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MAXIMAL NEAR-RINGS OF POLYNOMIAL FUNCTIONS ON GROUPS

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Abstract: For a group G, let $M(G)$ denote the near-ring of functions on G. We determine when the subnear-rings $\mathrm{Pol}(G)$ of polynomial functions on G and $\mathrm{Con}(G)$, the near-ring of congruence preserving functions on G are maximal subnear-rings of $M(G)$.

I. Introduction

Let $(G,+)$ be a group written additively, but not necessarily abelian, and as usual let $M(G)$ be the near-ring of functions on G and $M_0(G)$ the subnear-ring of $M(G)$ consisting of the zero preserving functions on G. The investigation of substructures of the near-rings $M(G)$ and $M_0(G)$ has been an ongoing project. The goal is to use information about the near-rings and their substructures to obtain information about the group G and conversely, using the structure of G to obtain information about $M(G)$ and $M_0(G)$ and their substructures.

One of the early topics in this investigation was to identify the ideals of $M(G)$ and $M_0(G)$. It was found, [1], that $M_0(G)$ has only the

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trivial ideals $\{0\}$ and $M_0(G)$ while $M(G)$ has an additional ideal.

Another substructure investigated was the near-ring, $\mathrm{Pol}(G)$ of polynomials on the group G, as well as the subnear-ring $\mathrm{Pol}_0(G)$ of zero preserving polynomials. For a detailed account of these early investigations, see the book [6] by Lausch and Nöbauer.

More recently, Christian Neumaier, [9], has characterized the maximal subnear-rings of $M_0(G)$, for a finite group G, and this has been extended to $M(G)$ and $M_0(G)$ for infinite groups in [7]. Moreover, there has been a new construction of rings of functions in $M_0(G)$. (Note a ring of functions in $M(G)$ is zero preserving so all rings S in $M(G)$ are in $M_0(G)$.) (See [3].)

We start the next section with a brief review of this construction and then use it to give a new proof of the characterization of those groups G such that $\mathrm{Pol}_0(G)$ is a ring in $M_0(G)$. In establishing this result, we also give a new proof of a covering result, first obtained in [2].

II. Results

For a group G, a cover C is a collection $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ of subgroups of G with the property $\cup C_\alpha = G$. Here we are interested only in covers by abelian subgroups henceforth referred to as abelian covers. Now let $C = \{C_\alpha\}$ be an abelian cover and let $\mathcal{R}(C) := \{f \in M_0(G) | f|_{C_\alpha} \in \in \mathrm{End}(C_\alpha),\ \forall \alpha \in \mathcal{A}\}$. One finds that $\mathcal{R}(C)$ is a ring of functions in $M_0(G)$, called the *ring determined by* C.

Before stating our first result we recall that the near-ring, $\mathrm{Pol}(G)$, of polynomials on G is the subnear-ring of $M(G)$ generated by the group, $\mathrm{Inn}(G)$, of inner automorphisms of G and the constant functions of G, ([6]). Hence $\mathrm{Pol}_0(G)$ is generated by $\mathrm{Inn}(G)$ and is also denoted by $I(G)$. We also recall that a group G is called a 2-Engel group if any two inner automorphisms commute, that is, G is 2-Engel if and only if for each $x \in G$, $\langle x^G\rangle := \langle i_g(x) | g \in G\rangle$ is an abelian subgroup of G, where i_g is the inner automorphism of G determined by g. (See [10].)

Theorem 1. *For any group G, the following are equivalent:*

i) *G has a cover by abelian normal subgroups;*

ii) $\mathrm{Pol}_0(G)$ *is a ring;*

iii) *G is a 2-Engel group.*

Proof. Let $C_{an} = \{C_\alpha\}_{\alpha \in \mathcal{A}}$ be an abelian normal cover of G. We note that $\mathrm{Inn}(G)$ is contained in $\mathcal{R}(C_{an})$ so $\mathrm{Pol}_0(G)$ is a ring. Hence for each $x \in G$, $\langle i_g(x) | g \in G \rangle$ is abelian so i) $\Rightarrow$ ii) $\Rightarrow$ iii).

Suppose now G is a 2-Engel and let x_β be a set of coset representatives of $Z(G)$ (center of G) in G. One shows that $\{\langle x_\beta^G \rangle + 2(G)\}$ is a set of abelian normal subgroups of G and that $\cup(\langle x_\beta^G \rangle + Z(G))$ is a cover of G. Hence we have the result. $\Diamond$

We remark that the equivalence of ii) and iii) in the above theorem was first established by Chandy, [4]. We also note that abelian normal covers were investigated in [2] with the restriction that there be only a finite number of subgroups in the cover. Their result was that G has a finite cover by abelian normal subgroups if and only if G is 2-Engel and central by finite, i.e., $G/Z(G)$ is a finite group.

