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SEQUENTIAL ORDER UNDER CH

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Abstract: Revisiting and completing a work due to A. I. Baškirov, we construct compact sequential spaces of any sequential order up to and including ω_1 as quotient spaces of $\beta\omega$ under CH.

1. Introduction

Let X be a topological space and $M \subseteq X$; the *sequential closure* of M is $\operatorname{seqcl}(M) = \{x \in X : \exists (x_n)_{n\in\omega} \subseteq M^1, \lim_{n\in\omega} x_n = x\}$. For every ordinal $\alpha \leq \omega_1$, the *α-sequential closure* of M is inductively defined as follows:

– $\operatorname{seqcl}_0(M) = M$ and $\operatorname{seqcl}_1(M) = \operatorname{seqcl}(M)$;
– $\operatorname{seqcl}_{\alpha+1}(M) = \operatorname{seqcl}(\operatorname{seqcl}_\alpha(M))$;
– $\operatorname{seqcl}_\alpha(M) = \bigcup_{\beta<\alpha} \operatorname{seqcl}_\beta(M)$ if α is a limit ordinal.

A topological space X is said to be *sequential* if

$$\operatorname{seqcl}_{\omega_1}(M) = \overline{M}, \qquad \forall M \subseteq X;$$

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[1]By the notation $(x_n)_{n\in\omega} \subset M$ we mean that $(x_n)_{n\in\omega}$ is a sequence and that $x_n \in M$ for every $n \in \omega$.

this definition is equivalent to that according to which a space X is said to be *sequential* if every sequentially closed subset of X is closed.

The *sequential order* of a sequential space X is an ordinal invariant of the space defined as

$$\sigma(X) = \min\{\alpha \leq \omega_1 : \forall M \subseteq X, \mathrm{seqcl}_\alpha(M) = \overline{M}\}.$$

While the problem naturally posed in the sixties concerning the possibility to produce examples of sequential spaces of any sequential order up to and including ω_1 in ZFC was completely solved in the affirmative by Arhangel'skiĭ and Franklin (cf. [1]), it turns out difficult to construct compact sequential spaces without additional assumptions of the Theory of the Sets, even of sequential order 3; indeed, up to now, 2 is the maximum order of sequentiality of a compact space in ZFC. In this context the work due to Baškirov and concisely presented in a Doklady article (see [3]) gathers a certain prominence: in this paper the author suggests a scheme of construction to produce compact sequential spaces of any order as quotient spaces of $\beta\omega$ under the assumption of the Continuum Hypothesis. Our aim is to reexamine completely his article completing and modifying in several point his construction in order to make clear all details and point out where CH is essentially used.

Other constructions were proposed by Kannan (cf. [6]) under CH and by Dow under MA (see [4] and [5]). In the last construction an example is given of a compact sequential space of order 4. While Baškirov suggests a construction from top to down, Kannan and Dow present a construction from down to top. Indeed Baškirov works in $\beta\omega$ and by assuming to have constructed all the spaces of sequential order a successor ordinal less than a fixed successor ordinal $\alpha + 1$ he gives a starting decomposition on $\beta\omega$; the Continuum Hypothesis guarantees him that in ω_1 steps he can purify the starting decomposition in such a way that in the space associated to the last decomposition there is a new point fit to produce a space of sequential order $\alpha + 1$. Instead Kannan and Dow start from the natural numbers with the discrete topology. If we want to summarize the idea of Kannan, we can say that he generalizes the construction of the one-point compactification of the Mrówka–Isbell space. On the other hand, Dow constructs by transfinite induction on $\mathfrak{c}$ three suitable families of subsets of ω in such a way that the Stone space associated to the Boolean algebra generated by the elements of these subsets admits a point of sequential order 4.

There is a remarkable reason to determine the maximum possible sequential order in the presence of the PFA which implies Martin's axiom and $\mathfrak{c} = \omega_2$; indeed in 1989 Balogh solved the Moore–Mrówka problem proving that each compact space of countable tightness is sequential under PFA (see [2]). If there is some finite bound on the sequential order of compact sequential spaces in models of PFA, it would mean that compact spaces of countable tightness are a few steps away from being Fréchet–Urysohn. In [5, Prop. 3.1], Dow points out that there are obstructions to extend his type of construction to produce compact sequential spaces of order greater than 4. The problem if there exists a bound on the sequential order of compact sequential spaces in models of PFA is still open. However Dow in an unpublished paper (Sequential Order under PFA) showed that no compact supersequential space has sequential order greater than ω. Here a space is called supersequential if it is compact scattered and points in X_α have sequential order α with respect to X_0.

2. Preliminary facts

We start with the construction of spaces of order 1 and 2 in ZFC as quotients of $\beta\omega$, in order to illustrate the scheme of the main general construction that will be presented later on in the following sections.

Let us consider the space $K_1 = \beta\omega/\omega^* = \beta\omega/\approx_1$ where $x \approx_1 y \Leftrightarrow (x = y \vee (x \in \omega^* \wedge y \in \omega^*))$ and let us denote by j_1 the natural quotient mapping from $\beta\omega$ to K_1. Trivially the one-point elements of the quotient under the relation of equivalence $\approx_1$ are images of the points of ω, while the natural quotient mapping collapses all the free ultrafilters to a single point P. The topology of the space $K_1 = \beta\omega/\omega^* = \omega \cup \{P\}$ with $P \notin \omega$ is a topology τ such that the points of ω are isolated while P has a fundamental system of (open) neighborhoods formed by $\{U_{P,F}\} = \big\{\{P\} \cup j_1(\omega\backslash F) : F \in [\omega]^{<\omega}\big\}$ as we state in the following lemma.

Lemma 2.1. *Let $P = j_1(\omega^*) \in K_1$; a fundamental system of neighborhoods of P is given by the collection $\{j_1(\beta\omega\backslash F) : F \in [\omega]^{<\omega}\}$.*

Notice that the fundamental neighborhoods $\{U_{P,F}\}$ of P are clopen subsets; we refer to these neighborhoods as *elementary*. It is easy to see that the space K_1 has the same topology as a convergent sequence and hence it is trivially a T_2 compact space.

Now we want to find a suitable relation of equivalence in $\beta\omega$ in such a way that the corresponding quotient space is a T_2 compact space of sequential order 2. Let $\mathcal{M}$ be an infinite MAD family on ω; to every element $M \in \mathcal{M}$ we can associate the unique element $M^* \subseteq \omega^*$ in the following way:

$$M \mapsto M^* = \{\mathcal{U} \in \omega^* : M \in \mathcal{U}\}.$$

It turns out that if $M_1, M_2 \in \mathcal{M}$ with $M_1 \neq M_2$ then $M_1^* \cap M_2^* = \emptyset$. Remark that the subset $\omega^* \setminus \bigcup_{M \in \mathcal{M}} M^*$ is not empty: if it was empty, then $\{M^* \cup \omega : M \in \mathcal{M}\}$ would be an infinite and open cover of $\beta\omega$ from which it would be impossible to extract a finite subcover.

Let us take into account the space $K_2 = \beta\omega / \approx_2$ where

$$x \approx_2 y \Longleftrightarrow \Big[x = y \vee (x \in M^* \wedge y \in M^* \text{ for some } M \in \mathcal{M}) \vee \vee \Big(\Big(x \in \omega^* \setminus \bigcup_{M \in \mathcal{M}} M^*\Big) \wedge \Big(y \in \omega^* \setminus \bigcup_{M \in \mathcal{M}} M^*\Big)\Big)\Big].$$

Then, if we denote by j_2 the natural quotient mapping from $\beta\omega$ to K_2, it holds that j_2 leaves the points of ω unaltered, while it collapses every M^* with $M \in \mathcal{M}$ to a single point and the non-empty subset $\omega^* \setminus \bigcup_{M \in \mathcal{M}} M^*$ to another single point too.

Let us set

$$L_0 = j_2(\omega), L_1 = \{j_2(M^*) : M \in \mathcal{M}\} \text{ and } x_\infty = j_2\Big(\omega^* \setminus \bigcup_{M \in \mathcal{M}} M^*\Big).$$

We say that the points in the set L_0 have level 0 while the points of L_1 have level 1 and the point x_∞ has level 2; it is easy to prove that the levels of the points coincide with their sequential order with respect to the set L_0.

Points in L_0 are isolated in K_2, while the following Lemma describes neighborhoods of points of level 1.

Lemma 2.2. *Let $y \in L_1$ with $y = j_2(M^*)$ and $M \in \mathcal{M}$; then the collection $\{U_{y,F}\} = \{\{y\} \cup j_2(M \setminus F) : F \in [M]^{<\omega}\}$ is a fundamental system of neighborhoods for y.*

We want to remark that the fundamental neighborhoods $U_{y,F}$ of a general point y of level 1 are clopen in K_2: indeed $M^* \cup (M \setminus F)$ is a saturated closed subset of $\beta\omega$ for each $F \in [M]^{<\omega}$; let us call *elementary* these neighborhoods.

The following remark is needed.

Remark 2.3. For every $D \in [\omega]^\omega$ the subfamily $\mathcal{M}_D = \{M \in \mathcal{M} : : |M \cap D| = \omega\}$ is such that $\bigcup \mathcal{M}_D \supseteq^* D$; suppose by contradiction that $\bigcup \mathcal{M}_D \not\supseteq^* D$, i.e. suppose that there exists a subset $E \subset D$ such that

$|E| = \omega$ and $(\bigcup \mathcal{M}_D) \cap E =^* \emptyset$. Since the family $\mathcal{M}$ is maximal, there exists a subset $A \in \mathcal{M} \setminus \mathcal{M}_D$ such that $|A \cap E| = \omega$ but $A \cap E \subset D$ and hence $|A \cap D| = \omega$; therefore A is an element of $\mathcal{M}_D$. A contradiction.

Let us fix an arbitrary $D \in [\omega]^\omega$; if for every finite union $\bigcup_{M \in \mathcal{F}} M$ with $\mathcal{F} \in [\mathcal{M}]^{<\omega}$ it turns out that $\bigcup_{M \in \mathcal{F}} M \not\supseteq^* D$, then $|\mathcal{M}_D| \geq \omega$. Indeed if $|\mathcal{M}_D| < \omega$, by taking $\mathcal{F} = \mathcal{M}_D$, it holds that $\bigcup \mathcal{F} \supseteq^* D$ because of the above note.

Lemma 2.4. *The collection of the clopen subsets $K_2 \setminus \bigcup_{x \in G} U_x$ (where G is a finite set and for every $x \in G$ the clopen subset U_x is an elementary neighborhood of the point x in K_2 that can have level 0 or 1) is a base at the point x_∞.*

Let us call *elementary* these clopen neighborhoods of the point x_∞. Now it is finally easy to prove the following lemma.

Lemma 2.5. *K_2 is a compact sequential T_2 space of sequential order 2.*

We want to remark that the space K_2 we have just constructed is trivially homeomorphic to the one-point compactification of the Mrówka–Isbell space $\Psi(\mathcal{M})$.

By referring to the type of construction of the space K_2, we could think that a good idea to construct a space with a larger order of sequentiality could be to associate a new infinite MAD family $\mathcal{H}_M$ to every $M \in \mathcal{M}$; we will prove that in this way we do not construct a space of higher sequential order.

Then let $\mathcal{M}$ be an infinite MAD family on ω and let us suppose to associate a new infinite MAD family $\mathcal{H}_M$ to every $M \in \mathcal{M}$; let us consider the partition $\mathcal{P} = \big\{\{n\} : n \in \omega\big\} \cup \big\{H^* : H \in \bigcup_{M \in \mathcal{M}} \mathcal{H}_M\big\} \cup$ $\cup \big\{M^* \setminus \bigcup_{H \in \mathcal{H}_M} H^* : M \in \mathcal{M}\big\} \cup \big\{\omega^* \setminus \bigcup_{M \in \mathcal{M}} M^*\big\}$. Let us set $K_{\mathrm{III}} = \beta\omega / \approx$ where $\approx$ is the relation of equivalence associated to $\mathcal{P}$ and let j_{III} be the natural quotient mapping from $\beta\omega$ to K_{III}. The space K_{III} consists of the following elements.

1st) The isolated points of the form $j_{\mathrm{III}}(n)$ for every $n \in \omega$.

2nd) The points of the form $x_H = j_{\mathrm{III}}(H^*)$ for every $H \in \mathcal{H}_M$ and every $M \in \mathcal{M}$; a fundamental system of neighborhoods for x_H is given by

$$\big\{\{x_H\} \cup j_{\mathrm{III}}(H \setminus F)\big\}_{F \in [H]^{<\omega}}.$$

We refer to these neighborhoods with the symbols $U_{x_H, F}$.

3rd) The points of the form $y_M = j_{\mathrm{III}}(M^* \setminus \bigcup_{H \in \mathcal{H}_M} H^*)$ for every $M \in \mathcal{M}$; a fundamental system of neighborhoods for y_M is given by

$$\left\{(\{y_M\}\cup j_{\mathrm{III}}(M))\setminus\bigcup_{x_H\in G}U_{x_H,F}\right\}_G$$

where G is a finite set; we refer to these neighborhoods with the symbols W_{y_M}.

