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ON THE WEYL PROJECTIVE CURVATURE TENSOR OF AN $N(k)$-CONTACT METRIC MANIFOLD

Ahmet **Yıldız**

Dept. of Mathematics, Dumlupınar University, Kütahya, Turkey

Uday Chand **De**

Dept. of Pure Mathematics, University of Calcutta, 35, Ballygunje Circular Road, Kolkata 700019, West Bengal, India

Cengizhan **Murathan**

Dept. of Mathematics, Uludağ University, 16059 Bursa, Turkey

Kadri **Arslan**

Dept. of Mathematics, Uludağ University, 16059 Bursa, Turkey

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Abstract: In the present paper we classify $N(k)$-contact metric manifolds which satisfy $P(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot P = 0$, $P(\xi, X) \cdot S = 0$, $P(\xi, X) \cdot P = 0$ and $P(\xi, X) \cdot Z = 0$ where P is the Weyl projective curvature tensor and Z is the concircular curvature tensor.

E-mail addresses: ahmetyildiz@dumlupinar.edu.tr, uc_de@yahoo.com, cengiz@uludag.edu.tr, arslan@uludag.edu.tr
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1. Introduction

A Riemannian manifold (M^{2n+1}, g) is said to be *semi-symmetric* if its curvature tensor R satisfies $R(X,Y) \cdot R = 0$, $X, Y \in \chi(M)$, where $R(X,Y)$ acts on R as a derivation (see [11] and [15]). In [16], S. Tanno showed that a semi-symmetric K-contact manifold M^{2n+1} $(2n+1 > 3)$ is locally isometric to the unit sphere $S^{2n+1}(1)$.

A contact metric manifold M^{2n+1} satisfying $R(X,Y)\xi = 0$, where ξ is the characteristic vector field of the contact structure, is locally isometric to the product $E^{n+1} \times S^n(4)$ for $2n+1 > 3$ and flat in dimension 3 ([4] or see [5]). In [14], D. Perrone studied a contact metric manifold $M^{2n+1}(2n+1 > 3)$ satisfying $R(\xi, X) \cdot R = 0$; he shows that under additional assumptions the manifold is either Sasakian (and of constant curvature +1) or $R(X,\xi)\xi = 0$.

Baikoussis and Koufogiorgos [2] showed that an $N(k)$-contact metric manifold M^{2n+1} satisfying $R(\xi, X) \cdot C = 0$, is either locally isometric to $S^{2n+1}(1)$ or locally isometric to the product $E^{n+1} \times S^n(4)$, where C is the Weyl conformal curvature tensor of M^{2n+1}. This generalizes a result of Chaki and Tarafdar [8] that a Sasakian manifold M^{2n+1} satisfying $R(\xi, X) \cdot C = 0$ is locally isometric to $S^{2n+1}(1)$. In [13], Papantoniou showed that a semi-symmetric contact metric manifold M^{2n+1} $(2n+1 > 3)$ with ξ belonging to the (k,μ)-nullity distribution is either locally isometric to $S^{2n+1}(1)$ or locally isometric to the product $E^{n+1} \times S^n(4)$. Both Perrone and Papantoniou also studied contact metric manifolds satisfying $R(\xi, X) \cdot S = 0$, where S denotes the Ricci tensor of M^{2n+1}. In [14], Perrone showed that if ξ belongs to the k-nullity distribution, where k is a function, with $R(\xi, X) \cdot S = 0$, then M^{2n+1} is either Einstein–Sasakian manifold or locally isometric to the product $E^{n+1} \times S^n(4)$. De, Kim and Shaikh [9] studied contact metric manifolds with characteristic vector field ξ belonging to the (k,μ)-nullity distribution satisfying $R(X,\xi) \cdot C = 0$.

Recently, in [6], the authors studied contact metric manifold M^{2n+1} satisfying the curvature conditions $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot Z = 0$, where Z is the *concircular curvature tensor* of M^{2n+1} defined by

$$Z(X,Y)W = R(X,Y)W - \frac{\tau}{2n(2n+1)}(X \wedge_g Y)W, \tag{1.1}$$

and τ is the scalar curvature.

