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STEINER'S ELLIPSES OF THE TRIANGLE IN AN ISOTROPIC PLANE

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Abstract: The concept of the Steiner's ellipse of the triangle in an isotropic plane is introduced. The connections of the introduced concept with some other elements of the triangle in an isotropic plane are also studied.

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, absolute line ω and one point on that line, the absolute point Ω. The lines through the point Ω are isotropic lines, and the points on the line ω are isotropic points (the points at infinity). Two lines through the same isotropic point are parallel, and two points on the same isotropic line are parallel points. Therefore, an

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isotropic plane is in fact the affine plane with the pointed direction of isotropic lines and where the principle of duality is valid.

Conics, which touch the absolute line ω at the absolute point Ω are circles. However, singular circles (the circles of the second kind) are the sets of points which are equidistant from the given point and they consist of pairs of isotropic lines.

The triangle in an isotropic plane is allowable if any two of its vertices are not parallel.

If G is the centroid of the allowable triangle ABC then the homothecies $(G, -\frac{1}{2})$ and $(G, -2)$ map any point or line to its complementary or anticomplementary point or line, and the triangle ABC to its complementary or anticomplementary triangle.

Each allowable triangle in an isotropic plane can be set, by a suitable choice of coordinates, in the so called *standard position*, i.e. that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ where $a + b + c = 0$. With the labels $p = abc$, $q = bc + ca + ab$ it can be shown that the equalities $q = bc - a^2$, $(c - a)(a - b) = 2q - 3bc$, $(a - b)(b - c) = 2q - 3ca$, $(b - c)(c - a) = 2q - 3ab$, $(b - c)^2 = -(q + 3bc)$ are valid (see [7]).

Now, we will introduce the concept of Steiner's ellipse in an isotropic plane. Firstly, we will prove the following theorem.

Theorem 1. *The conic $\mathcal{S}$ with the equation*

$$q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 = 0 \tag{1}$$

is an ellipse which passes through the vertices $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ of the triangle ABC and the center of this ellipse is the centroid G of the triangle ABC.

Proof. The triangle ABC has the centroid $G = (0, -\frac{2}{3}q)$ (see [7]). Symmetry with respect to the point G is the mapping

$$x \to -x, \qquad y \to -\left(y + \frac{4}{3}q\right).$$

If we apply this mapping equation (1) transforms to the equation

$$q^2x^2 - 9px\left(y + \frac{4}{3}q\right) - 3q\left(y + \frac{4}{3}q\right)^2 + 6pqx + 4q^2\left(y + \frac{4}{3}q\right) + 9p^2 = 0,$$

which coincides with (1). Therefore the conic $\mathcal{S}$ has the center G. If we eliminate variable y from (1) and from the equation $y = x^2$ of the circumscribed circle of the triangle ABC, after dividing by -3 we get the following equation in x

$$qx^4 + 3px^3 + q^2x^2 + 2pqx - 3p^2 = 0,$$

which can be also written in the form $(qx + 3p)(x^3 + qx - p) = 0$, i.e. in the form $(qx + 3p)(x - a)(x - b)(x - c) = 0$, which has the solutions $x = a$, $x = b$, $x = c$, $x = -\frac{3p}{q}$ for the abscissas of the intersections of the conic $\mathcal{S}$ with the circumscribed circle. It means that conic $\mathcal{S}$ passes through the points A, B, C. The ratio $x : y$ for the points at infinity of the conic (1) satisfies the equation $q^2x^2 - 9pxy - 3qy^2 = 0$. As

$$81p^2 + 12q^3 = 3(27p^2 + 4q^3) < 0,$$

these points at infinity are imaginary, i.e. conic $\mathcal{S}$ is ellipse. $\Diamond$

By the analogy to the Euclidean case the ellipse $\mathcal{S}$ from Th. 1 will be called *circumscribed Steiner's ellipse* of the triangle ABC, and the fourth intersection (except A, B, C) of that ellipse with the circumscribed circle of that triangle will be called *Steiner's point* of the triangle ABC.

Corollary 1. *The triangle ABC from Th. 1 has the Steiner's point*

$$S = \left(-\frac{3p}{q}, \frac{9p^2}{q^2}\right). \tag{2}$$

Homothecy $(G, -\frac{1}{2})$ maps the ellipse $\mathcal{S}$ to the ellipse $\mathcal{S}'$, which passes through the midpoints of the sides BC, CA, AB, and whose center is G. Its inverse homothecy is the homothecy

$$x \to -2x, \qquad y \to -2(y + q),$$

whose application will transform equation (1) to the equation

$$4q^2x^2 - 36px(y + q) - 12q(y + q)^2 + 12pqx + 8q^2(y + q) + 9p^2 = 0,$$

i.e. the equation

$$4q^2x^2 - 36pxy - 12qy^2 - 24pqx - 16q^2y + 9p^2 - 4q^3 = 0 \tag{3}$$

of the ellipse $\mathcal{S}'$. By the elimination of the variable y from equation (3) and from the equation $y = -ax - bc$ of the line BC we get for the abscissa x of their intersection the equation

$$4q^2x^2 + 36px(ax + bc) - 12q(ax + bc)^2 - \\ - 24pqx + 16q^2(ax + bc) + 9p^2 - 4q^3 = 0,$$

which can be written in the form $4ux^2 + 4uax + ua^2 = 0$ where $u = 4q^2 - 3bcq + 9ap$. This equation has the the double solution $x = -\frac{a}{2}$.

