

## QUANTUM FUNCTOR $\mathfrak{Mor}$

Maysam Maysami **Sadr**

*Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), P. O. Box 45195-1159 Zanjan, Iran*

*Received:* May 2009

*MSC 2000:* 46 L 05, 46 L 85, 46 M 20

*Keywords:* Category of C\*-algebras, quantum family of maps, non commutative algebraic topology.

**Abstract:** Let  $\mathbf{Top}_c$  be the category of compact spaces and continuous maps and  $\mathbf{Top}_f \subset \mathbf{Top}_c$  be the full subcategory of finite spaces. Consider the covariant functor  $Mor : \mathbf{Top}_f^{op} \times \mathbf{Top}_c \longrightarrow \mathbf{Top}_c$  that associates any pair  $(X, Y)$  with the space of all morphisms from  $X$  to  $Y$ . In this paper, we describe a non commutative version of  $Mor$ . More pricelessly, we define a functor  $\mathfrak{Mor}$ , that takes any pair  $(B, C)$  of a finitely generated unital C\*-algebra  $B$  and a finite dimensional C\*-algebra  $C$  to the quantum family of all morphism from  $B$  to  $C$ . As an application we introduce a non commutative version of path functor.

### 1. Introduction

Let  $\mathbf{Top}$  be the category of topological spaces and continuous maps. For every  $X, Y \in \mathbf{Top}$ , denote by  $Mor(X, Y)$  the set of all morphisms (continuous maps) from  $X$  to  $Y$ . Also for spaces  $X_1, X_2, Y_1, Y_2$  and morphisms  $f : X_2 \longrightarrow X_1, g : Y_1 \longrightarrow Y_2$ , denote by  $Mor(f, g)$  the map  $h \longmapsto ghf$  from  $Mor(X_1, Y_1)$  to  $Mor(X_2, Y_2)$ . Then  $Mor$  can be considered as a covariant functor from the product category  $\mathbf{Top}^{op} \times \mathbf{Top}$  to  $\mathbf{Top}$ , where  $Mor(X, Y)$  has the compact-open topology. Now, suppose that  $\mathbf{Top}_c \subset \mathbf{Top}$  and  $\mathbf{Top}_f \subset \mathbf{Top}$  are the full subcategories of compact Hausdorff spaces and finite discrete spaces, respectively. Then

---

*E-mail address:* sadr@iasbs.ac.ir

the restriction of  $Mor$  to  $\mathbf{Top}_f^{op} \times \mathbf{Top}_c$  takes its values in  $\mathbf{Top}_c$ :

$$Mor : \mathbf{Top}_f^{op} \times \mathbf{Top}_c \longrightarrow \mathbf{Top}_c.$$

The aim of this paper is description of a *quantum (non commutative)* version of  $Mor$ . We define a covariant functor

$$\mathfrak{Mor} : \mathbf{C}_{fg}^* \times \mathbf{C}_{fd}^{*op} \longrightarrow \mathbf{C}_{fg}^*,$$

where  $\mathbf{C}_{fg}^*$  is the category of finitely generated unital  $C^*$ -algebras and  $\mathbf{C}_{fd}^*$  is the full subcategory of finite dimensional  $C^*$ -algebras, such that for  $C^*$ -algebras  $B, C$ ,  $\mathfrak{Mor}(B, C)$  is the quantum family of all morphisms from  $B$  to  $C$ . For clearness of the idea behind this definition, we recall some basic terminology of Non Commutative Geometry.

Let  $\mathbf{C}^*$  be the category of unital  $C^*$ -algebras and unital  $*$ -homomorphisms and let  $\mathbf{C}_{com}^*$  be the full subcategory of commutative algebras. For every  $X \in \mathbf{Top}_c$  and  $A \in \mathbf{C}_{com}^*$ , let  $\mathfrak{f}X$  and  $\mathfrak{q}A$  be the  $C^*$ -algebra of continuous complex valued maps on  $X$  and the spectrum of  $A$  with  $w^*$  topology, respectively. Then the famous Gelfand Theorem says that  $\mathbf{Top}_c$  and  $\mathbf{C}_{com}^*$  are dual of each other, under functors  $\mathfrak{f}$  and  $\mathfrak{q}$ . Thus for every  $A \in \mathbf{C}^*$ , one can consider a *symbolic* notion  $\mathfrak{q}A$  of a quantum or non commutative space that its algebra of functions ( $\mathfrak{f}\mathfrak{q}A$ ) is  $A$ . In this correspondence, for  $A \in \mathbf{C}_{fg}^*$  and  $B \in \mathbf{C}_{fd}^*$ ,  $\mathfrak{q}A$  and  $\mathfrak{q}B$  are called *finite dimensional compact quantum space* and *discrete finite quantum space*, respectively.

Now, suppose that  $X, Y, Z$  are topological spaces and  $f : X \longrightarrow Y$  is a continuous map. One can redefine the map  $f$  in a useful manner:

$$f : X \times Z \longrightarrow Y \quad (x, z) \longmapsto f(x)(z).$$

Thus, one can consider a *family*  $\{f(x)\}_{x \in X}$  of maps from  $Z$  to  $Y$  *parameterized* by  $f$  and with *parameters*  $x$  in  $X$ , as a map from  $X \times Z$  to  $Y$ . Using this, Woronowicz [6] defined the notion of *quantum family of maps* from quantum space  $\mathfrak{q}C$  to quantum space  $\mathfrak{q}B$  as a pair  $(\mathfrak{q}A, \Phi)$ , where  $\Phi : B \longrightarrow C \otimes A$  is a  $*$ -homomorphism and  $\otimes$  denotes the minimal tensor product of  $C^*$ -algebras. (Note that Woronowicz has used the terminology “pseudo space” instead of “quantum space”. Our terminology here is from Soltan’s paper [4].) Also, using a universal property, he defined the notion of quantum family of *all* maps, that may or may not exist in general case, see Sec. 2.