4th) The point $p_\infty = j_{\mathrm{III}}\big(\omega^*\setminus\bigcup_{M\in\mathcal{M}}M^*\big)$ which has a fundamental system of neighborhoods given by $\big\{j_{\mathrm{III}}(\beta\omega)\setminus\bigcup_{y_M\in K}W_{y_M}\big\}_K$ where K is a finite set.

We could think that the points of the sets

$$\left\{j_{\mathrm{III}}(H^*) : H\in\bigcup_{M\in\mathcal{M}}\mathcal{H}_M\right\},\quad \left\{j_{\mathrm{III}}\Big(M^*\setminus\bigcup_{H\in\mathcal{H}_M}H^*\Big) : M\in\mathcal{M}\right\},$$

$$\left\{j_{\mathrm{III}}\Big(\omega^*\setminus\bigcup_{M\in\mathcal{M}}M^*\Big)\right\}$$

have sequential orders respectively 1, 2 and 3 with respect to the set $j_{\mathrm{III}}(\omega)$. We want to show that the point $j_{\mathrm{III}}(\omega^*\setminus\bigcup_{M\in\mathcal{M}}M^*)$ does not have sequential order 3 with respect to the set $j_{\mathrm{III}}(\omega)$. In an obvious way we point out that $x_\infty\in\overline{j_{\mathrm{III}}(\omega)}$: indeed there are always infinitely many points of ω out of the union of any finite number of neighborhoods of points of the third type; otherwise there would not be space enough for the other elements of the MAD family $\mathcal{M}$. Now let us fix a countably infinite set $\{M_n : n\in\omega\}\subseteq\mathcal{M}$ and, for every $n\in\omega$, let us fix an infinite subset $H_n\in\mathcal{H}_{M_n}$; moreover, for every $n\in\omega$, let us call z_n the unique point in K_{III} such that $j_{\mathrm{III}}^{-1}(z_n) = H_n^*$. We assert that for every $n\in\omega$ it is possible to extract a subsequence $\{m_{n_i}\}_{i\in\omega}$ from $j_{\mathrm{III}}(\omega)$ with $\{m_{n_i}\}_{i\in\omega}\to z_n$: indeed for every $n\in\omega$ it is enough to put into the subsequence the image under j_{III} of a countably infinite number of points belonging to H_n. Therefore for every $n\in\omega$ it turns out that $z_n\in\mathrm{seqcl}_1(j_{\mathrm{III}}(\omega))$. We claim that $(z_n)_{n\in\omega}\to p_\infty$: consider an open subset $\Omega\subseteq\beta\omega$ such that $\omega^*\setminus\bigcup_{M\in\mathcal{M}}M^*\subseteq\Omega$; we want to prove that the set $N=\{n\in\omega : H_n^*\nsubseteq\Omega\}$ is finite. Towards a contradiction, suppose that N is infinite; then, in particular, it turns out that the set $\mathcal{M}'=\{M\in\mathcal{M} : \exists H\in\mathcal{H}_M, H^*\nsubseteq\Omega\}$ is infinite and hence that the set $\mathcal{M}''=\{M\in\mathcal{M} : M^*\nsubseteq\Omega\}$ is infinite. Now consider the infinite open cover $\mathcal{A}=\{\Omega\}\cup\{M^*\cup\omega : M\in\mathcal{M}''\}$ of $\beta\omega$; from this open cover it is not possible to extract a finite subcover since the set $\mathcal{M}''$ is infinite. A contradiction. We can conclude that $p_\infty\in\mathrm{seqcl}_2(j_{\mathrm{III}}(\omega))$ and hence that it does not have sequential order 3 with respect to the set $j_{\mathrm{III}}(\omega)$; it follows that K_{III} does not have sequential order 3.

3. Baškirov's idea

In this section we want to explain the general scheme of the construction suggested by Baškirov in [3]; it works in a different way for successor and for limit ordinals. If we can construct compact T_2 spaces $K_{\alpha+1}$ of sequential order $\alpha+1<\omega_1$ for every successor ordinal less than ω_1, then we easily get a compact sequential space of order β for any limit $\beta \leq \omega_1$. It is enough to consider the disjoint sum

$$Z_\beta = \bigoplus_{\alpha+1<\beta} K_{\alpha+1}\ .$$

Trivially the sequential order of this space is β. It is easy to prove that also its one-point compactification $K_\beta = Z^*_\beta$ has sequential order β: indeed the added point ∞ has order 1 with respect to any subset $A \subseteq K_\beta$ such that $\infty \in \overline{A}$.

Therefore the problem reduces to construct compact spaces whose sequential order is a successor ordinal number. We will construct a compact space $K_{\alpha+1}$ of sequential order $\alpha+1$ for every successor ordinal number $\alpha+1 < \omega_1$; each $K_{\alpha+1}$ will be a quotient space of $\beta\omega$, i.e. $K_{\alpha+1} = \beta\omega/\approx_{\alpha+1}$ where the relation $\approx_{\alpha+1}$ is such that only natural numbers are one-point elements of the quotient. For every $\alpha+1<\omega_1$ we will denote by $j_{\alpha+1}$ the natural quotient mapping $j_{\alpha+1} : \beta\omega \to K_{\alpha+1}$. We will prove by transfinite induction that for each $\alpha+1<\omega_1$ the space $K_{\alpha+1}$ satisfies the following conditions.

S.1 The space $K_{\alpha+1}$ can be uniquely represented in the form of

$$K_{\alpha+1} = L_0 \bigsqcup \Big(\bigsqcup_{\gamma\leq\alpha} L_{\gamma+1} \Big).$$

The points of level $\gamma+1$ with $\gamma \in [0,\alpha]$, i.e. the points belonging to the set $L_{\gamma+1}$, have sequential order equal to $\gamma+1$ with respect to L_0, the subset consisting of the images of the points of ω under $j_{\alpha+1}$.

S.2 The set $L_{\alpha+1}$ consists of only one point.

S.3 Every point in $K_{\alpha+1}$ of nonzero level has a basis formed by clopen subsets called elementary; moreover if U is an elementary neighborhood of a point of level $\gamma+1$, then the relation $\approx_{\alpha+1}$ restricted to $\widetilde{U} = j^{-1}_{\alpha+1}(U)$ produces a compact space homeomorphic to $K_{\gamma+1}$.

S.4 For every $\gamma \leq \alpha$, if a nonconstant sequence $(x_n)_{n\in\omega}$ of points $x_n \in L_{\gamma_n+1}$, with nondecreasing levels, converges to a point $x \in L_{\gamma+1}$, then the sequence $(\gamma_n+1)_{n\in\omega}$ is such that $\sup\{\gamma_n+1\} = \gamma$.

S.5 For every $\gamma \leq \alpha$, from every injective sequence $(x_n)_{n\in\omega}$ of points $x_n \in L_{\gamma_n+1}$ with nondecreasing levels such that $\sup_{n\in\omega}\{\gamma_n+1\} = \gamma$, it is possible to extract a subsequence converging to a point of level $\gamma+1$.

S.6 If $\{N_i\}_{i\in\omega}$ is a countable family of pairwise disjoint infinite subsets N_i of ω and for every $i \in \omega$ a relation of type β_i+1 is given on $\overline{N_i}$ in such a way that the sequence of ordinals $(\beta_i+1)_{i\in\omega}$ is not decreasing and $\sup\{\beta_i+1\} = \alpha$, then it is possible to extend the relation obtained on $\bigcup_{i=1}^{\infty} \overline{N_i}$ to a relation of $\beta\omega$ of type $\alpha+1$.

From the first three conditions we trivially deduce other two properties.

S.7 If U is an elementary neighborhood of a point x of level $\gamma+1$ in $K_{\alpha+1}$, then its level in $U = \widetilde{U}/(\approx_{\alpha+1} \ |_{\widetilde{U}})$ is equal to $\gamma+1$.

S.8 If U is an elementary neighborhood of a point x of level $\gamma+1$ in $K_{\alpha+1}$, then $U \setminus \{x\} \subseteq \bigcup_{\gamma'<\gamma} L_{\gamma'+1}$.

The compact sequential spaces K_1 and K_2 will be taken as bases of the recursion. Properties S.1 to S.6 are easy fot K_1. Let us check for K_2.

S.1 The space K_2 can be uniquely represented in the form of

$$K_2 = L_0 \bigsqcup L_1 \bigsqcup L_2$$

where we denote by L_0 the one-point elements of the quotient that are images of the points of ω under j_2; the quotient mapping j_2 collapses every M^* with $M \in \mathcal{M}$ to a single point, while it collapses $\omega^* \setminus \bigcup_{M\in\mathcal{M}} M^*$ to another single point that gives L_2.

The points of L_1 have sequential order 1 with respect to L_0, while the unique point of L_2 has sequential order 2 with respect to L_0.

S.2 The set L_2 consists of the unique point $x_\infty = j_2\big(\omega^* \setminus \bigcup_{M\in\mathcal{M}} M^*\big)$.

S.3 Every point $y \in L_1$ has a basis formed by the clopen elementary neighborhoods $U_{y,F}$ and the space obtained by restricting the relation $\approx_2$ to $\widetilde{U}_{y,F} = j_2^{-1}(U_{y,F})$ is homeomorphic to the compact sequential space K_1. The point $x_\infty \in L_2$ has a basis given by the clopen elementary neighborhoods $K_2 \setminus \bigcup_{x\in G} U_x$; the space obtained by restricting the relation $\approx_2$ to $j_2^{-1}(K_2 \setminus \bigcup_{x\in G} U_x)$ is a compact space homeomorphic to K_2.

S.4 For K_2 Property S.4 is obvious.

S.5 From every noncostant sequence $(x_n)_{n\in\omega}$ of points with nondecreasing levels such that $\sup\{l(x_n)\} = 0$ it is possible to extract a subsequence converging to a point of level 1: indeed, since the family $\mathcal{M}$ is MAD, the sequence $(x_n)_{n\in\omega}$ has infinite intersection with at least one element M_1 of the family $\mathcal{M}$ and hence we can extract a subsequence converging to the point $j_2(M_1^*)$ of level 1. From every noncostant sequence $(x_n)_{n\in\omega}$ of points with nondecreasing levels such that $\sup\{l(x_n)\} = 1$ it is possible to extract a subsequence converging to x_∞: indeed the points of the sequence are eventually in every neighborhood of x_∞.

S.6 If $\{N_i\}_{i\in\omega}$ is a countable family of pairwise disjoint infinite subsets $N_i \subset \omega$ and if it holds that for every $i \in \omega$ a relation of type $\beta_i + 1$ is given on $\overline{N_i}$ in such a way that the sequence of ordinals $\beta_i + 1$ is nondecreasing and that $\sup\{\beta_i + 1\} = 1$ (and hence in such a way that $\beta_i + 1 = 1$ for every $i \in \omega$), then we can extend the so obtained relation on $\bigcup_{i=1}^{\infty} \overline{N_i}$ to a relation on $\beta\omega$ of type 2: indeed it is enough to complete the almost disjoint family $\{N_i\}_{i\in\omega}$ to a MAD family and then put into relation the elements of $\beta\omega$ in the way we have already seen when we constructed K_2.

Now we are sure we can take the compact sequential spaces K_1 and K_2 as bases of the induction. Moreover we will assume that for all $\beta + 1 < \alpha + 1$ the compact sequential spaces $K_{\beta+1}$ (in which properties S.1 to S.6 hold) have been constructed; then we will able to construct the compact space $K_{\alpha+1}$ with sequential order $\alpha + 1$ satisfying conditions S.1 to S.6.

4. Some propaedeutic lemmas

Before giving the construction in full details, we recall some technical lemmas, which are possibly known. Anyway we prove them for the sake of completeness.