It means that the ellipse $\mathcal{S}'$ touches the side BC at its midpoint, and analogously it is also valid for the sides CA and AB, i.e. the ellipse $\mathcal{S}'$ is inscribed in the triangle ABC. This ellipse will be called *inscribed Steiner's ellipse* of the triangle ABC. We have just proved the following theorem.

Theorem 2. *Inscribed Steiner's ellipse of the triangle ABC from Th. 1 has equation* (3).

Corollary 2. *The circumscribed Steiner's ellipse of the triangle ABC touches the lines parallel respectively to the lines BC, CA, AB at the vertices A, B, C.*

As the centroid is an affine property of a triangle, the affine properties of the circumscribed Steiner ellipse are the same as in the Euclidean case. In this sense Cor. 2 is obvious.

The tangents of the circumscribed circle of the allowable triangle ABC form the so called tangential triangle $A_tB_tC_t$ of the triangle ABC. The triangles ABC and $A_tB_tC_t$ are homologic. The center of its homology is the so called symmedian center K of the triangle ABC, and the axis of its homology is the Lemoine line L of that triangle. The lines AK, BK, CK are symmedians of the triangle ABC.

Theorem 3. *The symmedian center of the triangle lies on its inscribed Steiner's ellipse* (Tőlke [10]).

Proof. According to [6] the symmedian center of the triangle ABC is the point

$$K = \left(\frac{3p}{2q}, -\frac{q}{3}\right).$$

For this point the left side of (3) gets the value

$$9p^2 + 18p^2 - \frac{4}{3}q^3 - 36p^2 + \frac{16}{3}q^3 + 9p^2 - 4q^3 = 0. \quad \Diamond$$

If the inscribed circle of the allowable triangle ABC touches its sides at the points A_i, B_i, C_i, then the triangles ABC and $A_iB_iC_i$ are homologic, and the center of this homology is the so called Gergonne's point of the triangle ABC.

Owing to [3] Gergonne's point of the triangle is anticomplementary to its symmedian center, and according to the proof of Th. 2 the circumscribed Steiner's ellipse is anticomplementary to its inscribed ellipse.

Therefore we have the following

Corollary 3. *The Gergonne's point of the triangle lies on its circumscribed Steiner's ellipse.*

Owing to [3] for the triangle ABC in a standard position the Gergonne's point has the coordinates

$$\Gamma = \left(-\frac{3p}{q}, -\frac{4}{3}\,q\right). \tag{4}$$

Owing to (2) and (4) it follows straightforward.

Corollary 4. *Steiner's point S of the triangle is parallel to its Gergonne's point.*

With $y = -\frac{4}{3}\,q$ we get from (1) the equation $q^2x^2 + 6pqx + 9p^2 = 0$ with the double solution $x = -\frac{3p}{q}$, i.e. according to [11] the line with the equation $y = -\frac{4}{3}\,q$ is the Longchamps line of the triangle ABC. It is tangent to the ellipse (1) at the point Γ . In [1] that line is defined as the line anticomplementary to the orthic line of the triangle ABC, where that orthic line is the axis of homology of the triangle ABC and its orthic triangle $A_hB_hC_h$, and the points A_h, B_h, C_h are the points on the lines BC, CA, AB successively parallel to the points A, B, C. Owing to this it follows

Theorem 4. *The Longchamps line of the triangle touches its circumscribed Steiner's ellipse at its Gergonne's point.*

This theorem generalizes the Euclidean result (see Cesaro [5]).

The circumscribed circle of the orthic triangle and the complementary triangle of the allowable triangle ABC is Euler circle of the triangle ABC. It touches the inscribed circle of that triangle at its Feuerbach point, and the tangent of these two circles at that point is the Feuerbach line of the triangle ABC.

With $y = 0$ from (1) we get equation $q^2x^2 - 6pqx + 9p^2 = 0$ with the double solution $x = \frac{3p}{q}$. Therefore the line with the equation $y = 0$ touches the ellipse (1) at the point $\left(\frac{3p}{q}, 0\right)$. This is the analogy to the Feuerbach line (see [2]) of the tangential and anticomplementary triangle of the triangle ABC, it touches the ellipse (1) at the point $\left(\frac{3p}{q}, 0\right)$. The midpoint of this point and the point Γ from (4) is the centroid $G = \left(0, -\frac{2}{3}\,q\right)$ of the triangle ABC. So, we get the following.