Now for the definition of  $\mathfrak{Mor}$ , we let  $\mathfrak{Mor}(B, C)$  be the quantum family of all maps from  $\mathfrak{q}C$  to  $\mathfrak{q}B$ .

In the following sections, we use the terminology “quantum family of morphisms from  $B$  to  $C$ ” instead of “quantum family of maps from  $\mathfrak{q}C$  to  $\mathfrak{q}B$ ”.

## 2. Quantum family of morphisms

Let  $B, B', C, C'$  be unital  $C^*$ -algebras. We denote by  $B \otimes C$  and  $B \oplus C$  the minimal tensor product and direct sum, respectively. For  $*$ -homomorphisms  $\Phi : B \rightarrow C$  and  $\Phi' : B' \rightarrow C'$ ,  $\Phi \otimes \Phi'$  denotes the natural homomorphism from  $B \otimes B'$  to  $C \otimes C'$  defined by  $b \otimes b' \rightarrow \Phi(b) \otimes \Phi'(b')$  ( $b \in B, b' \in B'$ ). Also, let  $\Phi \oplus \Phi' : B \oplus B' \rightarrow C \oplus C'$  be the homomorphism defined by  $\Phi \oplus \Phi'(b, b') = (\Phi(b), \Phi'(b'))$ . Denote by  $C^\circ$ , the space of all bounded functionals on  $C$ , and by  $1_C$  the unite element of  $C$ .

A quantum family  $(A, \Phi)$  of morphisms from  $B$  to  $C$  consists of a unital  $C^*$ -algebra  $A$  and a unital  $*$ -homomorphism  $\Phi : B \rightarrow C \otimes A$ .

**Definition 1.** Let  $B, C$  and  $(A, \Phi)$  be as above. Then  $(A, \Phi)$  is called a quantum family of all morphisms from  $B$  to  $C$  if for every unital  $C^*$ -algebra  $D$  and any unital  $*$ -homomorphism  $\Psi : B \rightarrow C \otimes D$ , there is a unique unital  $*$ -homomorphism  $\Lambda : A \rightarrow D$  such that the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\Phi} & C \otimes A \\ \parallel & & \downarrow id_C \otimes \Lambda \\ B & \xrightarrow{\Psi} & C \otimes D \end{array}$$

If  $(A, \Phi)$  and  $(A', \Phi')$  are two quantum families of all morphisms from  $B$  to  $C$ , then by the above universal property there is a isometric  $*$ -isomorphism between  $A$  and  $A'$ .

**Theorem 2.** *Let  $B$  be a finitely generated unital  $C^*$ -algebra and  $C$  be a finite dimensional  $C^*$ -algebra. Then the quantum family  $(A, \Phi)$  of all morphisms from  $B$  to  $C$  exists. Also  $A$  is finitely generated (and unital) and the set  $G = \{(\omega \otimes id_A)\Phi(b) : b \in B, \omega \in C^\circ\}$  is a generator for  $A$  (the smallest closed  $*$ -subalgebra of  $A$  containing  $G$  is equal to  $A$ ).*

**Proof.** See [4].  $\diamond$

For examples of quantum families of morphisms see, [6] and [4].

### 3. Definition of the functor

As previously, let  $\mathbf{C}^*$  be the category of unital  $\mathbf{C}^*$ -algebras and unital homomorphisms. Also, let  $\mathbf{C}_{fg}^* \subset \mathbf{C}^*$  and  $\mathbf{C}_{fd}^* \subset \mathbf{C}^*$  be the full subcategories of finitely generated and finite dimensional  $\mathbf{C}^*$ -algebras, respectively. For  $B_1, B_2 \in \mathbf{C}^*$ , denote by  $Mor(B_1, B_2)$  the set of all morphisms from  $B_1$  to  $B_2$  in  $\mathbf{C}^*$ . For more details on the category of  $\mathbf{C}^*$ -algebras, see [6]. Note that by elementary results of the theory of  $\mathbf{C}^*$ -algebras, a morphism  $f \in Mor(B_1, B_2)$  is an isomorphism in the categorical sense (i.e. there is  $g \in Mor(B_2, B_1)$  such that  $gf = id_{B_1}$  and  $fg = id_{B_2}$ ) if and only if  $f$  is an isometric  $*$ -isomorphism from  $B_1$  onto  $B_2$ .

For any  $B \in \mathbf{C}_{fg}^*$  and every  $C \in \mathbf{C}_{fd}^*$ , let  $(\mathfrak{Mor}(B, C), \mathfrak{P}_{B,C})$  be the quantum family of all morphisms from  $B$  to  $C$  ( $\mathfrak{P}$  stands for “parameter”).

