof. Let $T = (x', y')$ be any point of the line $\mathcal{P}$ and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the tangents from the point $T = (x', y')$ to the circle $\mathcal{K}$. Denoting by $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ the points of contact of these two tangents and the circle, the slopes of the tangents are

$$k_i = 2ux_i + v, \quad i = 1, 2, \tag{7}$$

respectively. As the points of contact P_1 and P_2 can be written in the form $P_i = (x_i, u{x_i}^2 + vx_i + w)$, $i = 1, 2$, it is easy to show that the tangents $\mathcal{T}_i$, $i = 1$, 2 are of the form

$$y = (2ux_i + v)x - u{x_i}^2 + w, \quad i = 1, 2.$$

Coordinates of the point $T = (x', y')$ satisfy the upper equation under the condition

$$u{x_i}^2 - 2ux_i x' + y' - vx' - w = 0, \quad i = 1, 2$$

that is quadratic equation with x_i, $i = 1, 2$ as its roots. From Viete's formulae we get

$$\begin{aligned} x_1 + x_2 &= 2x', \\ x_1 \cdot x_2 &= \frac{y' - vx' - w}{u}. \end{aligned} \tag{8}$$

Let us now calculate the product $\angle(\mathcal{P}, \mathcal{T}_1) \cdot \angle(\mathcal{P}, \mathcal{T}_2)$ using the values given in (7) and the abscissae x_i, $i = 1, 2$ of the points P_i, $i = 1, 2$:

$$(k_1 - k)(k_2 - k) = k^2 - 2[u(x_1 + x_2) + v]k + 4u^2 x_1 x_2 + 2uv(x_1 + x_2) + v^2. \tag{9}$$

Inserting (8) into (9) we obtain that

$$(k_1 - k)(k_2 - k) = k^2 - 2(2ux' + v)k + 4uy' - 4uw + v^2.$$

Inserting $y' = kx' + l$ (that is also valid) in the upper equality, it turns into (6). $\Diamond$

The first part of the above theorem can be found in [7, p. 49], while its proof and the form $P_{\mathcal{P},\mathcal{K}}$ given by (6) are characteristic for our approach. The notion of an umbilic point is introduced by:

Theorem 4. *All lines that have the same power with respect to two circles form a pencil of lines.*

Proof. The line $\mathcal{P}$ with the equation $y = kx + l$ and the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ with the equations $y = u_i x^2 + v_i x + w_i$, $i = 1, 2$ are given. According to Th. 3, the powers of the line $\mathcal{P}$ with respect to the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ are

$$P_{\mathcal{P},\mathcal{K}_1} = k^2 + 4u_1 l - 2v_1 k + {v_1}^2 - 4u_1 w_1,$$

and

$$P_{\mathcal{P},\mathcal{K}_2} = k^2 + 4u_2 l - 2v_2 k + {v_2}^2 - 4u_2 w_2,$$

respectively.

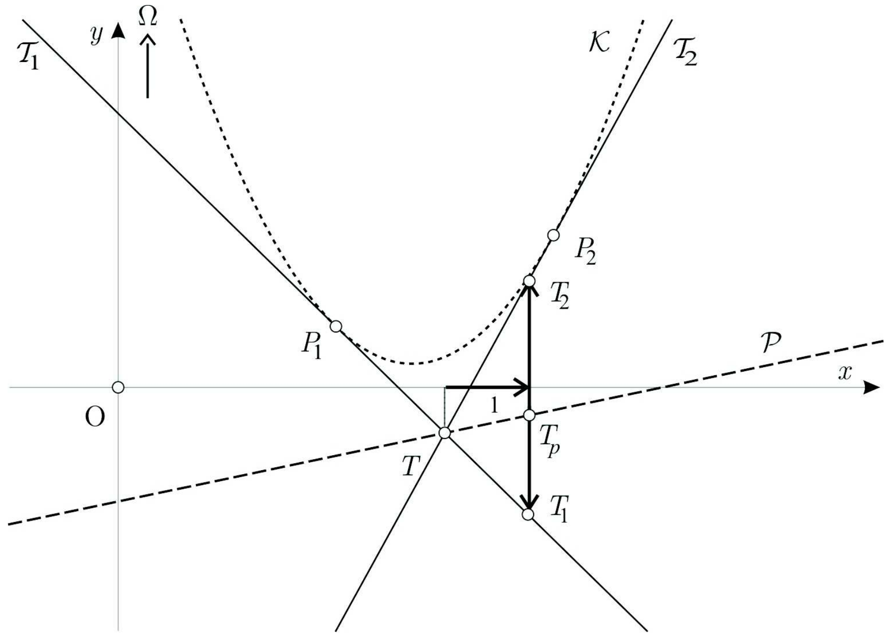

Figure 1. The power of the line $\mathcal{P}$ with respect to the circle $\mathcal{K}$