Lemma 4.1. *The intersection of any countable family of open subsets of ω^* is either empty or contains a non-empty open subset.*

Proof. Let $\{A_i\}_{i\in\omega}$ be a countable family of open subsets of ω^* whose intersection contains a point $\mathcal{U}$. For every $i \in \omega$ there exists a subset $N_i \subset \omega$ such that $\mathcal{U} \in N_i^* \subset A_i$. The intersection of any finite collection of the sets N_i^* is not empty and open and hence the intersection of any finite collection of the sets N_i is infinite. Then there exists an increasing sequence of integers n_i such that $n_i \in N_1 \cap N_2 \cap \ldots \cap N_i$. Let us set $N = \{n_i : i \in \omega\}$; since $N \backslash N_i$ is finite for each $i \in \omega$, it holds that $N^* \subset N_i^*$ for each $i \in \omega$ and then we can conclude that $N^* \subset \bigcap_i A_i$ where $N^* \neq \emptyset$ as $|N| = \omega$. $\Diamond$

Lemma 4.2. *Let $\{N_i^*\}_{i\in\omega}$ be a countably infinite family of clopen subsets of ω^*; let us suppose that $\omega^*\backslash\overline{\bigcup_{i\leq\bar\imath} N_i^*} \neq \emptyset$ for every $\bar\imath \in \omega$. Then there exists $\Delta \subset \omega$ such that $|\Delta| = \omega$ and $\Delta^* \cap \bigcup_{i\in\omega} N_i^* = \emptyset$.*

Proof. Let us set $\Delta_1 = N_1^*, \Delta_2 = N_1^* \cup N_2^*, \dots, \Delta_{\bar\imath} = N_1^* \cup N_2^* \cup \dots \cup N_{\bar\imath}^*$ and so on; it is clear that for every $\bar\imath \in \omega$, $\Delta_{\bar\imath}$ is a clopen subset of ω^*. Notice that $\omega^*\backslash\Delta_1 \supseteq \omega^*\backslash\Delta_2 \supseteq \dots \supseteq \omega^*\backslash\Delta_{\bar\imath} \supseteq \omega^*\backslash\Delta_{\bar\imath+1} \supseteq \dots$; for every $i \in \omega$, let $C_i = \omega^*\backslash\Delta_i$ and $\mathcal{E} = \{C_i : i \in \omega\}$. The family $\mathcal{E}$ satisfies the finite intersection property and, since ω^* is compact, $\bigcap \mathcal{E} \neq \emptyset$. Then we have

$$\emptyset \neq \bigcap_{i\in\omega} C_i = \bigcap_{i\in\omega} \Delta_i^C = \Big[\bigcup_{i\in\omega} (\cup_{j\leq i} N_j^*)\Big]^C = \Big(\bigcup_{i\in\omega} N_i^*\Big)^C = \omega^*\backslash \bigcup_{i\in\omega} N_i^*.$$

We can conclude that the family $\{\Delta_i^C\}_{i\in\omega}$ consisting of open subsets of ω^* has non-empty intersection and hence, by Lemma 4.1, this intersection contains an open subset $A \subseteq \omega^*\backslash \bigcup_{i\in\omega} N_i^*$. Then there exists $\Delta \subset \omega$ with $|\Delta| = \omega$ such that $\Delta^* \subseteq \omega^*\backslash \bigcup_{i\in\omega} N_i^*$ whence $\Delta^* \cap \bigcup_{i\in\omega} N_i^* = \emptyset$. $\Diamond$

Lemma 4.3. *Let $\mathcal{P} = \mathcal{Q} \cup \mathcal{R}$ be a family of infinite subsets of ω such that*

- *$\mathcal{Q}$ is an almost disjoint family;*
- *$|\mathcal{Q}| \leq \omega$ and $|\mathcal{R}| \leq \omega$;*
- *for every element $Q_i \in \mathcal{Q}$ and every element $R_n \in \mathcal{R}$ it turns out that $|Q_i \cap R_n| < \omega$.*

Then there exists $L \in [\omega]^\omega$ such that $L^* \supseteq \bigcup_{Q_i\in\mathcal{Q}} Q_i^*$ and $L^* \cap R_n^* = \emptyset$ for every $R_n \in \mathcal{R}$.

Proof. Let us set $\mathcal{Q} = \{Q_i : i \in \omega\}$ with $|Q_i \cap Q_j| < \omega$ for $i \neq j$. Obviously we can suppose $\mathcal{R} \neq \emptyset$ and then we can write $\mathcal{R} = \{R_n : : n \in \omega\}$ with $n \mapsto R_n$ not necessarily injective. If $|\mathcal{Q}| < \omega$, then we set $L = \bigcup Q_i$. If instead $|\mathcal{Q}| = \omega$, we set $L = \bigcup_{n\in\omega}(Q_n\backslash \cup_{n'<n} R_{n'})$. For every $\overline{n} \in \omega, R_{\overline{n}}$ intersects L only in those points in which $R_{\overline{n}}$, in case, intersects the subsets Q_n with $n = 0, \dots, \overline{n}$; these points are in a finite number. Furthermore for every $\overline{n} \in \omega$, $Q_{\overline{n}}\backslash L$ consists of a finite number of points and exactly of those in which $Q_{\overline{n}}$ intersects R_n with $n < \overline{n}$ (and these again are certainly in a finite number); therefore $Q_{\overline{n}}^* \subseteq L^*$ for every $\overline{n} \in \omega$ and hence $\bigcup_n Q_n^* \subseteq L^*$. $\Diamond$

In the following lemma we will take into account a countable family of infinite pairwise disjoint subsets of ω, $\{\tilde N_i\}_{i\in\omega}$ and a relation $\approx$ on $U = \bigsqcup \tilde N_i^* \subset \omega^*$. Let us set $H = U/\approx$ and let j be the quotient mapping $j : U \to H$. We assume that the subsets $\tilde N_i^*$ are *distinguished* *relative to* $\approx$, i.e. that $j^{-1}(j(\tilde{N}_i^*)) = \tilde{N}_i^*$ for every $i \in \omega$. Now let us prove the lemma.

Lemma 4.4. *Let* $\{\tilde{N}_i\}_{i\in\omega}$ *be a countable family of infinite pairwise disjoint subsets* $\tilde{N}_i \subset \omega$ *and let* $\approx$ *be a relation on* $U = \bigsqcup \tilde{N}_i^* \subset \omega^*$ *where the subsets* $\tilde{N}_i^*$ *are distinguished relative to* $\approx$ *and the spaces* $\tilde{N}_i^*/\approx$ *are zero-dimensional compact spaces. Let us suppose that the set* $B = \{x_n : n \in \omega\}$ *has no accumulation point in* H *and that for every* x_n *there exists an index* i_n *and a clopen neighborhood* $U(x_n)$ *such that* $U(x_n) \subseteq \tilde{N}_{i_n}^*/\approx$. *Moreover let us suppose that* $\bigcup_{n\in\omega} U(x_n) \neq H$. *Then*

i) *there exist pairwise disjoint clopen subsets* U_n *with* $n \in \omega$ *such that* $x_n \in U_n$ *for every* $n \in \omega$*;*

ii) *there exists* $\tilde{N}' \subset \omega$ *such that* $(\tilde{N}')^* \cap \bigcup_{i\in\omega} \tilde{N}_i^* = j^{-1}\big(\bigsqcup_{n\in\omega} U_n\big) = \bigsqcup_{n\in\omega} E_n^*$.

Proof. Since the subsets $\{\tilde{N}_i^*\}_i$ are disjoint and distinguished relative to $\approx$ it holds that $(\tilde{N}_i^*/\approx) \cap (\tilde{N}_j^*/\approx) = \emptyset$ for every $i, j \in \omega$ with $i \neq j$. Moreover $\tilde{N}_i^*/\approx$ is open and closed in H for every $i \in \omega$, since $j^{-1}(\tilde{N}_i^*/\approx) = \tilde{N}_i^*$ is open and closed in U. We need to remark that, for every $i \in \omega$, $\tilde{N}_i^*/\approx$ intersects only a finite number of the neighborhoods $\{U(x_n)\}_n$ because of the hypothesis that the subset B has no accumulation point in H.

By transfinite induction we are going to construct the subsets U_n with $n \in \omega$ such that $x_n \in U_n$ for every $n \in \omega$.

Let us consider the point x_1 and the clopen subset $U(x_1) \subseteq \tilde{N}_{i_1}^*/\approx$; since B has no accumulation point in H, there exists an open neighborhood $A_1 \subseteq H$ of x_1 such that $A_1 \cap B = \{x_1\}$. Now $x_1 \in \big[(A_1 \cap \tilde{N}_{i_1}^*/\approx) \cap U(x_1)\big] = D_1$: this subset is open in $\tilde{N}_{i_1}^*/\approx$ and hence, since $\tilde{N}_{i_1}^*/\approx$ is zero-dimensional, there exists a clopen subset $U_1 \subseteq D_1$ of $\tilde{N}_{i_1}^*/\approx$ with $x_1 \in U_1$; trivially U_1 is clopen also in H. Now $H\backslash U_1$ is an open subset of H and it contains $B\backslash\{x_1\}$ which has no accumulation point in $H\backslash U_1$; thus there exists an open neighborhood $A_2 \subseteq H$ of x_2 such that $A_2 \cap B = \{x_2\}$. Hence $x_2 \in [(H\backslash U_1) \cap A_2 \cap (\tilde{N}_{i_2}^*/\approx) \cap U(x_2)] = D_2$: this is an open subset of $\tilde{N}_{i_2}^*/\approx$ and then, since $\tilde{N}_{i_2}^*/\approx$ is zero-dimensional, there exists a clopen subset $U_2 \subseteq D_2$ of $\tilde{N}_{i_2}^*/\approx$ with $x_2 \in U_2$; trivially U_2 is clopen also in H and moreover it results that $U_1 \cap U_2 = \emptyset$. Notice that $H\backslash(U_1 \sqcup U_2)$ is a non-empty clopen subset of H and that $B\backslash\{x_1, x_2\} \subseteq H\backslash(U_1 \sqcup U_2)$.

Let us suppose that for every $n \leq \overline{n}$ there exists a clopen subset $U_n \subseteq H$ with $x_n \in U_n$ and $U_n \subseteq U(x_n)$ and that $U_n \cap U_{n'} = \emptyset$ for every $n', n \leq \overline{n}$; moreover suppose that for every $n \leq \overline{n}$ it holds that $B_n = B \backslash \{x_1, \ldots, x_n\} \subseteq H \backslash \bigsqcup_{j \leq n} U_j$. Let us prove that these properties hold also for $\overline{n}+1$. By inductive hypothesis $x_{\overline{n}+1} \in (H \backslash \bigsqcup_{j \leq \overline{n}} U_j)$ where $H \backslash \bigsqcup_{j \leq \overline{n}} U_j$ is open, since the finite union $\bigsqcup_{j \leq \overline{n}} U_j$ is clopen; moreover there exists an open neighborhood $A_{\overline{n}+1} \subseteq H$ of $x_{\overline{n}+1}$ such that $A_{\overline{n}+1} \cap \cap B = \{x_{\overline{n}+1}\}$. It turns out that $x_{\overline{n}+1} \in \left[\left(H \backslash \bigsqcup_{j \leq n} U_j\right) \cap A_{\overline{n}+1} \cap \tilde{N}^*_{i_{\overline{n}}+1}/\approx\right] \cap \cap U(x_{\overline{n}+1}) = D_{\overline{n}+1}$ and that $D_{\overline{n}+1}$ is open in $\tilde{N}^*_{i_{\overline{n}}+1}/\approx$. Now, since $\tilde{N}^*_{i_{\overline{n}}+1}/\approx$ is zero-dimensional, there exists a clopen subset $U_{\overline{n}+1} \subseteq D_{\overline{n}+1}$ of $\tilde{N}^*_{i_{\overline{n}}+1}/\approx$ with $x_{\overline{n}+1} \in U_{\overline{n}+1}$: we trivially remark that $U_{\overline{n}+1}$ is also clopen in H, that $U_{\overline{n}+1} \cap U_{n'} = \emptyset$ for every $n' < \overline{n}+1$ and that $H \backslash \bigsqcup_{j \leq \overline{n}+1} U_j \supseteq \supseteq B \backslash \{x_1, \ldots, x_{\overline{n}+1}\}$.

Therefore $U_n \subseteq \tilde{N}^*_{i_n}/\approx$ is a clopen neighborhood of x_n in H for every $n \in \omega$ and $j^{-1}(U_n)$ is a clopen subset of $\tilde{N}^*_{i_n}$; then for every $n \in \omega$ it holds that $j^{-1}(U_n) = E^*_n$ where E_n is an infinite subset of ω. Moreover we can assert that $\bigsqcup_{n \in \omega} U_n \neq H$ since $\bigsqcup_{n \in \omega} U_n \subseteq \bigcup_{n \in \omega} U(x_n)$. Now we want to prove that $\bigsqcup_{n \in \omega} U_n$ is clopen in $U/\approx$; trivially $\bigsqcup_{n \in \omega} U_n$ is open and now we show that it is also closed. If we take a point $z \in H \backslash \bigsqcup_{n \in \omega} U_n$ there exists an index i_z such that $z \in (\tilde{N}^*_{i_z}/\approx) \backslash \bigsqcup_{n \in \omega} U_n$. Since $\tilde{N}^*_{i_z}/\approx$ intersects only a finite number of the clopen subsets we have just constructed (we denote these clopen subsets by $U_{j_1}, \ldots, U_{j_{\overline{n}}}$), then $\bigsqcup_{i=1}^{\overline{n}} U_{j_i} \cap (\tilde{N}^*_{i_z}/\approx)$ is closed in $\tilde{N}^*_{i_z}/\approx$ since $\bigsqcup_{i=1}^{\overline{n}} U_{j_i}$ is closed in H; the subset $\tilde{N}^*_{i_z}/\approx \backslash (\bigsqcup_{i=1}^{\overline{n}} U_{j_i})$ is an open subset to which z belongs and hence there exists an open neighborhood of z in $\tilde{N}^*_{i_z}/\approx$ (and then in H) disjoint from $\bigsqcup U_n$.

Finally we can conclude that $j^{-1}(\bigsqcup U_n) = \bigsqcup j^{-1}(U_n)$ is clopen in U and then there exists a clopen subset $(\tilde{N}')^*$ of ω^* with $\tilde{N}' \subseteq \omega$ and $|\tilde{N}'| = \omega$ such that $(\tilde{N}')^* \cap \bigcup \tilde{N}^*_i = \bigsqcup j^{-1}(U_n) = \bigsqcup E^*_n$. $\Diamond$

Remark 4.5. We remark that Lemma 4.4 still holds when we consider a family $\left\{\tilde{N}_\gamma : \gamma \in \omega_1\right\}$ of infinite subsets $\tilde{N}_\gamma \subset \omega$ keeping all the other hypotheses.