Theorem 5. *The circumscribed Steiner's ellipse of the allowable triangle ABC is tangent to the Feuerbach line of its tangential triangle and* *its anticomplementary triangle at the point which is symmetric to the Gergonne's point of the triangle ABC with respect to its centroid.*

Th. 5 is also a consequence of Th. 4 and the fact that the centroid of the triangle is the center of its circumscribed Steiner's ellipse.

The lines with the equations $y = -\frac{q}{3}$ and $y = -q$ are complementary to the lines with the equations $y = -\frac{4}{3}q$, $y = 0$ since the centroid G has the ordinate $-\frac{2}{3}q$. The obtained lines are the orthic axis and the Feurbach line of the triangle ABC (see [7] and [2]). The points $\left(\frac{3p}{2q}, -\frac{q}{3}\right)$ and $\left(-\frac{3p}{2q}, -q\right)$, where the first point is the point K from the proof of Th. 3 and the second point is symmetrical to this point with regard to the point G, are complementary to the points $\left(-\frac{3p}{q}, -\frac{4}{3}q\right)$ and $\left(\frac{3p}{q}, 0\right)$. Therefore it follows

Theorem 6. *The inscribed Steiner's ellipse of the allowable triangle ABC touches its orthic axis at its symmedian center K and the Feuerbach line of that triangle at the point symmetric to the point K with respect to the centroid G of the triangle ABC.*

A short computation yields

Theorem 7. *The tangent of the circumscribed Steiner's ellipse of the standard triangle ABC at its Steiner's point has the equation*

$$y = -\frac{3p}{q}x. \tag{5}$$

Owing to [2] the tangential triangle of the triangle ABC and its anticomplementary triangle have the same Feuerbach point $(0, 0)$, which obviously lies on the line (5), i.e. the following statement is valid.

Corollary 5. *The tangent of the circumscribed ellipse of the allowable triangle at its Steiner's point passes through the Feuerbach point of its tangential and its anticomplementary triangle.*

A short computation gives the following statement.

Theorem 8. *The set of points $T = (x, y)$ on isotropic tangents of the ellipse* (3) *has the equation*

$$x^2 + \frac{q}{3} = 0. \tag{6}$$

The circle (of the second kind) with equation (6) is indeed the orthoptic circle of the ellipse (3) i.e. it is the set of points from which the tangents on that ellipse are mutually perpendicular, and in an isotropic plane the isotropic lines and nonisotropic lines are perpendicular. In Euclidean geometry this circle belongs to the so called *Griffiths' pencil of the circles*, whose potential axis is the orthic axis of the considered triangle. In [2] it is shown that in isotropic geometry there is an analogous pencil of the circles, in which there are for example the circumscribed and Euler circle of the triangle. In the case of the triangle ABC in a standard position circumscribed circle and the orthic axis of that triangle have the equations

$$y = x^2, \tag{7}$$

$$y = -\frac{q}{3}. \tag{8}$$

However, from (6) and (7) it follows (8), so the circle (6) belongs to the considered pencil of circles.

For the values x_1 and x_2 of the abscissa of x, which satisfy equation (6) the corresponding ordinates y_1 and y_2 of the points of contact T_1 and T_2 of the isotropic tangents of the ellipse (3) can be written in the form

$$y_i = -\frac{9px_i + 4q^2}{6q} = -\frac{3p}{2q}x_i - \frac{2}{3}q \quad (i = 1, 2).$$

Therefore the line T_1T_2, which is the polar line of the absolute point for the ellipse (3) has the equation

$$y = -\frac{3p}{2q}x - \frac{2}{3}q. \tag{9}$$

The center of conic as the pole of the absolute line is dual concept to the concept of *axis* of the conic as polar line of absolute point for that conic. The axis of the conic evidently passes through its center. Because of that the line (9) is the axis of ellipse (3). We have the same consideration for the ellipse (1), so the line (9) is its axis too. Therefore we have

Theorem 9. *The inscribed and the circumscribed Steiner's ellipses of the allowable triangle have the same axis, which passes through the centroid of that triangle and which in the case of a standard triangle has equation* (9).

The axis from Th. 9 will be called *Steiner's axis* of the considered triangle. In Euclidean geometry the triangle has two Steiner's axes and here it has only one Steiner's axis. In fact, for the second Steiner's axis we could use the isotropic line through the centroid of the triangle, i.e. its Euler line.

Corollary 6. *The Steiner's axis of the standard triangle ABC has equation* (9).

The points T_1 and T_2 from the previous consideration are the foci of the circumscribed Steiner ellipse.