This line $\mathcal{P}$ will have the same powers with respect to these two circles on condition that the equality

$$4(u_1 - u_2)l - 2(v_1 - v_2)k + {v_1}^2 - 4u_1w_1 - {v_2}^2 + 4u_2w_2 = 0$$

is fulfilled. Thus it follows that all such lines form a pencil of lines given by

$$\frac{4u_1w_1 - 4u_2w_2 - {v_1}^2 + {v_2}^2}{4(u_1 - u_2)} = -\frac{v_1 - v_2}{2(u_1 - u_2)}k + l.$$

Hence, the point that all such lines are incident with has coordinates of the form

$$P_{\mathcal{K}_1,\mathcal{K}_2} = \left(-\frac{v_1 - v_2}{2(u_1 - u_2)}, \frac{4u_1w_1 - 4u_2w_2 - {v_1}^2 + {v_2}^2}{4(u_1 - u_2)}\right). \qquad \Diamond \tag{10}$$

The point from Th. 4 we call *the umbilic point* of these two circles. According to its definition, it is dual to the radical axis. As a consequence, we have:

Corollary 1. *The umbilic point of the circles* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *given with the equations* $y = u_ix^2 + v_ix + w_i$, $i = 1, 2$ *is of the form* (10).

3.1. Homothecy and dilatation

The umbilic point has an interesting property. It represents a center of homothecy $\left(P_{\mathcal{K}_1,\mathcal{K}_2}, \dfrac{u_1}{u_2}\right)$ that transforms the two circles related to that point into each other. Indeed,

Theorem 5. *If $P_{\mathcal{K}_1,\mathcal{K}_2}$ represents the umbilic point of the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ given with the equations $y = u_i x^2 + v_i x + w_i$, $i = 1, 2$ then the homothecy $\left(P_{\mathcal{K}_1,\mathcal{K}_2}, \dfrac{u_1}{u_2}\right)$ transforms the circle $\mathcal{K}_1$ into the circle $\mathcal{K}_2$.*

Proof. Homothecy $\left(P_{\mathcal{K}_1,\mathcal{K}_2}, \dfrac{u_1}{u_2}\right)$ with the equation

$$\begin{aligned} x' &= \frac{u_1}{u_2} x + \frac{v_1 - v_2}{2u_2}, \\ y' &= \frac{u_1}{u_2} y - \frac{4u_1w_1 - 4u_2w_2 - {v_1}^2 + {v_2}^2}{4u_2}. \end{aligned} \tag{11}$$

transforms circle $\mathcal{K}_1$ into circle $\mathcal{K}_2$. $\Diamond$

Similar investigation can be carried out for the radical axis. Actually, it is an axis of dilatation that transforms the circles related to the radical axis into each other. In accordance with the above, we have:

Theorem 6. *If $\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_2}$ represents the radical axis of the circles $\mathcal{K}_1$ and $\mathcal{K}_2$ given with the equations $y = u_i x^2 + v_i x + w_i$, $i = 1, 2$ then the dilatation $\left(\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_2}, \dfrac{u_2}{u_1}\right)$ transforms the circle $\mathcal{K}_1$ into the circle $\mathcal{K}_2$.*

Proof. Dilatation $\left(\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_2}, \dfrac{u_2}{u_1}\right)$ with equation

$$\begin{aligned} x' &= x, \\ y' &= \frac{u_2}{u_1} y - \frac{v_1u_2 - v_2u_1}{u_1} x - \frac{w_1u_2 - w_2u_1}{u_1}. \end{aligned} \tag{12}$$

transforms circle $\mathcal{K}_1$ into the circle $\mathcal{K}_2$. $\Diamond$

The following theorem is the result dual to the radical center given in ([6, p. 204]).