5. Construction of a Baškirov's space of order an arbitrary successor ordinal

Finally we will show how to construct the space $K_{\alpha+1}$ by assuming that all compact sequential spaces $K_{\beta+1}$ (of sequential order $\beta+1$) with $\beta+1<\alpha+1$ have been constructed and that properties S.1 to S.6 hold in each of these space; moreover we will check that properties S.1 to S.6 hold in $K_{\alpha+1}$ too. We will carry out the construction when α is a successor ordinal, but we will remark from time to time what it is necessary to change if we have to work in the case in which α is a limit ordinal).

It will be very important to take the set Γ into account: it is the set of all the families $\mathcal{C}_\xi$ whose elements are countable pairwise disjoint clopen subsets of ω^*; under the Continuum Hypothesis, we can write Γ as

$$\Gamma = \{\mathcal{C}_\xi : \omega \leq \xi < \omega_1\}\,. \tag{1}$$

Roughly speaking, our type of construction ensures that, by a number of steps of cardinality equal to the cardinality of Γ, we are able to exhaust the whole Γ; moreover at each stage $\alpha<\omega_1$ of the inductive construction, it will be essential the fact that α is a countable ordinal in order to guarantee that we can continue the process and hence it is crucial that we can enumerate Γ as in (1).

Let us begin the construction. Let $\{N_i\}_{i\in\omega}$ be a family of pairwise disjoint infinite subsets $N_i\subset\omega$. For every $i\in\omega$ the closures of N_i in $\beta\omega$, namely $\overline{N_i}$, is a clopen subset of $\beta\omega$ which is homeomorphic to it; for every $i\in\omega$ let us set a decomposition of type β_i+1 on $\overline{N_i}$ taking care that the sequence of ordinals $S=(\beta_i+1)_{i\in\omega}$ is nondecreasing and such that $\sup\{\beta_i+1\}=\alpha$. Notice that it is possible to extract a subsequence $S'=(\beta_{i_n}+1)_{n\in\omega}\subseteq S$ in such a way that the sequence S' converges upwards to α; since α is a successor ordinal, this mean that there are infinite $n\in\omega$ such that the decomposition set on $\overline{N_{i_n}}$ is a relation of type α.[2]

For every $i\in\omega$, let $j_{\beta_i+1}:\overline{N_i}\to K_{\beta_i+1}$ be the quotient mapping. The following properties T.1, T.2 and T.3 are obvious for every $i<\bar{\imath}<\omega$,

[2] If α is a limit ordinal we will have to set decompositions of type β_i+1 on $\overline{N_i}$ in such a way that the sequence $(\beta_i+1)_{i\in\omega}$ is nondecreasing and $\sup\{\beta_i+1\}=\alpha$. Also in this case it is possible to extract a subsequence $S'=(\beta_{i_n}+1)_{n\in\omega}\subseteq S$ in such a way that the sequence S' converges upwards to α.

while a property T.4 does not apply for $\xi < \omega$.

T.1 $\overline{N^*_{\bar\imath} \setminus \bigcup_{i' < \bar\imath} N^*_{i'}} \neq \emptyset$.

T.2 $\overline{\bigcup_{i' \leq \bar\imath} N^*_{i'}} \neq \omega^*$.

T.3 For every $i' < \bar\imath$ it holds that $\overline{N_{i'}} \cap \overline{N_{\bar\imath}} = \emptyset$.

In view of T.3 and the relations set on each $\overline{N_i}$ with $i \in \omega$, we have defined a relation Q_ω on $U_\omega = \bigcup_{i\in\omega} N^*_i$. Let $j^\omega_{\alpha+1}$ be the quotient mapping $j^\omega_{\alpha+1} : U_\omega \to U_\omega/Q_\omega$.

We say that $\mathcal{C}_\xi \in \Gamma$ is an ω-family if $\mathcal{C}_\xi$ consists of elements that can be decomposed into two subfamilies $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfying the following conditions.

U.1 $\bigcup \mathcal{L}_0 \cap U_\omega = \emptyset$.

U.2 For every $c \in \mathcal{L}_1$ there exists $i < \omega$, a point $x_c \in \overline{N_i}/\approx_{\beta_i+1}$ of level $\gamma_c + 1$ and an elementary neighborhood U_c of x_c such that $c = \widetilde{U_c} \cap \omega^*$ where $\widetilde{U_c} = j^{-1}_{\beta_i+1}(U_c)$.

U.3 The set $\{x_c : c \in \mathcal{L}_1\}$ has no accumulation points in U_ω/Q_ω.

U.4 It holds that $\sup\{\gamma_c + 1 : c \in \mathcal{L}_1\} < \alpha$.

Let us rewrite these properties in order to make clear the new notion.

i) $\mathcal{L}_0$ consists of elements C^*_n where the subsets $C_n \subset \omega$ are transversal to the subsets N_i, i.e. every $C_n \subset \omega$ intersects every N_i in a finite number of points (in this way we are respecting U.1);

ii) $\mathcal{L}_1$ consists of elements C^*_m where for every m there exist $i \in \omega$, a point $x_m \in \overline{N_i}/\approx_{\beta_i+1}$ of level $l(x_m) < \alpha$ and an elementary neighborhood $U(x_m)$ such that $C^*_m = j^{-1}_{\beta_i+1} U(x_m) \cap \omega^*$. A further necessary requirement is that the set $\{x_m\}$ has no accumulation point in U_ω/Q_ω and that $\sup\{l(x_m)\} < \alpha$. We want to point out that by $l(x_j)$ we mean a successor ordinal. (In this way we are respecting U.2–U.3–U.4.)

Notice that it is possible to find an ω-family: for example, we can use Lemma 4.2 since the subsets N^*_i comply with the hypotheses; in this way we find an infinite subset $\Delta_\omega \subset \omega$ such that Δ_ω intersects every N_i in a finite number of points. We can decompose this infinite set in an infinite number of infinite subsets $T_n \subset \omega$ that again intersect every N_i in a finite number of points; we set $\mathcal{L}_0 = \{T^*_n : n \in \omega\}$. It is clear that $\mathcal{L} = \mathcal{L}_0$ is an ω-family.

Among all the ω-families let us take the one with the minimum index $\overline{\omega}$; we write it as $\mathcal{C}_{\overline{\omega}} = \mathcal{L}_0 \sqcup \mathcal{L}_1$ with $\mathcal{L}_0 = \{N^*_{\overline{\omega},n} : n \in J_0\}$, $\mathcal{L}_1 = \{N^*_{\overline{\omega},n} : n \in J_1\}$ and $J_0 \cap J_1 = \emptyset$. Of course, by construction, the ω-family $C_{\overline{\omega}}$ complies with the following properties.

U.1 $\bigcup \mathcal{L}_0 \cap U_\omega = \emptyset$.

U.2 For every $N^*_{\overline{\omega},n}$ with $n \in J_1$ there exist $i_n \in \omega$, a point $x_n \in \in \overline{N_{i_n}}/\approx_{\beta_{i_n}+1}$ of level $l(x_n) < \alpha$ and an elementary neighborhood $U(x_n)$ such that $N^*_{\overline{\omega},n} = j^{-1}_{\beta_{i_n}+1}(U(x_n)) \cap \omega^*$.

U.3 The set $\{x_n : n \in J_1\}$ has no accumulation point in U_ω/Q_ω.

U.4 It holds that $\sup\{l(x_n) : n \in J_1\} = \beta_\omega < \alpha$ with β_ω that can take up value from 1 to α not included. Without loss of generality we can always assume that the levels of the points are ordered in a nondecreasing way.

For every $n \in J_1$ it turns out that $\hat{U}(x_n) = U(x_n)\backslash\omega$ is a clopen neighborhood of x_n in $N^*_{i_n}/\approx_{\beta_{i_n}+1}$. We can apply Lemma 4.4 since the family $\{N_i : i \in \omega\}$, the points x_n with $n \in J_1$ and the relation Q_ω defined on $U_\omega = \bigsqcup N^*_i$ satisfy the hypotheses. We remark that $\bigcup \hat{U}(x_n) \neq U_\omega/Q_\omega$ since in U_ω/Q_ω there are points of level α which $\bigcup \hat{U}(x_n)$ does not cover.[3] Therefore it is possible to find pairwise elementary neighborhoods U_n with $x_n \in U_n$ and a subset $N'_\omega \subset \omega$ such that $(N'_\omega)^* \cap U_\omega = \bigsqcup_{n\in J_1}(j^\omega_{\alpha+1})^{-1}(U_n) = \bigsqcup_{n\in J_1} E^*_n$. Let us define $\mathcal{C}' = = \mathcal{L}_0 \cup \{(j^\omega_{\alpha+1})^{-1}(U_n) : n \in J_1\}$.

Now if we set $\mathcal{Q} = \{N_{\overline{\omega},n} : n \in J_0\}$ and $\mathcal{R} = \{N_i : i \in \omega\}$, then $\mathcal{P} = \mathcal{Q} \cup \mathcal{R}$ is a family of subsets with the following properties:

– $\mathcal{Q}$ is an almost disjoint family;

– $|\mathcal{Q}| \leq \omega$ and $|\mathcal{R}| \leq \omega$;

– for every $N_{\overline{\omega},n} \in \mathcal{Q}$ and every $N_i \in \mathcal{R}$ it holds that $|N_{\overline{\omega},n} \cap N_i| < \omega$.

Therefore, by Lemma 4.3, there exists a subset $N''_\omega \in [\omega]^\omega$ such that $\bigcup_{n\in J_0} N^*_{\overline{\omega},n} \subseteq (N''_\omega)^*$ and $(N''_\omega)^* \cap N^*_i = \emptyset$, $\forall N_i \in \mathcal{R}$. Trivially it follows that $\bigcup_{n\in J_0} N^*_{\overline{\omega},n} \subseteq \overline{N''_\omega}$ and $\overline{N''_\omega} \cap U_\omega = \emptyset$. Let us recapitulate:

1) $N''_\omega \supseteq^* N_{\overline{\omega},n}$ for every $n \in J_0$;

2) $|N''_\omega \cap N_i| < \omega$ for every $i \in \omega$.

Notice that $(N'''_\omega)^* = (N'_\omega)^* \cup (N''_\omega)^*$ is a clopen subset of ω^*. Now it turns out that

$$(N'''_\omega)^* \cap U_\omega = \left[(N'_\omega)^* \cup (N''_\omega)^*\right] \cap U_\omega = (j^\omega_{\alpha+1})^{-1}\left(\bigsqcup_{n\in J_1} U_n\right) \cup \emptyset = \bigsqcup_{n\in J_1} E^*_n$$

[3]Notice that $\bigcup \hat{U}(x_n) \neq U_\omega/Q_\omega$ also in the case in which α is a limit ordinal: indeed at the beginning of the construction we put decompositions of type $\beta_i + 1$ on the subsets $\overline{N}_i$ in such a way that $\sup\{\beta_i + 1\} = \alpha$; hence in U_ω/Q_ω there certainly exists a point of level $\beta_\omega + 1 < \alpha$ that $\bigcup \hat{U}(x_n)$ does not cover.

and then we can conclude that $N'''_\omega \supseteq^* E_n$ for every $n \in J_1$ and $N'''_\omega \supseteq^* N_{\overline{\omega},n}$ for every $n \in J_0$. Let us set

$$M_n = \begin{cases} N'''_\omega \cap E_n & \text{if } \ n \in J_1, \\ N'''_\omega \cap N_{\overline{\omega},n} & \text{if } \ n \in J_0. \end{cases}$$

Certainly $M_n^* = E_n^*$ for every $n \in J_1$ and $M_n^* = N_{\overline{\omega},n}^*$ for every $n \in J_0$.

For every $n \in \omega$ let us fix a point $l_n \in M_n \backslash \big(\bigcup_{j=0}^{n-1} M_j \cup \{l_0, ..., l_{n-1}\}\big)$ – it is possible since the family $\{M_n\}$ is almost disjoint – and let us set $L = \{l_i : i \in \omega\}$.

Let us define

$$N_\omega = \bigsqcup_{n\in\omega} \Big(M_n \backslash \bigcup_{j=0}^{n-1} M_j\Big) \backslash \{l_i : i \in \omega\} = \bigsqcup_{n\in\omega} H_n$$

where $H_n = (M_n \backslash \bigcup_{j=0}^{n-1} M_j) \backslash \{l_i : i \in \omega\}$. Notice that $N_\omega^* \supseteq \bigcup \mathcal{C}'$ (indeed from every M_n we removed only a finite number of points) and that $(N'''_\omega)^* \backslash N_\omega^* \neq \emptyset$ (since $N'''_\omega \backslash N_\omega = \{l_i : i \in \omega\}$) whence $|\omega \backslash N_\omega| = \omega$.

Now let us take into account $N_\omega = \bigsqcup_{n\in\omega} H_n$ where $M_n^* = H_n^*$ for every $n \in \omega$; we want to remark that on each $\overline{H_n}$ with $n \in J_1$ we have already a decomposition of type $l(x_n)$ by construction.