In the theory of the projective transformations of connections the Weyl projective curvature tensor plays an important role. The *Weyl projective curvature tensor* P in a Riemannian manifold (M^{2n+1}, g) is defined by

$$P(X,Y)W = R(X,Y)W - \frac{1}{2n}(X \wedge_S Y)W, \tag{1.2}$$

where S is the Ricci tensor.

In the present paper we give a full classification of the $N(k)$-contact metric manifold M^{2n+1} satisfying the curvature conditions $P(\xi,X)\cdot R=0$, $R(\xi,X)\cdot P=0$, $P(\xi,X)\cdot P=0$, $P(\xi,X)\cdot S=0$ and $P(\xi,X)\cdot Z=0$. In the conclusive part, we prove that an $N(k)$-contact metric manifold with non-vanishing recurrent Weyl curvature tensor does not exist.

2. Preliminaries

Let (M^{2n+1}, g) be a $(2n+1)$-dimensional Riemannian manifold of class C^∞. We denote *Riemannian–Christoffel curvature tensor* by

$$R(X,Y)W = \nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W, \tag{2.1}$$

where ∇ is the Levi–Civita connection and $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M.

A contact metric manifold M^{2n+1} is said to be *Einstein* if its Ricci tensor S is of the form

$$S(X,Y) = \gamma g(X,Y), \tag{2.2}$$

for any vector fields X, Y, where γ is a constant on M^{2n+1} [3].

We next define the endomorphism $X \wedge_A Y$ of $\chi(M)$ by

$$(X \wedge_A Y)W = A(Y,W)X - A(X,W)Y, \tag{2.3}$$

where $X, Y, W \in \chi(M)$ and A is a symmetric $(0,2)$-tensor field.

Now, the homomorphisms $R(X,Y)\cdot R$, $R(X,Y)\cdot S$ and the endomorphisms $(X \wedge_A Y)\cdot R$, $(X \wedge_A Y)\cdot S$ are defined by

$$\begin{aligned}(R(X,Y)\cdot R)(U,V)W = {} & R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ & - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W,\end{aligned} \tag{2.4}$$

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V), \tag{2.5}$$

$$\begin{aligned}&((X\wedge_A Y)\cdot R)(U,V)W = \\ &= (X\wedge_A Y)R(U,V)W - R((X\wedge_A Y)U,V)W - \\ &\quad - R(U,(X\wedge_A Y)V)W - R(U,V)(X\wedge_A Y)W,\end{aligned} \tag{2.6}$$

$$((X\wedge_A Y)\cdot S)(U,V) = -S((X\wedge_A Y)U,V) - S(U,(X\wedge_A Y)V), \tag{2.7}$$

respectively, where $X,Y,U,V,W \in \chi(M)$ and A is a symmetric $(0,2)$-tensor field on (M,g). For the case $A = S$ the last equation vanishes, i.e.

$$((X\wedge_A Y)\cdot S)(U,V) = 0. \tag{2.8}$$

From now on we assume that M^{2n+1} is an $(2n+1)$-dimensional Riemannian manifold of class C^∞. The manifold M^{2n+1} is said to admit an *almost contact structure*, sometimes called a (ϕ,ξ,η)-*structure*, if it admits a tensor field ϕ of type $(1,1)$ a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0,\ \eta\circ\phi = 0. \tag{2.9}$$

An almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $M^{2n+1}\times\mathbb{R}$ defined by

$$J\left(X,\lambda\frac{d}{dt}\right) = \left(\phi X - \lambda\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, where X is tangent to M^{2n+1}, t the coordinate of $\mathbb{R}$ and λ a smooth function on $M^{2n+1}\times\mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ,ξ,η), that is,

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.10}$$

or equivalently,

$$g(X,\phi Y) = -g(\phi X,Y) \quad \text{and} \quad \eta(X) = g(X,\xi)$$

for all $X,Y \in TM^{2n+1}$. Then, M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g).