Theorem 7. *Umbilic points $P_{\mathcal{K}_1,\mathcal{K}_2}$, $P_{\mathcal{K}_1,\mathcal{K}_3}$ and $P_{\mathcal{K}_2,\mathcal{K}_3}$ of the three circles $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are incident with one line.*

This line we call the *umbilic axis* of the circles $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and we will denote it by $\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3}$ (see Fig. 2).

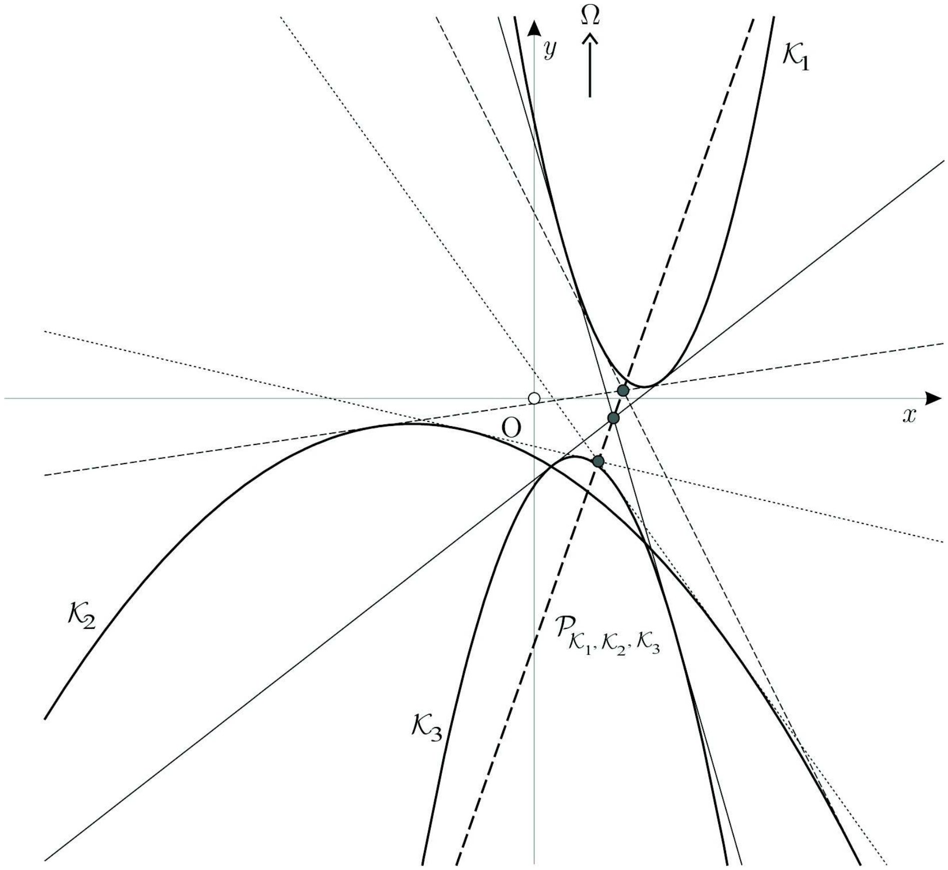

Figure 2. The umbilic axis $\mathcal{P}_{\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3}$ of the circles $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$

4. Properties of the umbilic point and the umbilic axis related to the non-cyclic quadrangle

In [2] we have introduced the geometry of a non-cyclic quadrangle in the isotropic plane. Before that, the geometry of a quadrangle has not been developed at all. Onwards we give new properties concerning the non-cyclic quadrangle related to its umbilic point and its umbilic axis.

A non-cyclic quadrangle is called *standard* if a special hyperbola with the equation $xy = 1$ is circumscribed to it. Such a standard quadrangle has vertices

$$A = \left(a, \frac{1}{a}\right), \quad B = \left(b, \frac{1}{b}\right), \quad C = \left(c, \frac{1}{c}\right), \quad D = \left(d, \frac{1}{d}\right), \tag{13}$$

and sides of the forms

$$(14)\qquad \begin{aligned} &AB \ldots y = -\frac{1}{ab}x + \frac{a+b}{ab}, && BC \ldots y = -\frac{1}{bc}x + \frac{b+c}{bc},\\ &AC \ldots y = -\frac{1}{ac}x + \frac{a+c}{ac}, && BD \ldots y = -\frac{1}{bd}x + \frac{b+d}{bd},\\ &AD \ldots y = -\frac{1}{ad}x + \frac{a+d}{ad}, && CD \ldots y = -\frac{1}{cd}x + \frac{c+d}{cd}. \end{aligned}$$