Now if $|J_1| = \omega$, on every $\overline{H_n}$ with $n \in J_0$ let us put a decomposition of type 1; then let us order the subsets $\overline{H_n}$ in such a way that the types of decomposition that we have put on them form a nondecreasing sequence with the supremum equal to $\beta_\omega < \alpha$.

If $|J_1| < \omega$, it turns out that $\sup\{l(x_n) : n \in J_1\} = \beta_\omega$ is a successor ordinal; let us put a decomposition of type β_ω on every $\overline{H_n}$ with $n \in J_0$[4] and then let us order the subsets $\overline{H_n}$ in such a way that the types of decomposition that we have put on them form a nondecreasing sequence whose supremum is equal to $\beta_\omega < \alpha$.

If $J_1 = \emptyset$, we choose to put a decomposition of type a successor ordinal $\beta_\omega < \alpha$ on every $\overline{H_n}$ with $n \in J_0$ and then we proceed as in the latter case.

This time let us apply property S.6 to the subsets H_n and to N_ω: it turns out that $\{H_n\}_{n\in\omega}$ is a countably infinite family of infinite pairwise disjoint subsets of N_ω and on every $\overline{H_n}$ is given a relation of some type in such a way that the supremum of the nondecreasing sequence consisting of the types of decomposition is β_ω with β_ω that can take up value from 1 to α not included. Then the relation on $\bigcup_{n=1}^{\infty} \overline{H_n}$ obtained in this way

[4] In this case $|J_0| = \omega$.

can be extended to a relation $\approx_{\beta_\omega+1}$ on $\overline{N_\omega}$ of type $\beta_\omega+1$ where $\beta_\omega+1$ can have value a successor ordinal from 2 up to α.[5]

Remark 5.1. We want to remark that for the points constructed by the decompositions on the $\overline{H}_n$ with $n \in J_0$ it is always possible to find a fundamental system of elementary neighborhoods contained in $\overline{N}_\omega/\approx_{\beta_\omega+1}$ and such that their inverse images through $j^\omega_{\alpha+1}$ have empty intersection with U_ω since $H^*_n \cap U_\omega = \emptyset$; from now on, we consider only these neighborhoods as elementary neighborhoods of those points.

Let us check the following properties:

T.1 $\overline{N^*_\omega \setminus \bigcup_{i\in\omega} N^*_i} \neq \emptyset$: indeed it turns out that $(N_\omega)^* \supseteq \bigcup \mathcal{L}_0$ while $\bigcup \mathcal{L}_0 \cap \left(\bigcup_{i\in\omega} N^*_i\right) = \emptyset$; hence, if $\mathcal{L}_0 \neq \emptyset$, it follows that $N^*_\omega \setminus \bigcup N^*_i \supseteq$ $\supseteq \bigcup \mathcal{L}_0 \neq \emptyset$. On the other hand if $\mathcal{L}_0 = \emptyset$, then the family $\{H_n : n \in J_1\}$ of pairwise disjoint subsets of ω is infinite; thus we can construct an infinite subset $T \subset N_\omega$ in this way: we choose a point $t_m \in H_m$ for every $m \in J_1$ and we set $T = \{t_m : m \in J_1\}$. The non-empty subset T^* is such that $T^* \subseteq N^*_\omega$, while $T^* \cap N^*_i = \emptyset$ for every $i \in \omega$. We want to check it: certainly $T^* \cap H^*_n = \emptyset$ for every $n \in J_1$ and hence $T^* \cap \left(\bigsqcup H^*_n\right) = \emptyset$; if there is an index $i \in \omega$ such that $|T \cap N_i| = \omega$, then $(T\cap N_i)^* \subset \left(N^*_\omega \cap \bigcup_{i\in\omega} N^*_i\right)$, while we know that $\left(N^*_\omega \cap \bigcup_{i\in\omega} N^*_i\right) = \bigsqcup H^*_n$.

T.2 $\overline{\bigcup_{i\leq\omega} N^*_i} \neq \omega^*$: notice that the set L is such that $L^* \cap N^*_\omega = \emptyset$ and that $L^* \cap N^*_i = \emptyset$ for every $i < \omega$ (indeed for every $i < \omega$ it holds that $|N_i \cap L| < \omega$; if there exists an index $i \in \omega$ such that $|L \cap N_i| = \omega$, then $(L \cap N_i)^* \subset \left((N'''_\omega)^* \cap \bigcup_{i\in\omega} N^*_i\right)$, while we know that $\left((N'''_\omega)^* \cap \bigcup_{i\in\omega} N^*_i\right) =$ $= \bigsqcup H^*_n$ and $L^* \cap \left(\bigsqcup H^*_n\right) = \emptyset$). Then we obtain that $\bigcup_{i\leq\omega} N^*_i \cap L^* = \emptyset$ where L^* is open in ω^* whence $\omega^* \setminus L^*$ is a closed subset that contains $\bigcup_{i\leq\omega} N^*_i$; so it contains its closure and it follows that $\overline{\bigcup_{i\leq\omega} N^*_i} \cap L^* = \emptyset$. At the end, we can conclude that $\overline{\bigcup_{i\leq\omega} N^*_i} \neq \omega^*$.

T.3 For every $i \in \omega$, the relations $\approx_{\beta_i+1}$ and $\approx_{\beta_\omega+1}$ coincide on $\overline{N_i} \cap \overline{N_\omega}$: indeed $N^*_\omega \cap N^*_i \subseteq \bigsqcup H^*_n$ (with $n \in J_1$), the relation on N^*_ω extends the relations placed on the subsets H^*_n (with $n \in J_1$) and these last relations coincide with the relations we put on the subsets N^*_i. Then a relation $Q_{\omega+1}$ is defined on $U_{\omega+1} = \bigcup_{i=1}^{\omega} N^*_i$.

T.4 A family $\mathcal{C}_\xi \in \Gamma$ with index $\xi \leq \omega$ is not an $(\omega+1)$-family. Notice that the families $\mathcal{C}_\xi$ with $\xi < \omega$ do not exist and hence we have only to prove that the family C_ω is not an $(\omega+1)$-family.

[5]In the case in which α is a limit ordinal, $\beta_\omega+1$ can take up value on the successor ordinals from 2 up to an ordinal strictly less than α.

We say that $\mathcal{C}_\xi \in \Gamma$ is an $(\omega+1)$-family if $\mathcal{C}_\xi$ can be decomposed into two subfamilies $\mathcal{L}_0^{\omega+1}$ and $\mathcal{L}_1^{\omega+1}$ satisfying the following conditions.

U.1 $\bigcup \mathcal{L}_0^{\omega+1} \cap U_{\omega+1} = \emptyset$.

U.2 *For every $c \in {\mathcal{L}_1}^{\omega+1}$ there exists $i \leq \omega$, a point $x_c \in \overline{N_i}/\approx_{\beta_i+1}$ of level γ_c+1 and an elementary neighborhood U_c of x_c such that $c = j_{\beta_i+1}^{-1}(U_c) \cap \omega^*$.*

U.3 *The set $\left\{x_c : c \in \mathcal{L}_1^{\omega+1}\right\}$ has no accumulation point in $U_{\omega+1}/Q_{\omega+1}$.*

U.4 *It holds that* $\sup\left\{\gamma_c+1 : c \in \mathcal{L}_1^{\omega+1}\right\} < \alpha$.

Remember that $\mathcal{C}_{\overline{\omega}}$ is the ω-family with minimum index we have just used in order to construct $\overline{N_\omega}/\approx_{\beta_\omega+1}$; moreover notice that if $\overline{\omega} > \omega$ the family $\mathcal{C}_\omega$ is not an ω-family and then it neither is an $(\omega+1)$-family. Towards a contradiction, suppose that it is an $(\omega+1)$-families: in $\mathcal{L}_0^\omega$ we put the elements that lie in $\mathcal{L}_0^{\omega+1}$ and all those elements $c \in \mathcal{L}_1^{\omega+1}$ such that ω is the only value of the index i for which U.2 is satisfied; these c are such that $c \cap U_\omega = \emptyset$ by Rem. 5.1. Instead in $\mathcal{L}_1^\omega$ we put all the other $c \in \mathcal{L}_1^{\omega+1}$ which are left: they obviously satisfy U.4 since we are estimating the supremum on a lesser number of elements; moreover they satisfy U.3 since if the points x_c had accumulation points in U_ω/Q_ω then they would have accumulation points in $U_{\omega+1}/Q_{\omega+1}$ due to the fact that the new relation respects the old ones.

If instead $\overline{\omega} = \omega$, then $\mathcal{C}_{\overline{\omega}} = \mathcal{L}_0 \cup \mathcal{L}_1$ is not an $(\omega+1)$-family since the elements of $\mathcal{C}_{\overline{\omega}}$ would have all to stay in $\mathcal{L}_1^{\omega+1}$ but the corresponding infinite points $\{x_c : c \in \mathcal{L}_1^{\omega+1}\}$, which are all in the compact space $N_\omega^*/\approx_{\beta_\omega+1}$, must have an accumulation point in $U_{\omega+1}/Q_{\omega+1} \supseteq$ $\supseteq N_\omega^*/\approx_{\beta_\omega+1}$.

Suppose to have constructed infinite subsets $N_\gamma \subset \omega$ (with $\gamma <$ $< \delta < \omega_1$) and relations $\approx_{\beta_\gamma+1}$ of type $\beta_\gamma+1 < \alpha+1$ on $\overline{N_\gamma}$ satisfying the following conditions.

T.1 $\overline{N_\gamma^* \setminus \bigcup_{\gamma'<\gamma} N_{\gamma'}^*} \neq \emptyset$.

T.2 $\overline{\bigcup_{\gamma'\leq\gamma} N_{\gamma'}^*} \neq \omega^*$.

T.3 For every $\gamma' < \gamma$, the relations $\approx_{\beta_{\gamma'}+1}$ and $\approx_{\beta_\gamma+1}$ coincide on $\overline{N_{\gamma'}} \cap \overline{N_\gamma}$.

T.4 A family $\mathcal{C}_\xi$ with index $\xi \leq \gamma$ is not a $(\gamma+1)$-family.

By T.3 and the decompositions set on the $\overline{N}_\gamma$ for every $\gamma < \delta$, a decomposition Q_δ is defined on $U_\delta = \bigcup_{\gamma<\delta} N_\gamma^*$. Let $j_{\alpha+1}^\delta$ be the quotient mapping $j_{\alpha+1}^\delta : U_\delta \to U_\delta/Q_\delta$.

We say that an element $\mathcal{C}_\xi \in \Gamma$ is a δ-family if $\mathcal{C}_\xi$ can be decomposed into two subfamilies $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfying the following conditions.

U.1 $\bigcup \mathcal{L}_0 \cap U_\delta = \emptyset$.

U.2 For every $c \in \mathcal{L}_1$ there exists $\gamma < \delta$, a point $x_c \in \overline{N_\gamma}/\approx_{\beta_\gamma+1}$ of level $\gamma_c + 1$ and an elementary neighborhood U_c of x_c such that $c = j^{-1}_{\beta_\gamma+1}(U_c) \cap \omega^*$.

U.3 The set $\{x_c : c \in \mathcal{L}_1\}$ has no accumulation point in U_δ/Q_δ.

U.4 It holds that $\sup\{\gamma_c + 1 : c \in \mathcal{L}_1\} < \alpha$.

We can rewrite these properties in the following way:

i) $\mathcal{L}_0$ has to consist of elements C_n^* where the subsets C_n are transversal to the subsets N_γ, i.e. every $C_n \subset \omega$ intersects every N_γ in a finite number of points (in this way we are respecting $U.1$);

ii) $\mathcal{L}_1$ has to consist of elements C_m^* where for every m there exists $\gamma \in \delta$, a point $x_m \in \overline{N_\gamma}/\approx_{\beta_\gamma+1}$ of level $l(x_m) < \alpha$ and an elementary neighborhood $U(x_m)$ such that $C_m^* = j^{-1}_{\beta_\gamma+1}(U(x_m)) \cap \omega^*$. A further necessary request is that the set $\{x_m\}$ has no accumulation point in U_δ/Q_δ and that $\sup\{l(x_m)\} < \alpha$. We want to remark that by $l(x_m)$ we mean a successor ordinal. (In this way we are respecting U.2–U.3–U.4.)

Let us show that it is possible to find a δ-family C_ξ: for example, we can use again Lemma 4.2, since the subsets N_γ^* with $\gamma < \delta < \omega_1$ comply with the hypotheses. We want to remark that here the fact that δ is a countable ordinal is essential in order to apply Lemma 4.2. In this way we are able to find a subset $\Delta_\delta \subset \omega$ with $|\Delta_\delta| = \omega$ and such that Δ_δ intersects every N_γ in a finite number of points. We can decompose this infinite set in an infinite number of infinite subsets $T_n \subset \omega$ which again intersect every N_γ in a finite number of points; we set $\mathcal{L}_0 = \{T_n^* : n \in \omega\}$. It is clear that $\mathcal{L} = \mathcal{L}_0$ is a δ-family.

