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ON A THEOREM OF DABOUSSI RELATED TO THE SET OF GAUSSIAN INTEGERS II

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Abstract: Let W stand for the union of finitely many convex bounded domains in $\mathbb{C}$. Given $x > 0$, we denote by xW the set $\{xz : z \in W\}$. Let $G = \mathbb{Z}[i]$ be the set of Gaussian integers and set $G^* := G \setminus \{0\}$. Given a complex number $z = u + iv$, where $u, v \in \mathbb{R}$, let $\{z\} = \{u\} + i\{v\}$, where $\{x\}$ stands for the fractional part of x. Let $E := \{w : 0 \leq \Re(w) < 1,\ 0 \leq \Im(w) < 1\}$. We say that the sequence of complex numbers $z_1, z_2, \ldots$ is uniformly distributed mod E if $\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : \Re(\{z_n\}) < u,\ \Im(\{z_n\}) < v\} = uv$ for every pair of real numbers $u, v \in]0, 1]$. Let $\mathcal{T}$ be the set of those functions $t : G^* \to \mathbb{C}$ for which $t(\alpha) + F(\alpha)$ is uniformly distributed mod E in limit on xW (as $x \to \infty$) for every additive arithmetical function F, and such that $t(\alpha) + F(\alpha)$ is uniformly distributed in F. We prove that if $P(z) \in \mathbb{C}[z]$ is a polynomial of positive degree, whose leading coefficient is a and such that the numbers 1, $\Re(a)$ and $\Im(a)$ are rationally independent, then $P \in \mathcal{T}$.

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1. Introduction

Let W stand for the union of finitely many convex bounded domains in $\mathbb{C}$. Given $x>0$, we denote by xW the set $\{xz : z\in W\}$, and observe that with the Lebesgue measure $|\cdot|$, we have $|xW|=x^2|W|$. Let $G=\mathbb{Z}[i]$ be the set of Gaussian integers and set $G^*:=G\setminus\{0\}$. Finally, let $\mathcal{M}$ be the set of multiplicative functions defined on G^* and let $\mathcal{M}^*$ be the subset of $\mathcal{M}$ made of those $g\in\mathcal{M}$ satisfying $|g(\alpha)|\le 1$ for all $\alpha\in G^*$. Let χ be an arbitrary additive character, that is a function $\chi : G\to\{z : |z|=1\}$ for which $\chi(0)=1$ and $\chi(\alpha_1+\alpha_2)=\chi(\alpha_1)\chi(\alpha_2)$ for all $\alpha_1,\alpha_2\in G$. Using the standard notation $e(u)=e^{2\pi iu}$, we set $\chi(1)=e(A)$ and $\chi(i)=e(B)$, and then denote by $\mathcal{A}$ the set of those χ's for which at least one of A and B is irrational. We proved in [1] that, given $\chi\in\mathcal{A}$ and $g\in\mathcal{M}^*$,

$$\lim_{x\to\infty}\frac{1}{|xW|}\sum_{\beta\in xW} g(\beta)\chi(\beta)=0,$$

where the convergence is uniform in g, thereby generalizing a previous result of Daboussi and Delange [2].

This paper is essentially a continuation of the results obtained in [1].

2. The main result

Given a complex number $z=u+iv$, where $u,v\in\mathbb{R}$, let $\{z\}=$ $=\{u\}+i\{v\}$, where $\{x\}$ stands for the fractional part of x. Let $E:=$ $:=\{w : 0\le\Re(w)<1,\ 0\le\Im(w)<1\}$. We say that the sequence of complex numbers $z_1,z_2,\dots$ is uniformly distributed mod E if

$$\lim_{N\to\infty}\frac{1}{N}\#\big\{n\le N : \Re(\{z_n\})<u,\ \Im(\{z_n\})<v\big\}=uv$$

for every pair of real numbers $u,v\in\,]0,1]$.

A result of H. Weyl states that (see [3]) that the sequence z_n is uniformly distributed mod E if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e(k\Re(z_n)+\ell\Im(z_n))=0$$

for each pair $(k,\ell)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$.