For every  $B_1, B_2 \in \mathbf{C}_{fg}^*$ ,  $C_1, C_2 \in \mathbf{C}_{fd}^*$  and  $f \in Mor(B_1, B_2)$ ,  $g \in Mor(C_2, C_1)$ , let  $\mathfrak{Mor}(f, g)$  be the unique morphism from  $\mathfrak{Mor}(B_1, C_1)$  to  $\mathfrak{Mor}(B_2, C_2)$  in  $\mathbf{C}_{fg}^*$  such that the following diagram is commutative:

$$(1) \quad \begin{array}{ccc} B_1 & \xrightarrow{\mathfrak{P}_{B_1, C_1}} & C_1 \otimes \mathfrak{Mor}(B_1, C_1) \\ \downarrow f & & \downarrow id_{C_1} \otimes \mathfrak{Mor}(f, g) \\ B_2 & \xrightarrow{(g \otimes id_{\mathfrak{Mor}(B_2, C_2)}) \mathfrak{P}_{B_2, C_2}} & C_1 \otimes \mathfrak{Mor}(B_2, C_2) \\ \parallel & & \uparrow g \otimes id_{\mathfrak{Mor}(B_2, C_2)} \\ B_2 & \xrightarrow{\mathfrak{P}_{B_2, C_2}} & C_2 \otimes \mathfrak{Mor}(B_2, C_2) \end{array}$$

Let  $B_3 \in \mathbf{C}_{fg}^*$ ,  $C_3 \in \mathbf{C}_{fd}^*$  and  $f' \in Mor(B_2, B_3)$ ,  $g' \in Mor(C_3, C_2)$ , then by the definition we have the following commutative diagram:

$$(2) \quad \begin{array}{ccc} B_2 & \xrightarrow{\mathfrak{P}_{B_2, C_2}} & C_2 \otimes \mathfrak{Mor}(B_2, C_2) \\ \downarrow f' & & \downarrow id_{C_2} \otimes \mathfrak{Mor}(f', g') \\ B_3 & \xrightarrow{(g' \otimes id_{\mathfrak{Mor}(B_3, C_3)}) \mathfrak{P}_{B_3, C_3}} & C_2 \otimes \mathfrak{Mor}(B_3, C_3). \end{array}$$

Then we have,

$$\begin{aligned}
& (id_{C_1} \otimes \mathfrak{Mor}(f', g')\mathfrak{Mor}(f, g))\mathfrak{P}_{B_1, C_1} \\
&= (id_{C_1} \otimes \mathfrak{Mor}(f', g'))(id_{C_1} \otimes \mathfrak{Mor}(f, g))\mathfrak{P}_{B_1, C_1} \\
&= (id_{C_1} \otimes \mathfrak{Mor}(f', g'))(g \otimes id_{\mathfrak{Mor}(B_2, C_2)})\mathfrak{P}_{B_2, C_2}f \quad (\text{by (1)}) \\
&= (g \otimes id_{\mathfrak{Mor}(B_3, C_3)})(id_{C_2} \otimes \mathfrak{Mor}(f', g'))\mathfrak{P}_{B_2, C_2}f \\
&= (g \otimes id_{\mathfrak{Mor}(B_3, C_3)})(g' \otimes id_{\mathfrak{Mor}(B_3, C_3)})\mathfrak{P}_{B_3, C_3}f'f \quad (\text{by (2)}) \\
&= ((gg') \otimes id_{\mathfrak{Mor}(B_3, C_3)})\mathfrak{P}_{B_3, C_3}(f'f).
\end{aligned}$$

Thus by the uniqueness property of the definition of  $\mathfrak{Mor}(f'f, gg')$  we have,

$$\mathfrak{Mor}(f'f, gg') = \mathfrak{Mor}(f', g')\mathfrak{Mor}(f, g).$$

Also, it is clear that

$$\mathfrak{Mor}(id_B, id_C) = id_{\mathfrak{Mor}(B, C)},$$

for every  $B \in \mathbf{C}_{fg}^*$  and  $C \in \mathbf{C}_{fd}^*$ . Thus we have defined a covariant functor

$$\mathfrak{Mor} : \mathbf{C}_{fg}^* \times \mathbf{C}_{fd}^{*op} \longrightarrow \mathbf{C}_{fg}^*$$

that is called Quantum Functor  $\mathfrak{Mor}$ . We often write  $\mathfrak{M}$  instead of  $\mathfrak{Mor}$ .

## 4. Some properties

In this section we prove some basic properties of the functor  $\mathfrak{Mor}$ . We need the following simple lemma.

**Lemma 3.** *Let  $A, A', B, B', C$  be  $C^*$ -algebras,  $\Phi, \Phi', \Lambda, \Gamma$  be  $*$ -homomorphisms and  $\omega \in C^\circ$  such that the following diagram is commutative:*

$$\begin{array}{ccc}
B & \xrightarrow{\Phi} & C \otimes A \\
\downarrow \Lambda & & \downarrow id_C \otimes \Gamma \\
B' & \xrightarrow{\Phi'} & C \otimes A'
\end{array}$$

Then for any  $b \in B$ , we have

$$\Gamma(\omega \otimes id_A)\Phi(b) = (\omega \otimes id_{A'})\Phi'\Lambda(b).$$

**Proof.** By commutativity of the diagram we have

$$(id_C \otimes \Gamma)\Phi(b) = \Phi'\Lambda(b),$$

and thus

$$(\omega \otimes id_{A'})(id_C \otimes \Gamma)\Phi(b) = (\omega \otimes id_{A'})\Phi'\Lambda(b).$$

Then the left-hand side of the latter equation is equal to  $\Gamma(\omega \otimes id_A)\Phi(b)$ , since  $(\omega \otimes id_{A'}) (id_C \otimes \Gamma) = \Gamma(\omega \otimes id_A)$ .  $\diamond$