An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y).$$

The 1-form η is then a contact form and ξ is its characteristic vector field. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [4]. On the other hand, we have on a Sasakian manifold [3]

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.11}$$

In [4], Blair, Koufogiorgos and Papantoniou considered the (k, μ)-nullity condition on a contact metric manifold M^{2n+1}. The (k, μ)-*nullity distribution* of a contact manifold M is a distribution

$$N(k, \mu) : p \longrightarrow N_p(k, \mu) =$$
$$= \left\{ \begin{array}{ll} W \in T_pM \mid R(X, Y)W & = k\,[g(Y, W)X - g(X, W)Y] \\ & +\mu\,[g(Y, W)hX - g(X, W)hY] \end{array} \right\},$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$ and $k \leq 1$. For more details see also [4], [13].

In particular a contact metric manifold M is Sasakian if and only if $k = 1$ and, consequently, $\mu = 0$ [4].

Furthermore, in a (k, μ)-contact manifold

$$S(X, \xi) = 2nk\eta(X), \tag{2.12}$$
$$Q\xi = 2nk\xi, \tag{2.13}$$
$$h^2 = (k-1)\phi^2, \tag{2.14}$$
$$R(X, Y)\xi = k\big\{\eta(Y)X - \eta(X)Y\big\} + \mu\left\{\eta(Y)hX - \eta(X)hY\right\}, \tag{2.15}$$
$$R(\xi, X)Y = k\big\{g(X, Y)\xi - \eta(Y)X\big\} + \mu\left\{g(hX, Y)\xi - \eta(Y)hX\right\}, \tag{2.16}$$

holds, where Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$.

If $\mu = 0$, the (k, μ)-nullity distribution $N(k, \mu)$ is reduced to the *k-nullity distribution* [16], where k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by

$$N(k) : p \longrightarrow N_p(k) = \{W \in T_pM \mid R(X, Y)W = k(X \wedge_g Y)W\}. \tag{2.17}$$

If $\xi \in N(k)$, then we call a contact metric manifold M an $N(k)$-*contact metric manifold*. For a $N(k)$-contact metric manifold the equations (2.15) and (2.16) reduce to

$$R(X, Y)\xi = k\left\{\eta(Y)X - \eta(X)Y\right\}, \tag{2.18}$$

$$R(\xi, X)Y = k\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.19}$$

respectively. If $k = 1$ then an $N(k)$-contact metric manifold is Sasakian and if $k = 0$ then an $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. In [1], $N(k)$-contact metric manifolds were studied in some detail. In particular, if $k < 1$, the scalar curvature is $\tau = 2n(2n - 2 + k)$.

Using (1.1), (1.2), (2.18) and (2.19) for an $N(k)$-contact metric manifold we have the followings:

$$P(\xi, X)Y = kg(X,Y)\xi - \frac{1}{2n}S(X,Y)\xi, \tag{2.20}$$

$$P(X,Y)\xi = 0, \tag{2.21}$$

$$Z(\xi, X)Y = \left(k - \frac{\tau}{2n(2n+1)}\right)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.22}$$

$$Z(X,Y)\xi = \left(k - \frac{\tau}{2n(2n+1)}\right)\{\eta(Y)X - \eta(X)Y\}. \tag{2.23}$$

The standard contact metric structure on the tangent sphere bundle T_1M satisfies the (k,μ)-nullity condition if and only if the base manifold M is of constant curvature. In particular if M has constant curvature c, then $k = c(2-c)$ and $\mu = -2c$.

We also recall the notion of a $\mathcal{D}$-*homothetic deformation.* For a given contact metric structure (φ, ξ, η, g), this is the structure defined by

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant. While such a change preserves the state of being contact metric, K-contact, Sasakian or strongly pseudo-convex CR, it destroys a condition like $R(X,Y)\xi = 0$ or

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\}.$$

However the form of the (k,μ)-nullity condition is preserved under a $\mathcal{D}$-homothetic deformation with

$$\bar{k} = \frac{k + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian (k,μ)-manifold M, E. Boeckx [7] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-k}}$$

and showed that for two non-Sasakian (k, μ)-manifolds $(M_i, \varphi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, we have $I_{M_1} = I_{M_2}$ if and only if up to a $\mathcal{D}$-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Thus we know all non-Sasakian (k, μ)-manifolds locally as soon as we have for every odd dimension $2n + 1$ and for every possible value of the invariant I, one (k, μ)-manifold $(M, \varphi, \xi, \eta, g)$ with $I_M = I$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c, where we have $I = \frac{1+c}{|1-c|}$. E. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I \leq -1$.