For each real positive number x, let $N(x):=\#\{\alpha\in xW\cap G^*\}$ and further let $h : G^*\to\mathbb{C}$. For $u,v\in\,]0,1]$, let

$$F_x(u,v):=\frac{1}{N(x)}\#\big\{z\in xW\cap G^* : \Re(\{h(z)\})<u,\ \Im(\{h(z)\})<v\big\}.$$

We say that h is uniformly distributed $\bmod E$ in limit on xW for $x \to \infty$ if

$$(2.1) \qquad \lim_{x\to\infty} F_x(u,v) = uv \quad \text{holds for } \ 0<u\le 1,\ 0<v\le 1.$$

Let $\mathcal{T}$ be the set of those functions $t : G^* \to \mathbb{C}$ for which $t(\alpha)+F(\alpha)$ is uniformly distributed mod E in limit on xW (as $x \to \infty$) for every additive arithmetical function F, and such that $t(\alpha)+F(\alpha)$ is uniformly distributed in F.

Theorem 1. *Let $P(z) \in \mathbb{C}[z]$ a polynomial of positive degree k. Let a be the coefficient of z^k in $P(z)$. Assume that the numbers 1, $\Re(a)$ and $\Im(a)$ are rationally independent. Then $P \in \mathcal{T}$.*

3. Preliminary lemmas

Lemma 1. *Let $\wp = \{\rho_1, \rho_2, \ldots, \rho_r\}$ be a finite set of Gaussian primes, with $|\rho_1| \le |\rho_2| \le \ldots \le |\rho_r|$ such that no two of them are associates. Let χ be an additive character. Set $T(x) := \sum_{\beta\in xW} g(\beta)\chi(P(\beta))$ and let*

$$T_1(x) := \sum_{\substack{\rho\gamma\in xW\\ \rho\in\wp}} g(\rho\gamma)\chi(P(\rho\gamma)), \qquad T_2(x) := \sum_{\substack{\rho\gamma\in xW\\ \rho\in\wp}} g(\rho)g(\gamma)\chi(P(\rho\gamma)).$$

Then,

$$|T_1(x) - T_2(x)| \le \frac{cx^2}{|\rho_1|^2} A_\wp,$$

where $A_\wp = \sum_{j=1}^{r} \frac{1}{|\rho_j|^2}$.

Lemma 2 (Weyl)**.** *Let $f(x) = \alpha_k x^k + \ldots + \alpha_1 x + \alpha_0$ be a polynomial with real coefficients $\alpha_0, \alpha_1, \ldots, \alpha_k$ and such that*

$$\left|\alpha_k - \frac{h}{q}\right| \le \frac{1}{q^2}, \qquad (h,q)=1.$$

Then,

$$\sum_{x=1}^{P} e(f(x)) \ll P^{1+\varepsilon} q^{\varepsilon} \left(\frac{1}{P} + \frac{1}{q} + \frac{q}{P^k}\right)^{2^{1-k}}.$$

Proof. This result is due to H. Weyl and is stated (and proved) as Lemma 3.6 in the book of Hua [3]. $\Diamond$

Lemma 3 (Erdős–Turán–Koksma). *Let (x_n), where $n = 1, 2, \ldots, N$, be a sequence of points in $\mathbb{R}^s$ and let G be an arbitrary positive integer. Then, the discrepancy $D_N(x_n)$ is less than*

$$2s^2 3^{s+1} \left(\frac{1}{G} + \sum_{0<\|h\|\le G} \frac{1}{R(h)} \left| \frac{1}{N} \sum_{n=1}^{N} e(\langle h, x_n \rangle) \right| \right),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^s$, $\|h\| = \max_{i=1,\ldots,s} |h_i|$ for integral lattice points $h = (h_1, \ldots, h_s)$, and $R(h) = \prod_{j=1}^{s} \max(|h_j|, 1)$.

Proof. For a proof of this result, see the book of Kuipers and Niederreiter [4]. $\Diamond$

4. The proof of the main result

The case $k = 1$ follows essentially from our Th. 1 proved in [1]. Hence, we may assume that $k \ge 2$.