**Theorem 4.** *Let  $B \in \mathbf{C}_{fg}^*$  and  $C_1, C_2 \in \mathbf{C}_{fd}^*$ . Then  $\mathfrak{Mor}(B, C_1 \otimes C_2)$  and  $\mathfrak{Mor}(\mathfrak{Mor}(B, C_1), C_2)$  are canonically isometric  $*$ -isomorphic.*

**Proof.** Let  $\Psi : \mathfrak{M}(B, C_1 \otimes C_2) \longrightarrow C_1 \otimes C_2 \otimes \mathfrak{M}(\mathfrak{M}(B, C_1), C_2)$  be the unique morphism such that the following diagram is commutative:

(3)

$$\begin{array}{ccc} B & \xrightarrow{\mathfrak{P}_{B, C_1 \otimes C_2}} & C_1 \otimes C_2 \otimes \mathfrak{M}(B, C_1 \otimes C_2) \\ \downarrow \mathfrak{P}_{B, C_1} & & \downarrow id_{C_1 \otimes C_2} \otimes \Psi \\ C_1 \otimes \mathfrak{M}(B, C_1) & \xrightarrow{id_{C_1} \otimes \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2}} & C_1 \otimes C_2 \otimes \mathfrak{M}(\mathfrak{M}(B, C_1), C_2). \end{array}$$

Suppose that the morphisms

$$\Gamma : \mathfrak{M}(B, C_1) \longrightarrow C_2 \otimes \mathfrak{M}(B, C_1 \otimes C_2)$$

and

$$\Psi' : \mathfrak{M}(\mathfrak{M}(B, C_1), C_2) \longrightarrow \mathfrak{M}(B, C_1 \otimes C_2)$$

are the unique morphisms such that the following diagrams are commutative.

$$(4) \quad \begin{array}{ccc} B & \xrightarrow{\mathfrak{P}_{B, C_1}} & C_1 \otimes \mathfrak{M}(B, C_1) \\ \parallel & & \downarrow id_{C_1} \otimes \Gamma \\ B & \xrightarrow{\mathfrak{P}_{B, C_1 \otimes C_2}} & C_1 \otimes C_2 \otimes \mathfrak{M}(B, C_1 \otimes C_2), \end{array}$$

$$(5) \quad \begin{array}{ccc} \mathfrak{M}(B, C_1) & \xrightarrow{\mathfrak{P}_{\mathfrak{M}(B, C_1), C_2}} & C_2 \otimes \mathfrak{M}(\mathfrak{M}(B, C_1), C_2) \\ \parallel & & \downarrow id_{C_2} \otimes \Psi' \\ \mathfrak{M}(B, C_1) & \xrightarrow{\Gamma} & C_2 \otimes \mathfrak{M}(B, C_1 \otimes C_2). \end{array}$$

Then commutativity of Diagram (5), implies that

$$(6) \quad id_{C_1} \otimes \Gamma = (id_{C_1 \otimes C_2} \otimes \Psi')(id_{C_1} \otimes \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2}).$$

Now, we have

$$\begin{aligned} & (id_{C_1 \otimes C_2} \otimes (\Psi' \Psi)) \mathfrak{P}_{B, C_1 \otimes C_2} \\ &= (id_{C_1 \otimes C_2} \otimes \Psi')(id_{C_1 \otimes C_2} \otimes \Psi) \mathfrak{P}_{B, C_1 \otimes C_2} \\ &= (id_{C_1 \otimes C_2} \otimes \Psi')(id_{C_1} \otimes \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2}) \mathfrak{P}_{B, C_1} \quad (\text{by (3)}) \\ &= (id_{C_1} \otimes \Gamma) \mathfrak{P}_{B, C_1} \quad (\text{by (6)}) \\ &= \mathfrak{P}_{B, C_1 \otimes C_2}. \quad (\text{by (4)}) \end{aligned}$$

Thus by the universal property of quantum family of all morphisms, we have

$$(7) \quad \Psi' \Psi = id_{\mathfrak{Mor}(B, C_1 \otimes C_2)}$$

Now, we show that

$$(8) \quad (id_{C_2} \otimes \Psi) \Gamma = \mathfrak{P}_{\mathfrak{Mor}(B, C_1), C_2}.$$

By Th. 2, it is sufficient to prove

$$(id_{C_2} \otimes \Psi) \Gamma(a) = \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2}(a),$$

where  $a = (\omega \otimes id_{\mathfrak{M}(B, C_1)}) \mathfrak{P}_{B, C_1}(b)$  for  $\omega \in C_1^\circ$  and  $b \in B$ . We have

$$\begin{aligned} & (id_{C_2} \otimes \Psi) \Gamma(a) \\ &= (id_{C_2} \otimes \Psi) \Gamma(\omega \otimes id_{\mathfrak{M}(B, C_1)}) \mathfrak{P}_{B, C_1}(b) \\ &= (id_{C_2} \otimes \Psi)(\omega \otimes \mathfrak{P}_{B, C_1 \otimes C_2})(b) && \text{(by (4))} \\ &= (\omega \otimes id_{C_2 \otimes \mathfrak{M}(B, C_1), C_2})(id_{C_1} \otimes \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2}) \mathfrak{P}_{B, C_1}(b) && \text{(by Lemma 3)} \\ &= \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2}(\omega \otimes id_{\mathfrak{M}(B, C_1)}) \mathfrak{P}_{B, C_1}(b) \\ &= \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2}(a). \end{aligned}$$