In Th. 6 of the present paper we need the following example.

Example 1 [7]. Since the Boeckx invariant for a $(1 - \frac{1}{n}, 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n + 1)$-dimensional manifold of constant curvature c, so chosen that the resulting $\mathcal{D}$-homothetic deformation will be a $(1 - \frac{1}{n}, 0)$-manifold. That is for $k = c(2 - c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a}$$

for a and c. The result is

$$c = \frac{(\sqrt{n} \pm 1)^2}{n - 1}, \quad a = 1 + c$$

and taking c and a to be these values we obtain an $N(1 - \frac{1}{n})$-contact metric manifold.

3. $N(k)$-contact metric manifolds satisfying some curvature conditions

In the present section we consider $N(k)$-contact metric manifold M^{2n+1} satisfying the curvature conditions $P(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot P = 0$, $P(\xi, X) \cdot S = 0$, $P(\xi, X) \cdot P = 0$ and $P(\xi, X) \cdot Z = 0$. First we recall the following result:

Theorem 1 [14]. *Let M be a contact Riemmanian manifold. If*

a) $R(X, \xi) \cdot S = 0$,

b) $R(X, Y)\xi = k \{\eta(Y)X - \eta(X)Y\}$,

then M is either locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$ or M is an Einstein–Sasakian manifold.

Now we give the following main results:

Theorem 2. *Let M be a $(2n+1)$-dimensional $N(k)$-contact metric manifold that satisfies*

$$P(\xi, X) \cdot R = 0$$

then M is locally isometric to the product $E^{n+1} \times S^n(4)$ or is an Einstein manifold. Furthermore if M is an Einstein manifold then $P(\xi, X) \cdot R = 0$.

Proof. Let M^{2n+1} be an $N(k)$-contact metric manifold satisfying $P(\xi, X) \cdot R = 0$.

By (2.4), we can write

$$(P(\xi,X)\cdot R)(U,V)W = \\ = P(\xi,X)R(U,V)W - R(P(\xi,X)U,V)W - \\ - R(U,P(\xi,X)V)W - R(U,V)P(\xi,X)W = 0, \tag{3.1}$$

where $X, U, V, W \in \chi(M)$. Developing the right-hand side of (3.1) and using the hypothesis and (2.18), (2.19), (2.12), (1.2), (2.3) we have

(3.2)

$$kg(R(U,V)W,X)\xi - \frac{1}{2n}S(X,R(U,V)W)\xi - k^2 g(X,U)g(V,W)\xi + \\ + k^2 g(X,U)\eta(W)V + \frac{k}{2n}S(X,U)g(V,W)\xi - \frac{k}{2n}S(X,U)\eta(W)V + \\ + k^2 g(X,V)g(U,W)\xi - k^2 g(X,V)\eta(W)U - \frac{k}{2n}S(X,V)g(U,W)\xi + \\ + \frac{k}{2n}S(X,V)\eta(W)U + k^2 g(X,W)\eta(U)V - k^2 g(X,W)\eta(V)U + \\ + \frac{k}{2n}S(X,W)\eta(V)U - \frac{k}{2n}S(X,W)\eta(U)V = 0.$$

Taking the inner product with ξ in (3.2), again using (2.18) and (2.19), we get

$$kg(R(U,V)W,X) - \frac{1}{2n}S(X,R(U,V)W) - \\ - k^2 g(X,U)g(V,W) + k^2 g(X,U)\eta(W)\eta(V) + \\ + \frac{k}{2n}S(X,U)g(V,W) - \frac{k}{2n}S(X,U)\eta(W)\eta(V) + \\ + k^2 g(X,V)g(U,W) - k^2 g(X,V)\eta(W)\eta(U) - \\ - \frac{k}{2n}S(X,V)g(U,W) + \frac{k}{2n}S(X,V)\eta(W)\eta(U) = 0. \tag{3.3}$$