The first part of the proof follows exactly the same reasoning as that of the proof of Th. 1 in [1].

Indeed, applying Lemma 1 with

$$a(\gamma) = g(\gamma) \quad \text{and} \quad b(\gamma) = \sum_{\substack{\rho \in \wp \\ \rho \in \frac{xW}{\gamma}}} g(\rho)\chi(P(\rho\gamma)),$$

we get that

$$T_2(x) = \sum_{\substack{\gamma \in G^* \\ \gamma \in \cup_{\rho \in \wp} \frac{xW}{\rho}}} a(\gamma) b(\gamma),$$

with

$$|T_1(x) - T_2(x)| \le \frac{cx^2}{|\rho_1|^2} A_\wp. \tag{4.1}$$

By the Cauchy–Schwarz Inequality, we obtain that

$$|T_2(x)| \le \left(\sum_{\gamma} |a(\gamma)|^2 \right)^{1/2} \cdot \left(\sum_{\gamma} |b(\gamma)|^2 \right)^{1/2} = \Sigma_1^{1/2} \cdot \Sigma_2^{1/2}, \tag{4.2}$$

say.

On the one hand, it is clear that

$$\Sigma_1 \ll x^2. \tag{4.3}$$

On the other hand, Σ_2 can be written as

$$\Sigma_2 = \sum_{\gamma} \sum_{\substack{\rho\in\wp \\ \rho\gamma\in xW}} 1 + \sum_{\nu\neq j} \sum_{\gamma\in S_{\nu,j}} g(\rho_\nu)\overline{g(\rho_j)}\chi(P(\rho_\nu\gamma))\overline{\chi(P(\rho_j\gamma))}, \tag{4.4}$$

where $S_{\nu,j} = \dfrac{xW}{\rho_\nu} \cap \dfrac{xW}{\rho_j}$. Now, since χ is an additive character, it follows that

$$\chi(P(\rho_\nu\gamma))\overline{\chi(P(\rho_j\gamma))} = \chi(P(\rho_\nu\gamma) - P(\rho_j\gamma)). \tag{4.5}$$

Assume that

$$\chi(z) := e(k\Re(z) + \ell\Im(z)), \qquad (k,\ell) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}.$$

Then, set

$$B_{\nu,j} := \sum_{\gamma\in S_{\nu,j}} \chi(P(\rho_\nu\gamma) - P(\rho_j\gamma)). \tag{4.6}$$

In light of the estimates (4.1) through (4.6), it is clear that it is sufficient to prove that

$$\lim_{x\to\infty} \frac{1}{N(x)} |B_{\nu,j}| = 0. \tag{4.7}$$

To do so, we argue as in the proof of Th. 1 of [1].

Let $\rho_\nu, \rho_j \in \wp$ be fixed, $\rho_\nu \neq \rho_j$. Further let $a = A + Bi$, $\rho_\nu^k - \rho_j^k = P + Qi$, $U = K(AP - QB) + L(AQ + BP)$. We must have that $U\neq 0$. Indeed, since

$$U = (KP + LQ)A + (LP - KQ)B,$$

and $KP + LQ = 0$, $LP - KQ = 0$ would imply that $\dfrac{P}{Q} = \dfrac{K}{L}$, $\dfrac{P}{Q} = -\dfrac{L}{K}$, that is either $K = L$ or $K = -L$.

If $K = L$, then $K \neq 0$, and $KP + LQ = 0$, $LP - KQ = 0$, which would apply that $P + Q = 0$, $P - Q = 0$, implying that $P = Q = 0$.

If $K = -L$, then $K \neq 0$, and so $P - Q = 0$, $P + Q = 0$ would follow, which is also impossible.

Since $A, B, 1$ are rationally independent, it follows that U is irrational and therefore that $k!U$ is an irrational number.

Let $0\leq\lambda<1$ be the unique (irrational) number such that $e(k!U) = e(\lambda)$ and let $q_1 < q_2 < \dots$ be a sequence of positive integers such that

$$q_\nu \|\lambda q_\nu\| < 1 \quad \text{for} \ \ \nu = 1, 2, 3, \ldots$$

holds.