Now, we have

$$\begin{aligned} & (id_{C_2} \otimes (\Psi \Psi')) \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2} \\ &= (id_{C_2} \otimes \Psi)(id_{C_2} \otimes \Psi') \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2} \\ &= (id_{C_2} \otimes \Psi) \Gamma && \text{(by (5))} \\ &= \mathfrak{P}_{\mathfrak{M}(B, C_1), C_2} && \text{(by (8)).} \end{aligned}$$

Thus by the universal property of  $\mathfrak{Mor}$ , we have

$$(9) \quad \Psi \Psi' = id_{\mathfrak{M}(\mathfrak{M}(B, C_1), C_2)}.$$

At last, (7) and (9) show that  $\mathfrak{Mor}(B, C_1 \otimes C_2)$ ,  $\mathfrak{Mor}(\mathfrak{Mor}(B, C_1), C_2)$  are isomorphic in  $\mathbf{C}^*$ .  $\diamond$

The above theorem corresponds to the following fact in  $\mathbf{Top}_c$ :

*Let  $X_1, X_2$  and  $Y$  be compact Hausdorff spaces. Then the map*

$$F : Mor(X_1 \times X_2, Y) \longrightarrow Mor(X_1, Mor(X_2, Y)),$$

*defined by  $(F(f))(x_1)(x_2) = f(x_1, x_2)$  for  $f \in Mor(X_1 \times X_2, Y)$ ,  $x_1 \in X_1$ ,  $x_2 \in X_2$ , is a homeomorphism of topological spaces.*

The proof of this topological fact is elementary. For some general results on this type, see [2].

Let  $X, Y_1, Y_2$  be in **Top** and let  $f : Y_1 \rightarrow Y_2$  be an injective continuous map. Then the morphism  $Mor(id_X, f) : Mor(X, Y_1) \rightarrow Mor(X, Y_2)$ , defined by  $g \mapsto fg$  is also injective. Analogously, in  $\mathbf{C}^*$  we have:

**Theorem 5.** *Let  $B_1, B_2 \in \mathbf{C}_{fg}^*$ ,  $C \in \mathbf{C}_{fd}^*$  and let  $f$  be in  $Mor(B_1, B_2)$ . Suppose that  $f$  is a surjective map. Then  $\mathfrak{Mor}(f, id_C)$  is also surjective.*

**Proof.** By Th. 2, the set

$$G = \{(\omega \otimes id_{\mathfrak{M}(B_2, C)})\mathfrak{P}_{B_2, C}(b) : b \in B_2, \omega \in C^\circ\}$$

is a generator for  $\mathfrak{M}(B_2, C)$ . For every  $b_2 \in B_2$  there is a  $b_1 \in B_1$  such that  $f(b_1) = b_2$ . Thus by Lemma 3, we have

$$G = \{\mathfrak{M}(f, id_C)(\omega \otimes id_{\mathfrak{M}(B_1, C)})\mathfrak{P}_{B_1, C}(b) : b \in B_1, \omega \in C^\circ\}.$$

Thus  $\mathfrak{Mor}(f, id_C) : \mathfrak{Mor}(B_1, C) \rightarrow \mathfrak{Mor}(B_2, C)$  is surjective, since the image of every \*-homomorphism between  $\mathbf{C}^*$ -algebras is closed.  $\diamond$

**Theorem 6.** *Let  $B_1, B_2 \in \mathbf{C}_{fg}^*$  and  $C_1, C_2 \in \mathbf{C}_{fd}^*$ , then there is a canonical morphism  $\Psi$  from  $\mathfrak{Mor}(B_1 \oplus B_2, C_1 \oplus C_2)$  onto  $\mathfrak{Mor}(B_1, C_1) \oplus \mathfrak{Mor}(B_2, C_2)$ .*

**Proof.** Let  $B = B_1 \oplus B_2$  and  $C = C_1 \oplus C_2$ . Let

$$\Gamma : (C_1 \otimes \mathfrak{M}(B_1, C_1)) \oplus (C_2 \otimes \mathfrak{M}(B_2, C_2)) \rightarrow C \otimes (\mathfrak{M}(B_1, C_1) \oplus \mathfrak{M}(B_2, C_2))$$

be the natural embedding. Then, suppose that  $\Psi$  is the unique morphism such that the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\mathfrak{P}_{B, C}} & C \otimes \mathfrak{M}(B, C) \\ \downarrow \mathfrak{P}_{B_1, C_1} \oplus \mathfrak{P}_{B_2, C_2} & & \downarrow id_C \otimes \Psi \\ (C_1 \otimes \mathfrak{M}(B_1, C_1)) \oplus (C_2 \otimes \mathfrak{M}(B_2, C_2)) & \xrightarrow{\Gamma} & C \otimes (\mathfrak{M}(B_1, C_1) \oplus \mathfrak{M}(B_2, C_2)). \end{array}$$

Now we prove that  $\Psi$  is surjective. By Th. 2, the  $\mathbf{C}^*$ -algebra  $\mathfrak{M}(B_1, C_1) \oplus \mathfrak{M}(B_2, C_2)$  is generated by the set  $G$  of all pairs  $(a_1, a_2)$  where