Substituting U by ξ in (3.3) and using (2.18), we obtain

$$\frac{k}{2n}S(X,V)\eta(W) - k^2 g(X,V)\eta(W) = 0. \tag{3.4}$$

If $k = 0$ then M^{2n+1} is locally isometric to the product $E^{n+1} \times S^n(4)$. If $k \neq 0$, again substituting W by ξ in (3.4), we get

$$S(X,V) = 2nkg(X,V). \tag{3.5}$$

Thus M^{2n+1} is an Einstein manifold.

Conversely, if M is an Einstein manifold then (3.5) holds by virtue of (2.13). Hence we substitute (3.5) in (2.20) and get $P(\xi,X)\cdot R = 0$. Our theorem is thus proved. $\lozenge$

Theorem 3. *Let a $(2n+1)$-dimensional $N(k)$-contact metric manifold M satisfies*

$$R(\xi,X)\cdot P = 0$$

then either M^{2n+1} is locally isometric to the product $E^{n+1}\times S^n(4)$ or M^{2n+1} is an Einstein manifold.

Proof. Let M^{2n+1} be an $N(k)$-contact metric manifold such that $R(\xi,X)\cdot P = 0$. By (2.4), we get

$$\begin{aligned} &R(\xi,X)P(U,V)W - P(R(\xi,X)U,V)W - \\ &- P(U,R(\xi,X)V)W - P(U,V)R(\xi,X)W = 0. \end{aligned}$$

Using (1.2), (2.18) and (2.19), we get

$$\begin{aligned} &kg(P(U,V)W,X)\xi - k\eta(P(U,V)W)X - kg(X,U)P(\xi,V)W + \\ &+ k\eta(U)P(X,V)W - kg(X,V)P(U,\xi)W + k\eta(V)P(U,X)W - \\ &- kg(X,W)P(U,V)\xi + k\eta(W)P(U,V)X = 0. \end{aligned} \tag{3.6}$$

By (2.21), we have

$$\begin{aligned} &kg(P(U,V)W,X)\xi - k\eta(P(U,V)W)X - kg(X,U)P(\xi,V)W + \\ &+ k\eta(U)P(X,V)W - kg(X,V)P(U,\xi)W + k\eta(V)P(U,X)W + \\ &+ k\eta(W)P(U,V)X = 0. \end{aligned} \tag{3.7}$$

Taking the inner product with ξ in (3.7), we get

$$\begin{aligned} &kg(P(U,V)W,X) - k\eta(P(U,V)W)\eta(X) - kg(X,U)\eta(P(\xi,V)W) + \\ &+ k\eta(U)\eta(P(X,V)W) - kg(X,V)\eta(P(U,\xi)W) + \\ &+ k\eta(V)\eta(P(U,X)W) + k\eta(W)\eta(P(U,V)X) = 0. \end{aligned} \tag{3.8}$$

Using (2.12) and (2.19) in (3.8), we get

$$\begin{aligned}(3.9)\qquad & kg(R(U,V)W,X)-k^2g(X,U)\eta(W)\eta(V)+\\ &+k^2g(X,V)\eta(W)\eta(U)-\frac{k}{2n}S(X,V)\eta(W)\eta(U)+\\ &+\frac{k}{2n}S(X,U)\eta(W)\eta(V)-k\eta(X)g(R(U,V)W,\xi)+\\ &+k\eta(U)g(R(X,V)W,\xi)+k\eta(V)g(R(U,X)W,\xi)+\\ &+k\eta(W)g(R(U,V)X,\xi)-kg(X,U)g(R(\xi,V)W,\xi)+\\ &+kg(X,V)g(R(\xi,U)W,\xi)-\frac{k}{2n}S(X,W)\eta(U)\eta(V)+\\ &+\frac{k}{2n}S(W,U)\eta(X)\eta(V)=0.\end{aligned}$$

Let $\{\tilde{e}_i : i = 1,\dots,2n+1\}$ be an orthonormal ϕ-basis of vector fields in M^{2n+1}. If we put $V = W = \tilde{e}_i$ in (3.9) and sum up with respect to i and using (2.12), (2.13), (2.18) and (2.19), then we get

$$(3.10)\qquad k\left[\left(1+\frac{1}{2n}\right)S(U,X)-(2n+1)kg(U,X)\right]=0.$$

If $k = 0$ then M is locally isometric to the product $E^{n+1} \times S^n(4)$. If $k \neq 0$, from (3.10), we have

$$(3.11)\qquad S(U,W)=2nkg(U,W),$$

which means that M^{2n+1} is an Einstein manifold.