Let

$$Y(x) = \max_{q_\nu < \log x} q_\nu$$

and

$$B_{\nu,j}^{(x)} := \frac{1}{Y(x)} \sum_{\gamma \in S_{\nu,j}} \sum_{\ell=0}^{Y(x)-1} \chi\left(P(\rho_\nu(\gamma+\ell)) - P(\rho_j(\gamma+\ell))\right).$$

First, letting $N(x) = \#\{\gamma \in S_{\nu,j}\}$, we observe that $|B_{\nu,j} - B_{\nu,j}^{(x)}| = = o(N(x))$ as $x \to \infty$. Thus in order to prove (4.7), we only need to prove that

$$\lim_{x\to\infty} \frac{1}{N(x)} \left| B_{\nu,j}^{(x)} \right| = 0. \tag{4.8}$$

Now let $N^{(0)}$ be the number of those γ for which $\gamma + \ell \in S_{\nu,j}$ $(\ell = 0, 1, \ldots, Y(x)-1)$. If $\gamma \in S_{\nu,j}$ and $\gamma + \ell \notin S_{\nu,j}$ for at least one $\ell \in \{0, 1, \ldots, Y(x)-1\}$, then either $\gamma\rho_\nu$ or $\gamma\rho_j$ is close to the boundary of xW. Since W is a finite union of convex domains, the length of the boundary of xW is $O(x)$, which implies that

$$0 \le N(x) - N^{(0)}(x) \ll xY(x) = o(N(x)) \qquad (x \to \infty).$$

We shall now prove that

$$\max_{\gamma \in S_{\nu,j}} \frac{1}{Y(x)} \left| \sum_{\ell=0}^{Y(x)-1} \chi(Q(\ell)) \right| \to 0 \qquad (x \to \infty), \tag{4.9}$$

where

$$Q(\ell) = Q_j(\ell) = P(\rho_\nu(\gamma+\ell)) - P(\rho_j(\gamma+\ell)).$$

To prove (4.9), we shall use Lemma 2. But in order to do so, we first observe that $Q(\ell) = a(\rho_\nu^\ell - \rho_j^\ell)\ell^k + \ldots$.

Let $R(\ell) = K\Re Q(\ell) + L\Im Q(\ell)$. Then $R(\ell)$ is a polynomial of degree k, of which the coefficient of the main term is $K\Re a(\rho_\nu^k - \rho_j^k) + L\Im a(\rho_\nu^k - \rho_j^k)$.

Thus,

$$T := \sum_{\ell=0}^{Y(x)-1} \chi(Q(\ell)) = \sum_{\ell=0}^{Y(x)-1} e(R(\ell)).$$

Applying Lemma 2 with $P = Y(x)$, $f = R$, $\alpha_k = \lambda$, we may conclude that

$$|T| \ll Y(x)^{1+2\varepsilon} \left(\frac{1}{Y(x)} \right)^{2^{1-k}},$$

where $\varepsilon > 0$ is an arbitrary small constant. Thus, $T/Y(x) \to 0$ as $x \to \infty$ uniformly in γ, which completes the proof of (4.9) and therefore of (4.8). The estimates being uniform in t, Th. 1 then follows from Lemma 3.

References

[1] BASSILY, N., DE KONINCK, J. M. and KÁTAI, I.: On a theorem of Daboussi related to the set of Gaussian integers, *Mathematica Pannonica* **14** (2003), 267–272.

[2] DABOUSSI, H. and DELANGE, H.: Quelques propriétés des fonctions multiplicatives de module au plus égal à 1, *C.R. Acad. Sci. Paris Série A* **278** (1974), 657–660.

[3] HUA, L. K.: *Additive Theory of Prime Numbers*, Vol. 13, Translations of Mathematical Monographs, American Mathematical Society, Rhode Island, 1965.

[4] KUIPERS, L. and NIEDERREITER, H. *Uniform Distribution of Sequences*, John Wiley and Sons, New York, 1974.