$a_1 = (\omega_1 \otimes id_{\mathfrak{M}(B_1, C_1)})\mathfrak{P}_{B_1, C_1}(b_1)$  and  $a_2 = (\omega_2 \otimes id_{\mathfrak{M}(B_2, C_2)})\mathfrak{P}_{B_2, C_2}(b_2)$  for  $b_1 \in B_1, b_2 \in B_2, \omega_1 \in C_1^\circ, \omega_2 \in C_2^\circ$ . It is easily checked that such pairs are in the form of

$$(a_1, a_2) = ((\omega_1 \oplus \omega_2) \otimes id_{\mathfrak{M}(B_1, C_1) \oplus \mathfrak{M}(B_2, C_2)})\Gamma(\mathfrak{P}_{B_1, C_1} \oplus \mathfrak{P}_{B_2, C_2})(b_1, b_2),$$

and thus by Lemma 3, we can write

$$(a_1, a_2) = \Psi((\omega_1 \oplus \omega_2) \otimes id_{\mathfrak{M}(B, C)})\mathfrak{P}_{B, C}(b_1, b_2).$$

Thus the generator set  $G$  is in the image of  $\Psi$  and therefore  $\Psi$  is surjective.  $\diamond$



Similarly one can prove the following.

**Theorem 7.** *Let  $B_1, \dots, B_n \in \mathbf{C}_{fg}^*$  and let  $C_1, \dots, C_n \in \mathbf{C}_{fd}^*$ . Then there is a canonical surjective \*-homomorphism*  

$$\Psi : \mathfrak{Mor}(B_1 \oplus \dots \oplus B_n, C_1 \oplus \dots \oplus C_n) \longrightarrow \mathfrak{Mor}(B_1, C_1) \oplus \dots \oplus \mathfrak{Mor}(B_n, C_n).$$

The construction appearing in the following theorem, is a special case of the notion of *composition* of quantum families of maps, defined in [4].

**Theorem 8.** *Let  $B \in \mathbf{C}_{fg}^*$  and  $C, D \in \mathbf{C}_{fd}^*$ . Then there is a canonical morphism*

$$\Psi : \mathfrak{Mor}(B, C) \longrightarrow \mathfrak{Mor}(C, D) \otimes \mathfrak{Mor}(B, C).$$

**Proof.** The desired morphism  $\Psi$  is the unique morphism such that the following diagram becomes commutative:

$$\begin{array}{ccc} B & \xrightarrow{\mathfrak{P}_{B,D}} & D \otimes \mathfrak{M}(B, D) \\ \downarrow \mathfrak{P}_{B,C} & & \downarrow id_D \otimes \Psi \\ C \otimes \mathfrak{M}(B, C) & \xrightarrow{\mathfrak{P}_{C,D} \otimes id_{\mathfrak{M}(B,C)}} & D \otimes \mathfrak{M}(C, D) \otimes \mathfrak{M}(B, C). \quad \diamond \end{array}$$

By the latter assumptions, let  $B = C = D = M$  be in  $\mathbf{C}_{fd}^*$ . Then it is proved in [4], that the map

$$\Psi : \mathfrak{Mor}(M, M) \longrightarrow \mathfrak{Mor}(M, M) \otimes \mathfrak{Mor}(M, M)$$

is a coassociative comultiplication, i.e.

$$(id_{\mathfrak{M}(M,M)} \otimes \Psi) \Psi = (\Psi \otimes id_{\mathfrak{M}(M,M)}) \Psi,$$

and thus the pair  $(\mathfrak{M}(M, M), \Psi)$  is a compact quantum semigroup.

**Theorem 9.** *Let  $(\{B_i\}, \{\Phi_{ij}\})_{i \leq j \in I}$  be a directed system in  $\mathbf{C}_{fg}^*$ , on the directed set  $I$ , such that  $B = \varinjlim B_i \in \mathbf{C}_{fg}^*$ . Suppose that  $C \in \mathbf{C}_{fd}^*$ , then the canonical morphism from  $A = \varinjlim \mathfrak{Mor}(B_i, C)$  to  $\mathfrak{Mor}(B, C)$  is surjective.*

**Proof.** For every  $i \in I$ , let  $f_i : B_i \longrightarrow B$  and  $g_i : \mathfrak{M}(B_i, C) \longrightarrow A$  be the canonical morphisms of direct limit structures. Consider the directed system

$$(\{\mathfrak{M}(B_i, C)\}, \{\mathfrak{M}(\Phi_{ij}, id_C)\})_{i \leq j \in I}.$$

Then, for every  $i \leq j$ , we have

$$\mathfrak{M}(f_i, id_C) = \mathfrak{M}(f_j, id_C) \mathfrak{M}(\Phi_{ij}, id_C).$$

Thus by the universal property of direct limit, there is a unique morphism  $\Psi : A \longrightarrow \mathfrak{M}(B, C)$  such that

$$(10) \quad \Psi g_i = \mathfrak{M}(f_i, id_C)$$

for every  $i \in I$ . We must prove that  $\Psi$  is surjective. By Th. 2, the set

$$G = \{(\omega \otimes id_{\mathfrak{M}(B,C)})\mathfrak{P}_{B,C}(b) : b \in B, \omega \in C^\circ\}$$

is a generator for  $\mathfrak{M}(B, C)$ . Let  $b \in B$  and  $\omega \in C^\circ$  be arbitrary and fixed. There are  $i \in I$  and  $b_i \in B_i$  such that  $f_i(b_i) = b$ . Let

$$a_i = (\omega \otimes id_{\mathfrak{M}(B_i,C)})\mathfrak{P}_{B_i,C}(b_i) \in \mathfrak{M}(B_i, C),$$

then by Lemma 3,  $\mathfrak{M}(f_i, id_C)(a_i) = b$ . Now, by (10), we have  $\Psi g_i(a_i) = b$ . Thus  $G$  is in the image of  $\Psi$ , and  $\Psi$  is surjective.  $\diamond$