Let us recall from [2] that for the study of the non-cyclic quadrangle the following symmetric functions of the numbers a, b, c, d will be useful:

$$(15)\qquad \begin{aligned} s &= a + b + c + d,\\ q &= ab + ac + ad + bc + bd + cd,\\ r &= abc + abd + acd + bcd,\\ p &= abcd. \end{aligned}$$

We begin with:

Theorem 8. *Let $ABCD$ be a non-cyclic quadrangle and $\mathcal{O}_a, \mathcal{O}_b, \mathcal{O}_c$ and $\mathcal{O}_d$ be the circumscribed circles of the triangles BCD, ACD, ABD and ABC. If $\mathcal{B}_a$ is tangent to the circle $\mathcal{O}_a$ at the point B and $\mathcal{A}_b$ is tangent to the circle $\mathcal{O}_b$ at the point A, then the bisector of the lines $\mathcal{A}_b$ and $\mathcal{B}_a$ is incident with the umbilic point of the circles $\mathcal{O}_c, \mathcal{O}_d$.*

Proof. It is easy to show that a circumscribed circle of the triangle BCD has the equation

$$(16)\qquad bcdy = x^2 - (b + c + d)x + bc + bd + cd.$$

In general, the tangent to this circle $\mathcal{O}_a$ at a point (x_0, y_0) has the equation

$$bcd(y + y_0) = 2xx_0 - (b + c + d)(x + x_0) + 2bc + 2bd + 2cd.$$

Therefore, the tangent $\mathcal{B}_a$ to the circle $\mathcal{O}_a$ at the point B given in (13) has the equation

$$bcdy = (b - c - d)x + bc + bd + cd - b^2.$$

Analogously, $\mathcal{A}_b$, the tangent to the circle $\mathcal{O}_b$ at the point A, is

$$acdy = (a - c - d)x + ac + ad + cd - a^2.$$

After multiplying these two equations by a, b, respectively and adding them up the bisector $\mathcal{S}_{ab}$ of $\mathcal{B}_a$ and $\mathcal{A}_b$ has the equation

$$2py = [2ab - (a + b)(c + d)]x + 2ab(c + d) - (a + b)(ab - cd).$$

According to (16), circles $\mathcal{O}_c$ and $\mathcal{O}_d$ are of the forms

$$\mathcal{O}_c \ldots abdy = x^2 - (a+b+d)x + ab + ad + bd,$$
$$\mathcal{O}_d \ldots abcy = x^2 - (a+b+c)x + ab + ac + bc.$$

Referring (10) their umbilic point has coordinates

$$U_{cd} = \left(\frac{a+b}{2}, \frac{2r-(a^2+b^2)(c+d)}{4p}\right). \tag{17}$$

Out of the equality

$$2r - a^2c - a^2d - b^2c - b^2d = -(a-b)^2(c+d) + 2(a+b)cd$$

follows that the bisector $\mathcal{S}_{ab}$ is incident with the umbilic point from (17). $\Diamond$

We proceed with some interesting results concerning the circumscribed circles of the triangles of the non-cyclic quadrangle.

Theorem 9. *Let $ABCD$ be a non-cyclic quadrangle and $\mathcal{O}_a, \mathcal{O}_b, \mathcal{O}_c$ and $\mathcal{O}_d$ be the circumscribed circles of the triangles BCD, ACD, ABD and ABC. The umbilic axes of three out of four circumscribed circles touch one circle.*

Proof. According to (16), the equations of the circumscribed circles are

$$\mathcal{O}_a \ldots bcdy = x^2 - (b+c+d)x + bc + bd + cd,$$
$$\mathcal{O}_b \ldots acdy = x^2 - (a+c+d)x + ac + ad + cd,$$
$$\mathcal{O}_c \ldots abdy = x^2 - (a+b+d)x + ab + ad + bd,$$
$$\mathcal{O}_d \ldots abcy = x^2 - (a+b+c)x + ab + ac + bc.$$