Theorem 4. *If a $(2n+1)$-dimensional $N(k)$-contact metric manifold M satisfies*

$$P(\xi,X)\cdot S=0$$

then either M is locally isometric to the product $E^{n+1} \times S^n(4)$ or M is an Einstein–Sasakian manifold.

Proof. Let M^{2n+1} be an $N(k)$-contact metric manifold. By the equations (2.8) and (1.2) the condition $P(\xi,X)\cdot S = 0$ turns into

$$(R(\xi,X)\cdot S)(U,V)=\frac{1}{2n}((\xi\wedge_S X)\cdot S)(U,V)=0.$$

Hence using Th. 1, we get the result. Thus, our theorem is proved. ♢

Theorem 5. *If a $(2n+1)$-dimensional $N(k)$-contact metric manifold M satisfies*

$$P(\xi, X) \cdot P = 0$$

then the condition

$$S^2(X, U) = 4nkS(X, U) - 4n^2k^2g(X, U),$$

holds on M.

Proof. Let M^{2n+1} be an $N(k)$-contact metric manifold satisfying $P(\xi, X) \cdot P = 0$. Then we get

(3.12)
$$(P(\xi, X) \cdot P)(U, V)W = P(\xi, X)P(U, V)W - P(P(\xi, X)U, V)W - \\ - P(U, P(\xi, X)V)W - P(U, V)P(\xi, X)W = 0.$$

Using (2.4), (2.18) and (2.19) in (3.12), and taking the inner product with ξ we get

$$kg(X, R(U, V)W) - \frac{1}{2n}S(X, P(U, V)W) + \frac{k}{2n}g(X, V)S(U, W) - \\ - k^2g(X, U)g(V, W) + \frac{k}{2n}S(X, U)g(V, W) + k^2g(X, V)g(U, W) - \\ - \frac{k}{2n}g(X, V)S(U, W) - \frac{k}{2n}S(X, V)g(U, W) = 0. \tag{3.13}$$

Let $\{\tilde{e}_i : i = 1, \ldots, 2n + 1\}$ be a orthonormal ϕ-basis of vector fields in M^{2n+1}. If we put $V = W = \tilde{e}_i$ in (3.13) and sum up with respect to i and using (2.12) and (2.13), then we get

$$S^2(X, U) = 4nkS(X, U) - 4n^2k^2g(X, U). \tag{3.14}$$

Lemma 1 [10]. *Let A be a symmetric $(0, 2)$-tensor at point x of a semi-Riemannian manifold (M, g), $\dim(M) \geq 3$, and let $T = g \barwedge A$ be the Kulkarni–Nomizu product of g and A. Then, the relation*

$$T \cdot T = \alpha Q(g, T), \quad \alpha \in \mathbb{R}$$

is satisfied at x if and only if the condition

$$A^2 = \alpha A + \lambda g, \quad \lambda \in \mathbb{R}$$

holds at x.

Corollary 1. *Let M^{2n+1} be a $N(k)$-contact metric manifold satisfying the condition $P(\xi, X) \cdot P = 0$ then $T \cdot T = \alpha Q(g, T)$, where $T = g \barwedge S$ and $\alpha = 4nk$.*

Regarding the concircular curvature tensor we have:

Theorem 6. *If a $(2n+1)$-dimensional non-Sasakian $N(k)$-contact metric manifold M satisfies*

$$P(\xi, X)\cdot Z = 0$$

then either M is locally isometric to the manifold of Ex. 1 or M is an Einstein manifold.