**Question 10.** Is the map  $\Psi$ , constructed in Th. 9, injective?

**Theorem 11.** *Let  $C$  be a commutative finite dimensional  $C^*$ -algebra and let  $B_1, B_2 \in \mathbf{C}_{fg}^*$ . Then there is a canonical surjective morphism*

$$\Psi : \mathfrak{Mor}(B_1 \otimes B_2, C) \longrightarrow \mathfrak{Mor}(B_1, C) \otimes \mathfrak{Mor}(B_2, C).$$

**Proof.** Let  $m : C \otimes C \longrightarrow C$  be defined by  $m(c_1, c_2) = c_1 c_2$  for  $c_1, c_2 \in C$ . Since  $C$  is commutative,  $m$  is a morphism. Also, let

$$F : C \otimes \mathfrak{M}(B_1, C) \otimes C \otimes \mathfrak{M}(B_2, C) \longrightarrow C \otimes C \otimes \mathfrak{M}(B_1, C) \otimes \mathfrak{M}(B_2, C)$$

be the flip map (i.e.  $c_1 \otimes x_1 \otimes c_2 \otimes x_2 \longmapsto c_1 \otimes c_2 \otimes x_1 \otimes x_2$ ) and let  $\Psi$  be the unique morphism such that the following diagram is commutative:

$$\begin{array}{ccc} B_1 \otimes B_2 & \xrightarrow{\mathfrak{P}_{B_1 \otimes B_2, C}} & C \otimes \mathfrak{M}(B_1 \otimes B_2, C) \\ \downarrow \mathfrak{P}_{B_1, C} \otimes \mathfrak{P}_{B_2, C} & & \downarrow id_C \otimes \Psi \\ C \otimes \mathfrak{M}(B_1, C) \otimes C \otimes \mathfrak{M}(B_2, C) & \xrightarrow{mF} & C \otimes \mathfrak{M}(B_1, C) \otimes \mathfrak{M}(B_2, C). \end{array}$$

Now, we show that  $\Psi$  is surjective. By Th. 2, the set  $G_1$  and  $G_2$ , defined by

$$G_1 = \{(\omega \otimes id_{\mathfrak{M}(B_1,C)})\mathfrak{P}_{B_1,C}(b_1) : b_1 \in B_1, \omega \in C^\circ\}$$

and

$$G_2 = \{(\omega \otimes id_{\mathfrak{M}(B_2,C)})\mathfrak{P}_{B_2,C}(b_2) : b_2 \in B_2, \omega \in C^\circ\}$$

are generator sets of  $\mathfrak{M}(B_1, C)$  and  $\mathfrak{M}(B_2, C)$ , respectively. Thus the set

$$G = \{a_1 \otimes 1_{\mathfrak{M}(B_2,C)}, 1_{\mathfrak{M}(B_1,C)} \otimes a_2 : a_1 \in G_1, a_2 \in G_2\}$$

is a generator for  $\mathfrak{M}(B_1, C) \otimes \mathfrak{M}(B_2, C)$ . On the other hand, for  $b_1 \in B_1$  and  $\omega \in C^\circ$ , we have

$$\begin{aligned} & ((\omega \otimes id_{\mathfrak{M}(B_1,C)})\mathfrak{P}_{B_1,C}(b_1)) \otimes 1_{\mathfrak{M}(B_2,C)} \\ &= (\omega \otimes id_{\mathfrak{M}(B_1,C) \otimes \mathfrak{M}(B_2,C)})mF(\mathfrak{P}_{B_1,C} \otimes \mathfrak{P}_{B_2,C})(b_1 \otimes 1_{\mathfrak{M}(B_2,C)}) \\ &= \Psi(\omega \otimes id_{\mathfrak{M}(B_1 \otimes B_2, C)})\mathfrak{P}_{B_1 \otimes B_2, C}(b_1 \otimes 1_{\mathfrak{M}(B_2,C)}) \quad (\text{by Lemma 3}). \end{aligned}$$

Thus for every  $a_1 \in G_1$ ,  $a_1 \otimes 1_{\mathfrak{M}(B_2,C)}$  is in the image of  $\Psi$ . Similarly, for every  $a_2 \in G_2$ ,  $1_{\mathfrak{M}(B_1,C)} \otimes a_2$  is also in the image of  $\Psi$ . Therefore  $G$  is in the image of  $\Psi$ , and  $\Psi$  is surjective.  $\diamond$

The above result, is similar to the following trivial fact in **Top**:

Let  $Y_1, Y_2, X$  be topological spaces, then the spaces  $Mor(X, Y_1 \times Y_2)$  and  $Mor(X, Y_1) \times Mor(X, Y_2)$  are homeomorphic.