In [3] (Th. 5 and Th. 8) it is proved that the triangle ABC and its contact triangle $A_iB_iC_i$ are homologic with respect to Gergonne's point Γ of the triangle ABC, and the axis of this homology is the harmonic polar of the point Γ for the triangle ABC and for that point, in the case of a standard triangle ABC, equation (9) is obtained. Thus we have

Corollary 7. *The Steiner's axis of an allowable triangle is the harmonical polar of its Gergonne's point for that triangle.*

In an isotropic plane the isogonality with respect to the allowable triangle ABC is defined in the same manner as in Euclidean plane, i.e. the points T and T' are isogonal if the pairs of lines AT, AT'; BT, BT'; CT, CT' are symmetrical with respect to the angle bisectors of the angles A, B, C of that triangle. The centroid and the symmedian center of the triangle are mutually isogonal points with respect to this triangle.

According to [8] isogonality with respect to the triangle ABC in a standard position maps the line with the equation $y = kx + l$ into the conic, circumscribed to the triangle ABC, whose equation is

$$lx^2 - kxy - y^2 + (p - kq)x - (q + l)y + kp = 0. \tag{10}$$

If $k = \frac{3p}{q}$, $l = \frac{q}{3}$, then owing to [6] we have the Lemoine line of the triangle ABC, and equation (10) gets the form

$$\frac{q}{3}x^2 - \frac{3p}{q}xy - y^2 - 2px - \frac{4}{3}qy + \frac{3p^2}{q} = 0,$$

and indeed it is equation (1) divided by $\frac{q}{3}$. Therefore we get

Theorem 10. *The circumscribed Steiner's ellipse of the triangle is the isogonal image of its Lemoine line with respect to this triangle.*

According to [8] isogonality maps the circumscribed circle of the triangle to the line at infinity. Therefore Th. 10 implies

Corollary 8. *Steiner's point of the triangle is isogonal to the point at infinity of its Lemoine line with respect to that triangle.*

The polar line of the point (x_o, y_o) with respect to the ellipse (1) has the equation

$2q^2x_ox - 9p(x_oy + y_ox) - 6qy_oy - 6pq(x + x_o) - 4q^2(y + y_o) + 18p^2 = 0,$
i.e. the equation $y = kx + l$, where

$$k = \frac{2q^2x_o - 9py_o - 6pq}{9px_o + 6qy_o + 4q^2}, \qquad l = -\frac{6pqx_o + 4q^2y_o - 18p^2}{9px_o + 6qy_o + 4q^2}. \tag{11}$$

The equalities (11) can be solved for x_o and y_o. This yields

$$x_o = \frac{4qk + 6p}{3l + 2q}, \qquad y_o = -\frac{6pk + 2ql}{3l + 2q}. \tag{12}$$

Therefore we have

Theorem 11. *With respect to the circumscribed Steiner's ellipse of the standard triangle ABC the polar line of the point (x_o, y_o) has the equation $y = kx + l$ given by formulae* (11)*, and conversely the line with the equation $y = kx + l$ has the pole (x_o, y_o) defined by formulae* (12)*.*

For the diameters of the ellipse (1), i.e the lines through the centroid $G = (0, -\frac{2}{3}q)$, the equality $l = -\frac{2}{3}q$ is valid, thus the point (x_o, y_o) from (12) is a point at infinity. For this point we have

$$k' = \frac{y_o}{x_o} = -\frac{6pk + 2ql}{4qk + 6p} = -\frac{9pk + 3ql}{6qk + 9p} = -\frac{9pk - 2q^2}{6qk + 9p},$$

This point is a point at infinity for all lines with the slope k' i.e. the equality

$$6qkk' + 9p(k + k') - 2q^2 = 0 \tag{13}$$

is valid. It means that we obtain the following theorem.

Theorem 12. *Diameters of circumscribed (and inscribed) Steiner's ellipse of the standard triangle ABC are its conjugate diameters if and only if for their slopes k and k' holds equality* (13)*.*

In [4, Th. 14] for the inscribed conic of the triangle the following statement is proved: the joint lines of the focus of that conic with one vertex of the triangle and with point of contact with the opposite side are isogonal with respect to the joint lines of that focus with the two remaining vertices of that triangle. In the case of the inscribed Steiner's ellipse the point of contact with the side BC is the midpoint A_m of that side. So if F is the focus of that ellipse, then the line FA is isogonal to the median FA_m of the triangle FBC with respect to F i.e. FA is symmedian of that triangle. Foci of the inscribed Steiner's ellipse of that triangle will be called *Steiner's foci* of that triangle. We have just proved the following theorem.

Theorem 13. *If F is a Steiner's focus of the allowable triangle ABC, then the lines FA, FB, FC are symmedians from the vertex F in the triangles FBC, FCA, FAB* ([9] and [1, p. 140] have the Euclidean analogue of this theorem).

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