Proof. Let M^{2n+1} be an $N(k)$-contact metric manifold satisfying $P(\xi, X)\cdot Z = 0$. Then we can write

$$\begin{aligned}(P(\xi, X)\cdot Z)(U,V)W &= \\ &= P(\xi,X)Z(U,V)W - Z(P(\xi,X)U,V)W \\ &\quad - Z(U,P(\xi,X)V)W - Z(U,V)P(\xi,X)W = 0,\end{aligned} \tag{3.15}$$

where $X, U, V, W \in \chi(M)$. Using (2.18) in (3.15), we have

$$\begin{aligned}& kg(X, Z(U,V)W)\xi - \frac{1}{2n}S(X,Z(U,V)W)\xi - \\ & - kg(X,U)Z(\xi,V)W + \frac{1}{2n}S(X,U)Z(\xi,V)W - \\ & - kg(X,V)Z(U,\xi)W + \frac{1}{2n}S(X,V)Z(U,\xi)W - \\ & - kg(X,W)Z(U,V)\xi + \frac{1}{2n}S(X,W)Z(U,V)\xi = 0.\end{aligned} \tag{3.16}$$

Taking the inner product with ξ in (3.16) and substituting V by ξ and using (2.22) and (2.23), we get

$$\tilde{A}kg(X,U)\eta(W) - \frac{\tilde{A}}{2n}S(X,U)\eta(W) = 0, \tag{3.17}$$

where $\tilde{A} = k - \frac{\tau}{2n(2n+1)}$. Again substituting W by ξ in (3.17) , we get

$$\tilde{A}[kg(X,U) - \frac{1}{2n}S(X,U)] = 0.$$

Therefore either $\tilde{A} = 0$ or

$$S(X,U) = 2nkg(X,U).$$

Thus M^{2n+1} is an Einstein manifold if $\tilde{A} \neq 0$.

If $\tilde{A} = 0$, then we have $\tau = 2n(2n+1)k$. Thus, we recall that the scalar curvature of an $N(k)$-contact metric manifold is $\tau = 2n(2n-2+k)$. Comparing $k = 1 - \frac{1}{n}$ and hence M is locally isometric to the manifold of Ex. 1 for $n > 1$ and to the flat case if $n = 1$.

4. Weyl projective recurrent contact metric manifolds

A non-flat Riemannian manifold M is said to be *Weyl projective recurrent* if the Weyl projective curvature tensor P satisfies the condition

$$\nabla P = A \otimes P, \tag{4.1}$$

where A is an everywhere non-zero 1-form [12].

Then, we prove the following theorem:

Theorem 7. *An $N(k)$-contact metric manifold with non-vanishing recurrent Weyl curvature tensor does not exist.*

Proof. If possible, let M^{2n+1} be an $N(k)$-contact metric manifold with non-vanishing recurrent Weyl projective curvature tensor. Then from (4.1), we get

$$\nabla_X \nabla_Y P = (XA(Y) + A(X)A(Y))P,$$

which implies that

$$R(X,Y) \cdot P = 2dA(X,Y)P. \tag{4.2}$$

We define a function f on M^{2n+1} by $f^2 = g(P,P)$, where g is the usual extension to the inner product between the tensor fields . Since Riemannian metric tensor is parallel, by (4.1) and (4.2) it follows that

$$f(Xf) = f^2 A(X),$$

or,

$$Xf = fA(X). \tag{4.3}$$

By (4.3), it follows that

$$\begin{aligned} 2dA(X,Y)f &= (XA(Y) - YA(X) - A([X,Y]))f \\ &= (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})f \\ &= 0. \end{aligned} \tag{4.4}$$

Since f is non-vanishing by assumption, the 1-form A has to be closed. Thus, by (4.2) and (4.3) we get $R(X,Y) \cdot P = 0$, which in view of Th. 3 and the assumption non-vanishing of P, shows that M^{2n+1} is locally isometric to the product $E^{n+1} \times S^n(4)$. But $E^{n+1}(0) \times S^n(4)$ satisfies $\nabla P = 0$ [5], hence our assumption is not possible. $\Diamond$

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