**Question 12.** Is the map  $\Psi$  of Th. 11, injective?

## 5. Symbolic quantum family of paths

In this section we make a suggestion for more research. Let  $B \in \mathbf{C}_{fg}^*$  and  $C \in \mathbf{C}_{fd}^*$ . We have a contravariant functor

$$\mathfrak{Mor}(B, -) : \mathbf{C}_{fd}^* \longrightarrow \mathbf{C}_{fg}^*,$$

and a covariant functor

$$\mathfrak{Mor}(-, C) : \mathbf{C}_{fg}^* \longrightarrow \mathbf{C}_{fd}^*.$$

Thus one can consider any contravariant functor  $\mathfrak{F} : \mathbf{C}_{fd}^* \longrightarrow \mathbf{C}_{fg}^*$ , (resp. covariant functor  $\mathfrak{G} : \mathbf{C}_{fg}^* \longrightarrow \mathbf{C}_{fd}^*$ ) as a generalized notion of a unital (resp. finite dimensional)  $\mathbf{C}^*$ -algebra. Note that since  $\mathfrak{Mor}(B, \mathbb{C}) \cong B$ , where  $\mathbb{C}$  denotes the  $\mathbf{C}^*$ -algebra of complex numbers, one can recover the  $\mathbf{C}^*$ -algebra  $B$  from the data of the functor  $\mathfrak{Mor}(B, -)$ .

Recall from elementary Algebraic Topology ([5]), that for every  $X \in \mathbf{Top}$  the *cylinder* space  $\mathbf{c}X$  of  $X$  is the space  $X \times I$  with product topology and the *path* space  $\mathbf{p}X$  of  $X$  is the space  $Mor(I, X)$  of all continuous maps from  $I$  to  $X$  with compact open topology, where  $I$  is the interval  $0 \leq r \leq 1$ . The covariant functors  $\mathbf{c}$  and  $\mathbf{p}$  are *adjoint* functors on **Top**:

$$(11) \quad Mor(\mathbf{c}X, Y) \cong Mor(X, \mathbf{p}Y),$$

for every topological spaces  $X$  and  $Y$ . For every unital  $\mathbf{C}^*$ -algebra  $A$  the cylinder  $\mathbf{c}A$  of  $A$  is the  $\mathbf{C}^*$ -algebra of all continuous maps  $f : I \longrightarrow A$ , or equivalently  $\mathfrak{f}I \otimes A$ . The cylinder functors  $\mathbf{c} : \mathbf{Top}_c \longrightarrow \mathbf{Top}_c$  and  $\mathbf{c} : \mathbf{C}^* \longrightarrow \mathbf{C}^*$  are *compatible* with respect to Gelfand's duality, that is for every  $X \in \mathbf{Top}_c$  and  $A \in \mathbf{C}_{com}^*$  we have:

$$\mathbf{c}\mathfrak{f}X = \mathfrak{f}\mathbf{c}X, \quad \mathbf{c}\mathfrak{q}A = \mathfrak{q}\mathbf{c}A.$$

As is indicated by Alain Connes there is no interesting notion of *path* or *loop* in non commutative (quantum) spaces (page 544 of [1]). In fact, the natural notion for paths in a quantum space  $\mathfrak{q}A$  is nonzero  $*$ -homomorphisms from  $A$  to  $\mathfrak{f}I$ . But such  $*$ -homomorphisms do not exist for almost all noncommutative spaces. Using the functors  $\mathfrak{Mor}$  and  $\mathbf{c}$ , the generalized notion of  $\mathbf{C}^*$ -algebras as functors, and Identity

(11), we *symbolically* define the concept of quantum space of *paths* in a finite quantum space  $\mathfrak{q}C$ : If we replace  $Mor$ ,  $X$ ,  $Y$  and  $\mathfrak{c}X$  with the functor  $\mathfrak{Mor}$ , unital  $C^*$ -algebra  $B$ , finite dimensional  $C^*$ -algebra  $C$ , and the cylinder  $\mathfrak{c}B$  of  $B$ , respectively, then we have *formally*,

$$(12) \quad \mathfrak{Mor}(\mathfrak{c}B, C) \cong \mathfrak{Mor}(B, \mathfrak{p}C).$$

Therefore one can consider the covariant functor

$$\mathfrak{Mor}(\mathfrak{c}-, C) : \mathbf{C}^* \longrightarrow \mathbf{C}^*$$

as a *symbolic* definition of  $\mathfrak{p}C$ .

**Remark 13.** i) In the above construction, we lost the natural duality between  $\mathbf{Top}_c$  and  $\mathbf{C}_{com}^*$  (note with a more care the replacement of the notations in Formulas (11) and (12)). In the literature of Non Commutative Algebraic Topology, this type of constructions are called *wrong-way* functorial.

ii) Using suspension of  $C^*$ -algebras instead cylinder functor in (12), one can derive a formal definition for the non commutative loop space. But there is a gap in the transformation from commutative to non commutative case. The problem is that, in the commutative case, the (reduced) suspension functor and the loop functor are adjoint on the category of pointed topological spaces and point preserving maps instead of  $\mathbf{Top}$ . Also the standard notion of suspension for  $C^*$ -algebras involves non unital  $C^*$ -algebras.

iii) There are some other notions of loop and path space for algebras [3], that are very far from our construction.

## References

- [1] CONNES, A.: Noncommutative Geometry, Academic Press, 1994.
- [2] JACKSON, J. R.: Spaces of mappings on topological products with applications to homotopy theory, *Proc. Amer. Math. Soc.* **3** (1952), 327–333.
- [3] KAROUBI, M.: K-Théorie, Les presses de l’université de Montreal, Montréal, 1971.
- [4] SOLTAN, P. M.: Quantum families of maps and quantum semigroups on finite quantum spaces, *J. Geom. Phys.* **3**/59 (2009), 354–368.
- [5] SPANIER, E. H.: Algebraic Topology, Springer, 1981.
- [6] WORONOWICZ, S. L.: Pseudogroups, pseudospaces and Pontryagin duality, *Proceedings of the International Conference on Mathematical Physics, Lausanne* **116** (1979), 407–412.