The umbilic axes of three out of four circumscribed circles are studied now. Let us find e.g. an umbilic axis of the circles $\mathcal{O}_b$, $\mathcal{O}_c$ and $\mathcal{O}_d$. Denoted by $\mathcal{P}_a$, the umbilic axis of $\mathcal{O}_b, \mathcal{O}_c, \mathcal{O}_d$ has the equation

$$y = \frac{a^2 - bc - bd - cd}{2p}x - \frac{a^2s - 3r}{4p}. \tag{18}$$

As a matter of fact, out of the equality

$$-(a^2+b^2)(c+d) + 2r = (a+b)(a^2 - bc - bd - cd) - a^2s + 3r$$

follows that the point U_{cd} from (17) is incident with $\mathcal{P}_a$. The same is true for the umbilic points U_{bc} and U_{bd} of the circles $\mathcal{O}_b, \mathcal{O}_c$ and $\mathcal{O}_b, \mathcal{O}_d$. Then the umbilic axes of three circles out of four, $\mathcal{P}_a$, $\mathcal{P}_b$, $\mathcal{P}_c$, $\mathcal{P}_d$ respectively, touch a circle with the equation

$$4py = sx^2 - 2qx + 3r. \tag{19}$$

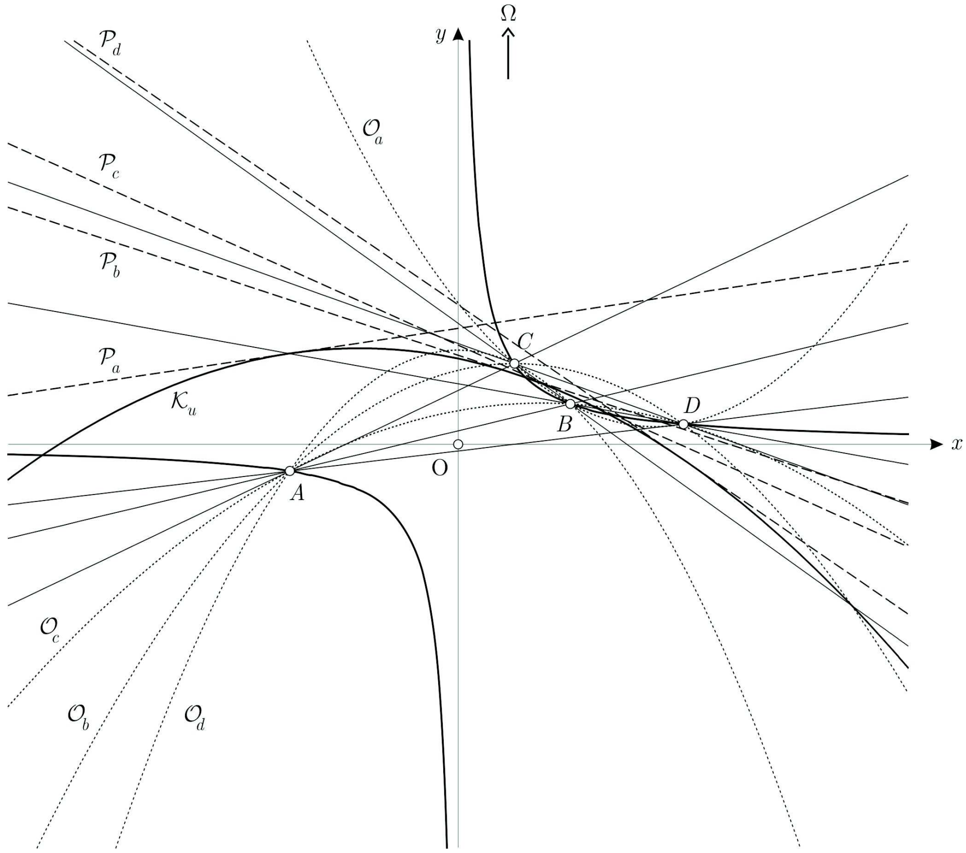

Figure 3. The visualization of Th. 9

Namely, from (18) and (19) it follows

$$sx^2 - 2(q + a^2 - bc - bd - cd)x + a^2 s = 0, \quad \text{i.e.} \quad (x-a)^2 = 0,$$

and therefore the umbilic axis $\mathcal{P}_a$ from (18) touches the circle (19). $\Diamond$

The visualization of Th. 9 is given in the Fig. 3 where by $\mathcal{K}_u$ we denote the circle (19).

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