Among all the δ-families in Γ, let us take the one with the minimum index $\overline{\delta}$: we can write it as $\mathcal{C}_{\overline{\delta}} = \mathcal{L}_0 \sqcup \mathcal{L}_1$ where $\mathcal{L}_0 = \{N^*_{\overline{\delta},n} : n \in J_0\}$, $\mathcal{L}_1 = \{N^*_{\overline{\delta},n} : n \in J_1\}$ and $J_0 \cap J_1 = \emptyset$. Of course, by construction, the δ-family $\mathcal{C}_{\overline{\delta}}$ will comply with the following properties.

U.1 $\bigcup \mathcal{L}_0 \cap U_\delta = \emptyset$.

U.2 For every $N^*_{\overline{\delta},n}$ with $n \in J_1$ there exist an index $\gamma_n \in \delta$, a point $x_n \in$ $\in \overline{N_{\gamma_n}}/\approx_{\beta_{\gamma_n}+1}$ of level $l(x_n) < \alpha$ and an elementary neighborhood $U(x_n)$ such that $N^*_{\overline{\delta},n} = j^{-1}_{\beta_{\gamma_n}+1}(U(x_n)) \cap \omega^*$.

U.3 The set $\{x_n : n \in J_1\}$ has no accumulation point in U_δ/Q_δ.

U.4 It holds that $\sup\{l(x_n) : n \in J_1\} = \beta_\delta < \alpha$ with β_δ that can take up value from 1 to α not included. Without loss of generality, we can always assume that the levels of the points are ordered in a nondecreasing way.

For every $n \in J_1$ it holds that $\hat{U}(x_n) = U(x_n)\backslash\omega$ is a clopen neighborhood of x_n in $N^*_{\gamma_n}/\approx_{\beta_{\gamma_n}+1}$. In order to apply Lemma 4.4, we need to rewrite $\bigcup_{\gamma\in\delta} N^*_\gamma$ as a disjoint union; notice that, since δ is a countable ordinal, we can enumerate $\{N^*_\gamma\}_{\gamma\in\delta}$ as $\{N^*_{\gamma_i}\}_{i\in\omega}$. Let us set $\tilde{N}_{\gamma_1} = N_{\gamma_1}, \tilde{N}_{\gamma_2} = N_{\gamma_2}\backslash N_{\gamma_1}, ..., \tilde{N}_{\gamma_k} = N_{\gamma_k}\backslash \bigcup_{i=1}^{k-1} N_{\gamma_i}$. It holds that $\bigsqcup_{k\in\omega} \tilde{N}_{\gamma_k} = \bigcup_{k\in\omega} N_{\gamma_k}$, whence $\bigsqcup_{k\in\omega} \tilde{N}^*_{\gamma_k} = \bigcup_{k\in\omega} N^*_{\gamma_k}$; we have only to prove the non-trivial inclusion $\bigsqcup_{k\in\omega} \tilde{N}^*_{\gamma_k} \supseteq \bigcup_{k\in\omega} N^*_{\gamma_k}$: if $x \in \bigcup_{k\in\omega} N^*_{\gamma_k}$, then there is $\bar{\imath} \in \omega$ such that $x \in N^*_{\gamma_{\bar{\imath}}} = \left[\bigsqcup_{k\leq\bar{\imath}} \tilde{N}_{\gamma_k}\right]^* = \bigsqcup_{k\leq\bar{\imath}} \tilde{N}^*_{\gamma_k} \subseteq \bigsqcup_{k\in\omega} \tilde{N}^*_{\gamma_k}$. Notice that, for every $k \in \omega$, $\tilde{N}^*_{\gamma_k}$ is distinguished relative to Q_δ. Finally we can apply Lemma 4.4 since the countable family $\{\tilde{N}_\gamma\}_{\gamma<\delta}$, the points $\{x_n\}_{n\in J_1}$ and the relation Q_δ defined on $U_\delta = \bigsqcup_{\gamma<\delta} \tilde{N}^*_\gamma = \bigcup_{\gamma<\delta} N^*_\gamma$ satisfy the hypotheses.[6] We remark that $\bigcup \hat{U}(x_n) \neq U_\delta/Q_\delta$ since in U_δ/Q_δ there are points of level α that $\bigcup \hat{U}(x_n)$ does not cover.[7]

Therefore it is possible to find pairwise disjoint elementary neighborhoods U_n with $x_n \in U_n$ and a subset $N'_\delta \subset \omega$ such that $(N'_\delta)^* \cap U_\delta = \bigsqcup (j^\delta_{\alpha+1})^{-1}(U_n) = \bigsqcup E^*_n$. Let us set $\mathcal{C}' = \mathcal{L}_0 \cup \{(j^\delta_{\alpha+1})^{-1}(U_n) : n \in J_1\}$, $\mathcal{Q} = \{N_{\bar{\delta},n} : n \in J_0\}$ and $\mathcal{R} = \{N_\gamma : \gamma \in \delta\}$.

From now on, by replacing everywhere the index ω by δ and following exactly the same steps we did for the construction of N_ω and the relation $\approx_{\beta_\omega+1}$, we are able to find a suitable N_δ and a relation $\approx_{\beta_\delta+1}$ on it of type $\beta_\delta + 1$ with $2 \leq \beta_\delta + 1 \leq \alpha$[8] which satisfies properties T.1 to T.3: the proof of these properties follows the one given on page 193.

Then a relation $Q_{\delta+1}$ is defined on $U_{\delta+1} = \bigcup_{\gamma=1}^{\delta} N^*_\gamma$.

Let us check property T.4 which states that a family $\mathcal{C}_\xi$ with index $\xi \leq \delta$ is not a $(\delta + 1)$-family.

A family $\mathcal{C}_\xi \in \Gamma$ is a $(\delta + 1)$-family if it can be decomposed into

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[6]At most we have to restrict the neighborhoods of the points $\{x_n\}$ in such a way that each of them belongs to some $\tilde{N}^*_\gamma/Q_\delta$ for some $\gamma < \delta$.

[7]Notice that $\bigcup \hat{U}(x_n) \neq U_\delta/Q_\delta$ also in the case in which α is a limit ordinal: indeed at the beginning of the construction we put decompositions of type $\beta_i + 1$ on the subsets $\overline{N_i}$ in such a way that $\sup\{\beta_i + 1\} = \alpha$; hence in U_δ/Q_δ there certainly exists a point of level $\beta_\delta + 1 < \alpha$ that $\bigcup \hat{U}(x_n)$ does not cover.

[8]In the case in which α is a limit ordinal it turns out that $2 \leq \beta_\delta + 1 < \alpha$.

two subfamilies $\mathcal{L}_0^{\delta+1}$ and $\mathcal{L}_1^{\delta+1}$, which satisfy conditions of the type U.1 to U.4 at level $\delta+1$.

Remember that $\mathcal{C}_{\overline{\delta}}$ is the δ-family with minimum index we have just used in the construction of $\overline{N}_\delta/\approx_{\beta_\delta+1}$. If $\overline{\delta}>\delta$ the families $\mathcal{C}_\xi$ with $\xi\leq\delta$ are not δ-families and then they neither are $(\delta+1)$-families. Towards a contradiction, suppose that they are $(\delta+1)$-families; then in $\mathcal{L}_0^\delta$ we put the elements that lie in $\mathcal{L}_0^{\delta+1}$ and all those elements $c\in\mathcal{L}_1^{\delta+1}$ such that δ is the only value of the index γ for which U.2 is satisfied; these c are such that $c\cap U_\delta=\emptyset$ (this result follows from the extension of Rem. 5.1 to the case δ). Instead in $\mathcal{L}_1^\delta$ we put all the other $c\in\mathcal{L}_1^{\delta+1}$ that are left: they obviously satisfy U.4 since we are estimating the supremum on a lesser number of elements; moreover they satisfy U.3, since if the points x_c had accumulation points in U_δ/Q_δ, then they would have accumulation points in $U_{\delta+1}/Q_{\delta+1}$, due to the fact that the new relation respects the old ones.

On the other hand, if $\overline{\delta}=\delta$ then the families $\mathcal{C}_\xi$ with $\xi<\delta$ are not δ-families and by what we have just remarked they neither are $(\delta+1)$-families; moreover $\mathcal{C}_{\overline{\delta}}=\mathcal{L}_0^\delta\cup\mathcal{L}_1^\delta$ is not a $(\delta+1)$-family, since the elements of $C_{\overline{\delta}}$ would have all to stay in $\mathcal{L}_1^{\delta+1}$ but the corresponding infinite points x_c, which are all in the compact space $N_\delta^*/\approx_{\beta_\delta+1}$, must have an accumulation point in $U_{\delta+1}/Q_{\delta+1}\supseteq N_\delta^*/\approx_{\beta_\delta+1}$.

Therefore, by transfinite induction, we have defined a relation Q_{ω_1} on $\bigcup_{\gamma<\omega_1}\overline{N_\gamma}$ which coincides with $\approx_{\beta_\gamma+1}$ on each $\overline{N_\gamma}$.

Let us prove the following lemma.

Lemma 5.2. *If a family $\mathcal{C}_\xi\in\Gamma$ is not a ϑ-family then it is not a δ-family for every $\delta>\vartheta$.*

Proof. We prove that if $\mathcal{C}_\xi\in\Gamma$ is a δ-family then it is also a ϑ-family. Let us suppose that $\mathcal{C}_\xi$ is a δ-family; then it can be decomposed into two subfamilies $\mathcal{L}_0^\delta$ and $\mathcal{L}_1^\delta$ satisfying the following conditions.

U.1 $\bigcup\mathcal{L}_0^\delta\cap U_\delta=\emptyset$.

U.2 For every $c\in\mathcal{L}_1^\delta$ there exist $\gamma<\delta$, a point $x_c\in\overline{N_\gamma}/\approx_{\beta_\gamma+1}$ of level γ_c+1 and an elementary neighborhood U_c of x_c such that $c=j^{-1}_{\beta_\gamma+1}(U_c)\cap\omega^*$.

U.3 The set $\{x_c:c\in\mathcal{L}_1^\delta\}$ has no accumulation point in U_δ/Q_δ.

U.4 It holds that $\sup\{\gamma_c+1:c\in\mathcal{L}_1^\delta\}<\alpha$.

We want to show that $\mathcal{C}_\xi$ is also a ϑ-family. In $\mathcal{L}_0^\vartheta$ we put the elements that lie in $\mathcal{L}_0^\delta$ and all those elements $c\in\mathcal{L}_1^\delta$ for which the only ordinals that fit for U.2 are larger than or equal to ϑ; these c are the inverse images of elementary neighborhoods of points constructed by starting from some $\overline{F}_n$ where $F_n^* \in \mathcal{L}_0^\zeta$ with $\zeta \geq \vartheta$. We know that the elementary neighborhoods of these points are contained in $\overline{F}_n/\approx_{\beta_\zeta+1}$ and then, by Rem. 5.1 and its extension to the case δ, it follows that for each of these c it holds that $c \cap U_\vartheta = \emptyset$. On the other hand in $\mathcal{L}_1^\vartheta$ we put all the other $c \in \mathcal{L}_1^\delta$ that are left. They obviously satisfy U.4, since we are estimating the supremum on a lesser number of elements; moreover they satisfy U.3, since if the points x_c had accumulation points in U_ϑ/Q_ϑ, then they would have accumulation points in U_δ/Q_δ since the new relations respect the old ones. $\Diamond$

Now we can prove the following fundamental remark.

Remark 5.3. For every $\vartheta < \omega_1$, $\mathcal{C}_\vartheta \in \Gamma$ is not an ω_1-family. By contradiction suppose that there exists an index $\vartheta < \omega_1$ such that $\mathcal{C}_\vartheta$ is an ω_1-family. By transfinite induction we proved that, for every $\vartheta < \omega_1$, a family $\mathcal{C}_\xi$ with $\xi \leq \vartheta$ is not a $(\vartheta+1)$-family and hence $\mathcal{C}_\vartheta$ is not a $(\vartheta+1)$-family. On the other hand we supposed that $\mathcal{C}_\vartheta$ is an ω_1-family and then by Lemma 5.2 it is a $(\vartheta+1)$-family. A contradiction. Let us point out that there can not exist ω_1-families in Γ, since the elements of the set Γ have indices that go from ω included to ω_1 not included.

Finally we define the relation $\approx_{\alpha+1}$ on $\beta\omega$ in this way:

– it coincides with Q_{ω_1} on $\bigcup_{\gamma<\omega_1} \overline{N_\gamma}$;

– two free ultrafilter belonging to $\omega^* \setminus \bigcup_{\gamma<\omega_1} N_\gamma^*$ are equivalent under the relation $\approx_{\alpha+1}$.

Let us call $K_{\alpha+1}$ the space obtained by the quotient of $\beta\omega$ with this relation and $j_{\alpha+1}$ the natural quotient mapping. Let us remark that, by property U.2, $\omega^* \setminus \bigcup_{\gamma<\omega_1} N_\gamma^*$ is not empty: indeed for every $\gamma \in \omega_1$ it holds that $B_\gamma = \omega^* \setminus \bigcup_{\gamma' \leq \gamma} N_{\gamma'}^*$ is a closed subset of ω^* and the subsets B_γ (with $\gamma \in \omega_1$) are such that $B_{\gamma_1} \supseteq B_{\gamma_2}$ for every $\gamma_1 < \gamma_2$; moreover the family of closed subsets $\{B_\gamma\}_{\gamma\in\omega_1}$ has the finite intersection property by T.2 proved for every step $\gamma \in \omega_1$. Thus, due to the compactness of ω^*, it follows that $\bigcap_{\gamma\in\omega_1} B_\gamma = \bigcap_{\gamma\in\omega_1}(\omega^* \setminus \bigcup_{\gamma'\leq\gamma} N_{\gamma'}^*) = \omega^* \setminus \bigcup_{\gamma\in\omega_1} N_\gamma^* \neq \emptyset$.

Then $j_{\alpha+1}$ collapses $\omega^* \setminus \bigcup_{\gamma<\omega_1} N_\gamma^*$ to a single point which we call x_∞.

If an element of $K_{\alpha+1}$ is a point of the decompositions $\approx_{\beta_\delta+1}$ and $\approx_{\beta_\gamma+1}$, then the point lies in the same level in $\overline{N_\delta}/\approx_{\beta_\delta+1}$ and $\overline{N_\gamma}/\approx_{\beta_\gamma+1}$: indeed whenever we reconsider a point that was in some previous decompositions we take care that there exists an elementary neighborhood of it that accompanies the point in the new decomposition; in this way the level of the point is preserved and the definition of $L_{\beta+1}$ as the set of the points that lie in the level $\beta+1$ in some $\overline{N_\gamma}/\approx_{\beta_\gamma+1}$ is correct. If a point of the space $K_{\alpha+1}$ is a point of the decompositions $\approx_{\beta_\delta+1}$ and $\approx_{\beta_\gamma+1}$ with $\delta>\gamma$, then the problem reduces to examine what happens in $\overline{N_\delta}/\approx_{\beta_\delta+1}$ as regards its elementary neighborhoods. We have to remark that in the construction of the space $K_{\alpha+1}$ we paid attention to the fact that for every level $0<\beta+1\leq\alpha$[9] every point of level $\beta+1$ had a basis of clopen subsets homeomorphic to the space $K_{\beta+1}$ which is compact and sequential by inductive hypothesis.

Now we have to understand which are the elementary neighborhoods of the unique point of level $\alpha+1$ in $K_{\alpha+1}$, i.e. of the point $x_\infty = j_{\alpha+1}(\omega^*\setminus\bigcup_{\gamma<\omega_1} N^*_\gamma)$. On this subject let us prove the following lemma.

Lemma 5.4. *The collection of the clopen subsets $K_{\alpha+1}\setminus\bigcup_{x\in G} U_x$ (where G is a finite set and for every $x\in G$ the clopen subset U_x is an elementary neighborhood in $K_{\alpha+1}$ of the point x that can have level equal to a successor ordinal smaller than or equal to α) is a basis at the point x_∞.*

Proof. In an obvious way $K_{\alpha+1}\setminus\bigcup_{x\in G} U_x$ is a clopen subset of $K_{\alpha+1}$ containing x_∞. Let A be an open subset of $K_{\alpha+1}$ containing x_∞ and let $C=K_{\alpha+1}\setminus A$ be the complementary closed subset. For every $x\in C$, let U_x be an elementary clopen neighborhood of x; trivially, by taking all the clopen neighborhoods U_x, with $x\in C$, we cover C. Let us consider $j^{-1}_{\alpha+1}(C)$: it is a closed subset of $\beta\omega$ and then it is compact. If we take all the open subsets $j^{-1}_{\alpha+1}(U_x)$ (with $x\in C$) they form an open cover of $j^{-1}_{\alpha+1}(C)$; then there exists a finite subcover $\bigcup_{x\in G} j^{-1}_{\alpha+1}(U_x)\supseteq j^{-1}_{\alpha+1}(C)$. Hence it turns out that $j_{\alpha+1}\big(\bigcup_{x\in G} j^{-1}_{\alpha+1}(U_x)\big)=\bigcup_{x\in G} j_{\alpha+1}(j^{-1}_{\alpha+1}(U_x))=$ $=\bigcup_{x\in G} U_x\supseteq j_{\alpha+1}(j^{-1}_{\alpha+1}(C))=C$ and, by passing to the complementary subsets, we can conclude that $K_{\alpha+1}\setminus\bigcup_{x\in G} U_x\subseteq K_{\alpha+1}\setminus C=A$. $\Diamond$

We call *elementary* each of these neighborhoods of the point x_∞.

6. Check of the properties of $K_{\alpha+1}$

Now we want to check that the space $K_{\alpha+1}$ satisfies all the requested properties.

Lemma 6.1. *$K_{\alpha+1}$ is a T_2 space and it is compact.*

[9] strictly smaller than α in the case in which α is a limit ordinal.

Proof. Trivially the points of L_0 can be separated from every other point since they are isolated. Moreover, if we want to separate $x_\infty = = j_{\alpha+1}(\omega^* \setminus \bigcup_{\gamma<\omega_1} N^*_\gamma)$ from any other point x, it is enough to take respectively the open disjoint elementary neighborhoods $K_{\alpha+1} \setminus U_x$ and U_x.

Suppose now to have to part two points x_1 and x_2 of level smaller than $\alpha + 1$; it is possible to face up with two different situations.

1) There exists $\vartheta \in \omega_1$ such that $x_1, x_2 \in j_{\alpha+1}(\overline{N_\vartheta})$; notice that $j_{\alpha+1}(\overline{N_\vartheta}) \simeq K_{\beta+1}$ (with $\beta + 1 < \alpha + 1$) which is a T_2 space by inductive hypothesis. Then in $j_{\alpha+1}(\overline{N_\vartheta})$ there are two open neighborhoods V_{x_1} and V_{x_2} with empty intersection; they are also open in $K_{\alpha+1}$ and hence V_{x_1} and V_{x_2} are open neighborhoods of x_1 and x_2 respectively with empty intersection.

2) There is no $\vartheta \in \omega_1$ such that $x_1, x_2 \in j_{\alpha+1}(\overline{N_\vartheta})$; therefore there are $\vartheta_1, \vartheta_2 \in \omega_1$ such that $x_1 \in j_{\alpha+1}(\overline{N_{\vartheta_1}}) \simeq K_{\beta+1}$ with $\beta + 1 < \alpha + 1$ and $x_2 \in j_{\alpha+1}(\overline{N_{\vartheta_2}}) \simeq K_{\gamma+1}$ with $\gamma+1 < \alpha+1$. Now $I = j_{\alpha+1}(\overline{N_{\vartheta_1}}) \cap j_{\alpha+1}(\overline{N_{\vartheta_2}})$ is a clopen subset of $K_{\alpha+1}$ and hence $j_{\alpha+1}(\overline{N_{\vartheta_1}}) \setminus I$ and $j_{\alpha+1}(\overline{N_{\vartheta_2}}) \setminus I$ are disjoint open neighborhoods of x_1 and x_2 respectively.

Therefore it turns out immediately that $K_{\alpha+1}$ is compact since $j_{\alpha+1}$ is a continuous function from the compact space $\beta\omega$ to the T_2 space $K_{\alpha+1}$. $\Diamond$

Before proving the sequentiality of the space $K_{\alpha+1}$ we need to demonstrate that properties S.4 and S.5 hold.

Remark 6.2. In $K_{\alpha+1}$, if a nonconstant sequence $(x_n)_{n\in\omega}$ of points $x_n \in L_{\gamma_n+1}$ with nondecreasing levels converges to a point $x \in L_{\gamma+1}$, then for the sequence (γ_n+1) of ordinal numbers it holds that $\sup\{\gamma_n+1\} = \gamma$. (Properties S.4.)

Proof. For every $\gamma + 1 < \alpha + 1$ we apply the inductive hypothesis, since we have supposed that property S.4 holds in $K_{\gamma+1}$ for every $\gamma+1 < \alpha+1$.

Now we have to prove that for a non-constant sequence of points $x_n \in L_{\gamma_n+1}$ (where the sequence (γ_n+1) is not decreasing) that converges to the point $x_\infty \in L_{\alpha+1}$ it holds that $\sup\{\gamma_n + 1\} = \alpha$. Towards a contradiction, let us suppose that $\sup\{\gamma_n + 1\} < \alpha$. In principle there are two different cases we have to analyse:

1) from the sequence $(x_n)_{n\in\omega}$ we can extract an injective subsequence $(x_{n_i})_{i\in\omega}$;

2) from the sequence $(x_n)_{n\in\omega}$ we can not extract any injective subsequence $(x_{n_i})_{i\in\omega}$.

We can avoid considering the latter case: indeed, since $(x_n)_{n\in\omega}$ is a non-constant sequence, there are at least two points that appear infinite times and then the sequence is not convergent to any point against the hypothesis.

In the former case the sequence $(x_{n_i})_{i\in\omega}$ has to converge to x_∞ too. If $\{x_{n_i}\}_{i\in\omega}$ had no accumulation point in $U_{\omega_1}/Q_{\omega_1}$, then by Rem. 4.5 it would be possible to find a countable infinity of pairwise disjoint clopen subsets of ω^*; moreover these clopen subsets would satisfy the properties to be an ω_1-family (notice that $\sup\{\gamma_n + 1\} < \alpha$) and this would be inconsistent with Rem. 5.3. Then the subset $S = \{x_{n_i} : i \in \omega\}$ has at least an accumulation point in $U_{\omega_1}/Q_{\omega_1}$; thus there exists a point $y \in \in U_{\omega_1}/Q_{\omega_1}$ (where certainly $l(y) = \delta+1 < \alpha+1$) such that $y \in \overline{\{x_{n_i} : i \in \omega\}}$. Then let us consider an elementary neighborhood of y, U_y, which has to be homeomorphic to the space $K_{\delta+1}$; we can assert that infinite points of S such that the supremum of their levels is equal to an ordinal number $\eta < \alpha$ are in U_y. We denote this set of points by $S' \subseteq S$; we know that property S.5 holds in $K_{\delta+1}$ and then from the injective sequence S' it is possible to extract a sequence converging to a point of level $\eta+1 < \alpha+1$. Therefore the sequence $(x_n)_{n\in\omega}$ admits a subsequence which converges to a point of level strictly smaller than $\alpha + 1$ against the hypothesis. $\Diamond$

Remark 6.3. In $K_{\alpha+1}$, from every injective sequence $S = (x_n)_{n\in\omega}$ of points with nondecreasing levels such that $\sup\{l(x_n)\} = \eta \leq \alpha$ it is possible to extract a subsequence converging to a point of level $\eta + 1$. (Property S.5.)

Proof. If $\eta = 0$ then the sequence $(x_n)_{n\in\omega}$ is formed by points of ω; therefore there is an index $\gamma \in \omega_1$ such that $|\{x_n\}_{n\in\omega} \cap N_\gamma| = \omega$: otherwise, if it turns out that $|\{x_n\}_{n\in\omega} \cap N_\gamma| < \omega$ for every $\gamma \in \omega_1$, then from $N_{\omega_1} = \{x_n\}_{n\in\omega}$ we are able to construct an ω_1-family and this is a contradiction. Then in $\overline{N_\gamma}/\approx_{\alpha+1}$ there are infinite points of the above sequence but $\overline{N_\gamma}/\approx_{\alpha+1} \simeq K_{\beta+1}$ with $\beta+1 < \alpha+1$ and hence, since property S.5 holds in $K_{\beta+1}$ by inductive hypothesis, it is possible to extract a subsequence converging to a point of level 1 from the starting sequence.

Suppose now that $0 < \eta < \alpha$; let us choose an injective subsequence $S' = (x_{n_i})_{i\in\omega} \subseteq S$ in such a way that the sequence of the levels of the points converges upwards to η; if S' had no accumulation point in $U_{\omega_1}/Q_{\omega_1}$, then by Rem. 4.5 it would be possible to find a countable infinity of pairwise disjoint clopen subsets of ω^*; moreover these clopen subsets would satisfy the properties to be an ω_1-family (notice that $\sup\{l(x_{n_i})\} < \alpha$) and this would be inconsistent with Rem. 5.3. Thus S' must have at least an accumulation point in $U_{\omega_1}/Q_{\omega_1}$ and hence there exists a point $y \in U_{\omega_1}/Q_{\omega_1}$ with $l(y) = \delta + 1 < \alpha + 1$ such that $y \in \overline{\{x_{n_i} : i \in \omega\}}$. Then let us consider an elementary neighborhood of y, U_y, which has to be homeomorphic to the space $K_{\delta+1}$; we can assert that infinite points of the set $\{x_{n_i} : i \in \omega\}$ such that the limit and hence the supremum of their levels is equal to $\eta < \alpha$ are in U_y. We denote this set of points by $S'' \subseteq S'$; we know that property S.5 holds in $K_{\delta+1}$ and then from the injective sequence S'' it is possible to extract a sequence converging to a point of level $\eta + 1 < \alpha + 1$.

If $\eta = \alpha$ then let us choose again an injective subsequence $S' = (x_{n_i})_{i\in\omega} \subseteq S$ in such a way that the levels of the points x_{n_i} converges upwards to α; the sequence S' converges to x_∞, since its points fall eventually in every neighborhood of x_∞. $\Diamond$

Now we are able to prove the sequentiality of $K_{\alpha+1}$.

Lemma 6.4. *$K_{\alpha+1}$ is sequential.*

Proof. Let us begin by proving that $B_{\alpha+1} = K_{\alpha+1}\backslash\{x_\infty\}$ is sequential, i.e. by showing that if F is a sequentially closed subset of $B_{\alpha+1}$ then it is closed. Let us suppose that F is sequentially closed and let us show that for every $x \in B_{\alpha+1}\backslash F$ there exists an elementary neighborhood $\hat{U}_x$ of x such that $\hat{U}_x \subseteq B_{\alpha+1}\backslash F$. If $x \in B_{\alpha+1}\backslash F$, then there exists an open neighborhood of x, $U_x \subseteq B_{\alpha+1}$ with the peculiarity that $U_x \simeq K_{\beta+1}$ (with $\beta+1 < \alpha+1$) which is a compact sequential space. Notice that $x \notin F \cap U_x$; if $F \cap U_x = \emptyset$, then U_x is an elementary neighborhood containing x such that $U_x \subseteq B_{\alpha+1}\backslash F$. If instead $F \cap U_x \neq \emptyset$, since F is sequentially closed in $B_{\alpha+1}$, then $F \cap U_x$ is sequentially closed in U_x (otherwise, if $F \cap U_x$ is not sequentially closed in U_x, hence we have a sequence in $F \cap U_x$ with its limit point in $U_x\backslash F$; we can see this sequence as a sequence in F with its limit point out of F and then F is not sequentially closed against the hypothesis). It follows that $F \cap U_x$ is closed in U_x since U_x is sequential and hence it is compact; let us consider the open cover of $F \cap U_x$ formed by elementary neighborhoods of points in $F \cap U_x$ not containing x. From this open cover it is possible to extract a finite subcover $\bigcup_{i=1}^{n} U_{y_i} \supseteq F \cap U_x$. Then $\hat{U}_x = U_x \backslash \bigcup_{i=1}^{n} U_{y_i}$ is an open neighborhood of x which has empty intersection with F. Thus we can conclude that $B_{\alpha+1}$ is sequential.

Now we have still to demonstrate that, if F is sequentially closed in $K_{\alpha+1}$ and $x_\infty \notin F$, then $x_\infty \notin \overline{F}$. Towards a contradiction, suppose that $x_\infty \notin F$ and, at the same time, $x_\infty \in \overline{F}$. Since $x_\infty \notin F$ then either

F is finite (and in this case the point $x_\infty \notin \overline{F}$ against the hypothesis) or F is infinite and in this second case from F it is not possible to extract any injective sequence of points with nondecreasing levels such that the supremum of the levels is equal to α; indeed if such a sequence existed, by Rem. 6.3 from this sequence it would possible to extract a subsequence converging to x_∞ and then x_∞ would stay in F (since F is sequentially closed) against the hypothesis. Now if α is a successor ordinal, there exists at most a finite number of points of level $\alpha = \gamma_0$ in F that we call $z_1, z_2, \ldots, z_m$; let us consider an elementary neighborhood U_{z_i} for each of these points and let us set $G_1 = F \setminus \bigcup_{i=1}^{m} U_{z_i} \subseteq F$. We assert that either G_1 is finite (and in this case it turns out that $x_\infty \notin \overline{F}$ against the hypothesis) or G_1 is infinite and in this second case from G_1 it is not possible to extract any injective sequence of points with nondecreasing levels such that the supremum of the levels is equal to $\alpha - 1 = \gamma_1$; indeed if such a sequence existed, by Rem. 6.3 from this sequence it would possible to extract a subsequence converging to a point of level α different from $z_1, z_2, \ldots, z_m$ and then also this point would stay in F against our assumption. If instead α is a limit ordinal, it is not true that for every $\gamma \in \alpha$ there exists $x \in F$ such that $l(x) > \gamma + 1$ (otherwise $x_\infty \in F$ which is sequentially closed) and hence there exists an index $\overline{\gamma} \in \alpha$ such that for every $x \in F$ it turns out that $l(x) \leq \overline{\gamma} + 1 < \alpha$. Therefore we can assert that in F there are at most a finite number of elements of level $\overline{\gamma} + 1$ that we call $y_1, y_2, \ldots, y_k$: indeed if we had an infinite number of these points, it would possible to extract a subsequence converging to a point of level $(\overline{\gamma} + 1) + 1$ and this point would stay again in F but this is against what we have just remarked. Let us consider an elementary neighborhood U_{y_i} for each of these points and let us call $G_1 = F \setminus \bigcup_{i=1}^{k} U_{y_i} \subseteq F$. We can say that either G_1 is finite (and in this case the point $x_\infty \notin \overline{F}$) or G_1 is infinite and in this second case from G_1 it is not possible to extract any injective sequence of points with nondecreasing levels such that the supremum of the levels is equal to $\gamma_1 = \overline{\gamma} < \alpha$; indeed if such a sequence existed, by Rem. 6.3 from this sequence it would be possible to extract a subsequence converging to a point of level $\overline{\gamma} + 1$ different from $y_1, y_2, \ldots, y_k$ and then also this point would stay in F against what we have assumed.

In each case we have constructed a sequentially closed subset G_1 from which it is not possible to extract any injective sequence of points with nondecreasing levels such that the supremum of the levels is equal to $\gamma_1 < \alpha$; moreover G_1 is the complement of a finite number of elementary neighborhoods in F. Then it is possible to repeat the procedure and to find step by step a decreasing sequence of ordinals $\gamma_0 > \gamma_1 > \gamma_2 > > \ldots > \gamma_n > \ldots$ and corresponding subsets $G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq \ldots$. This sequence has to be finite and then we find a finite set $G_{\overline{n}}$ after a finite number of steps. Trivially we can cover $G_{\overline{n}}$ by a finite number of elementary neighborhoods; moreover $G_{\overline{n}}$ has been constructed as complement of a finite number of elementary neighborhoods in F. Then it turns out that it is possible to cover F with finitely many elementary neighborhoods of points of level smaller than $\alpha + 1$ and hence it follows that $x_\infty \notin \overline{F}$. A contradiction. $\lozenge$

Now we want to show that every point in $K_{\alpha+1}$ belongs to the closure of L_0, i.e that the set L_0 is dense in $K_{\alpha+1}$.

Remark 6.5. For every $x \in K_{\alpha+1}$ it holds that $x \in \overline{L_0}$, i.e. $\overline{L_0} = K_{\alpha+1}$.

Proof. Let x be a point in $K_{\alpha+1}$ with $l(x) = \beta + 1 < \alpha + 1$ and let V be a non-empty neighborhood of x; then there exists an open elementary neighborhood $U_x \simeq K_{\beta+1} \subseteq V$ and it turns out that $j^{-1}_{\alpha+1}(U_x)$ is a non-empty open subset in $\beta\omega$. Therefore there exists a free or a fixed ultrafilter $\mathcal{U}$ such that $\mathcal{U} \in j^{-1}_{\alpha+1}(U_x)$. If $\mathcal{U}$ is fixed we trivially finish; if $\mathcal{U}$ is a free ultrafilter, since $j^{-1}_{\alpha+1}(U_x)$ is an open subset, there is an infinite subset U' of ω with $U' \in \mathcal{U}$ such that $(U')^* \cup U' \subseteq j^{-1}_{\alpha+1}(U_x)$; then $U' \subseteq \omega$ (with $|U'| = \omega$) is such that $U' \subseteq j^{-1}_{\alpha+1}(U_x)$ and hence $W = j_{\alpha+1}(U') \subseteq U_x$; we can conclude that $U_x \cap L_0 \supseteq W \cap L_0 \neq \emptyset$.

Now let us consider $x_\infty \in K_{\alpha+1}$ and let U be an open neighborhood of x_∞. By Lemma 5.4 there exists an open subset $A_{x_\infty} = K_{\alpha+1} \setminus \bigcup U_x \subseteq U$; therefore $j^{-1}_{\alpha+1}(A_{x_\infty})$ is a non-empty open subset of $\beta\omega$ and hence we can proceed as above. $\lozenge$

Since we have proved that the space $K_{\alpha+1}$ is sequential and that $\overline{L_0} = K_{\alpha+1}$, Rem. 6.2 allows us to conclude that the level of each point is larger or equal to its order of sequentiality with respect to L_0. We have to prove a last remark before concluding that the level of each point is exactly equal to its sequential order with respect to the set L_0.

Remark 6.6. Let A be a closed subset in $\bigcup_{\gamma+1 \leq \eta} L_{\gamma+1}$ with $\eta \leq \alpha$. Then it follows that $\overline{A} \cap \bigcup_{\gamma+1 \leq \eta+1} L_{\gamma+1} = \operatorname{seqcl}(A)$.

Proof. Since A is closed in $\bigcup_{\gamma+1 \leq \eta} L_{\gamma+1}$, then A is sequentially closed in $\bigcup_{\gamma+1 \leq \eta} L_{\gamma+1}$ and hence there is no sequence in A converging to some point of $\bigcup_{\gamma+1 \leq \eta} L_{\gamma+1} \setminus A$. Notice that by Rem. 6.2 it turns out that $\operatorname{seqcl}(A) \subseteq$

$\subseteq \overline{A} \cap \bigcup_{\gamma+1\leq\eta+1} L_{\gamma+1}$. We want to prove that $\overline{A} \cap \bigcup_{\gamma+1\leq\eta+1} L_{\gamma+1} \subseteq \mathrm{seqcl}(A)$. Let x be a point in $(\overline{A} \setminus A) \cap (\bigcup_{\gamma+1\leq\eta+1} L_{\gamma+1})$; trivially it holds that $l(x) = \eta + 1$. Since $K_{\alpha+1}$ is sequential and $x \in \overline{A}$, it turns out that there exists an index $\beta \in \omega_1$ such that $x \in \mathrm{seqcl}_\beta(A)$; we state that $\beta = 1$. Towards a contradiction, let us suppose that $x \notin \mathrm{seqcl}_1(A)$, i.e. let us suppose that no sequence in A converges to x. Then x is the limit point of a sequence whose elements are in some sequential closure of A and not in A, i.e. x is the limit point of a sequence $(y_{\beta_i+1})_{i\in\omega}$ with $\sup\{\beta_i+1\} \geq \eta+1$ but this is absurd since $l(x) = \eta + 1$: indeed if it was correct, in $K_{\alpha+1}$ there would exist a sequence $(y_{\beta_i+1})_{i\in\omega}$ with $\sup\{\beta_i + 1\} \neq \eta$ converging to a point of level $\eta + 1$ and this is inconsistent with Rem. 6.2. $\Diamond$

Finally we can prove the following crucial lemma.

Lemma 6.7. *In $K_{\alpha+1}$ the order of sequentiality of a point of level $\beta + 1$ with respect to L_0 is $\beta+1$ and $K_{\alpha+1}$ is a space with sequential order $\alpha+1$.*

Proof. Notice that the points of level 0 and 1 have sequential order respectively 0 and 1 with respect to the set L_0. Now consider the set $\bigcup_{\gamma+1\leq\eta} L_{\gamma+1}$ with $\eta \leq \alpha$; it complies with the hypotheses of Rem. 6.6 since it is closed in $\bigcup_{\gamma+1\leq\eta} L_{\gamma+1}$ and hence it holds that

$$\overline{\bigcup_{\gamma+1\leq\eta} L_{\gamma+1}} \bigcap \bigcup_{\gamma+1\leq\eta+1} L_{\gamma+1} =$$

$$= \Big\{ y \in \bigcup_{\gamma+1\leq\eta+1} L_{\gamma+1} : \forall U_y, \Big(U_y \cap \bigcup_{\gamma+1\leq\eta} L_{\gamma+1} \Big) \neq \emptyset \Big\} =$$

$$= \bigcup_{\gamma+1\leq\eta+1} L_{\gamma+1} = \mathrm{seqcl}\Big(\bigcup_{\gamma+1\leq\eta} L_{\gamma+1} \Big).$$

This result together with Rem. 6.5 allows us to conclude that the level of each point is smaller or equal to its order of sequentiality with respect to the set L_0. But we have already remarked that the level of each point is larger or equal to its order of sequentiality with respect to the set L_0 and hence we can conclude that the level of each point is exactly equal to its order of sequentiality with respect to the set L_0.

Then the space $K_{\alpha+1}$ has sequential order equal to $\alpha + 1$, since $x_\infty \in \overline{L_0}$ and x_∞ has sequential order equal to $\alpha + 1$ with respect to L_0. $\Diamond$

From the previous lemmas and remarks it follows that the space $K_{\alpha+1}$ satisfies conditions S.1 to S.6 presented in Sec. 3.

Remark 6.8. Notice that every Baškirov's space of sequential order a successor ordinal is a scattered space such that the sequential order of each point is equal to its scattering level.

Finally we can state the following theorem.

Theorem 6.9 (CH)**.** *Let α be any ordinal less than or equal to ω_1. There exists a compact sequential T_2 quotient space of $\beta\omega$ with sequential order α.*

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