We next turn to the problem of characterizing when $\mathrm{Pol}(G)$ is a maximal subnear-ring of $M(G)$. To this end we introduce two additional subnear-rings of $M(G)$. Let $\mathrm{Con}(G)$ be the near-ring of congruence preserving functions on G. Thus, $\mathrm{Con}(G) = \{f \in M(G) | \forall x, y \in G, \forall$ normal subgroup H of G, $x + H = y + H \Rightarrow f(x) + H = f(y) + H\}$ and let $\mathrm{Con}_0(G)$ be the zero symmetric part of $\mathrm{Con}(G)$, i.e. the zero preserving functions in $\mathrm{Con}(G)$.

Theorem 2. *Let G be a group.*

1) $\mathrm{Con}(G)$ *is a maximal subnear-ring of* $M(G)$ *if and only if* $\mathrm{Con}_0(G)$ *is a maximal subnear-ring of* $M_0(G)$*.*

2) $\mathrm{Pol}(G)$ *is a maximal subnear-ring of* $M(G)$ *if and only if* $\mathrm{Pol}_0(G)$ *is a maximal subnear-ring of* $M_0(G)$*.*

Proof. 1) Let $\mathrm{Con}(G)$ be maximal as a subnear-ring of $M(G)$. By definition, $\mathrm{Con}(G) \subsetneqq M(G)$ and from [7], $\mathrm{Con}(G) = \mathrm{Fix}(H)$ for some subgroup $H \lneqq G$ or $\mathrm{Con}(G) = N + M_c(G)$ where N is a maximal subnear-ring of $M_0(G)$ and, as usual, $M_c(G)$ is the collection of constant functions on G. Since we know $M_c(G) \subseteq \mathrm{Con}(G)$, the first case is impossible, so $\mathrm{Con}(G) = N + M_c(G)$. But $N = \mathrm{Con}_0(G)$ so $\mathrm{Con}_0(G)$ is a maximal subnear-ring of $M_0(G)$.

For the converse, if $\mathrm{Con}_0(G)$ is maximal in $M_0(G)$, then from ([7]), $\mathrm{Con}_0(M) + M_c(G)(= \mathrm{Con}(G))$ is a maximal subnear-ring of $M(G)$.

2) The proof of the second statement is similar and omitted. $\Diamond$

We consider $\mathrm{Con}_0(G)$ for any group G. Suppose first that G is not simple, say $\{0\} \subsetneqq H \subsetneqq G, H$ a normal subgroup of G. Then $\mathrm{Con}_0(G) \subseteq$ $\subseteq \mathrm{Fix}(H) \subsetneqq M_0(G)$. We show $\mathrm{Con}(G) \subsetneqq \mathrm{Fix}(H)$. Let $x, y \in G$, $y \notin H$, $x \in y + H$, i.e., $x \equiv y(H)$. Define $f\colon\ G \to G$ by $f(H) = \{0\}$, $f(x) = 0$, and $f(w) = y$ if $w \notin H \cup \{x\}$. Then $f \in \mathrm{Fix}(H)$ but $0 = f(x) \not\equiv y = f(y)$. Hence $\mathrm{Con}_0(G)$ is not a maximal subnear-ring of $M_0(G)$.

If G is simple then $\mathrm{Con}_0(G) = M_0(G)$ so by definition, $\mathrm{Con}_0(G)$ is not maximal. We have established the next result.

Theorem 3. *For any group G, $Con_0(G)$ is not a maximal subnear-ring of $M_0(G)$ and hence, $Con(G)$ is not a maximal subnear-ring of $M(G)$.*

We now turn to $\mathrm{Pol}_0(G)$. If G is not simple we again let H be a proper normal subgroup of G, $\{0\} \subsetneqq H \ntrianglelefteq G$. We have $\mathrm{Pol}_0(G) \subseteq$ $\subseteq \mathrm{Con}_0(G) \subsetneqq \mathrm{Fix}_0(H) \subsetneqq M_0(G)$ so $\mathrm{Pol}_0(G)$ is not a maximal subnear-ring of $M_0(G)$.

Suppose G is simple and infinite. Then $|\mathrm{Pol}_0(G)| = |I(G)| = |G|$ while $|M_0(G)| \gneqq |G|$. Let $f \in M_0(G) \backslash \mathrm{Pol}_0(G)$ and let K denote the subnear-ring of $M_0(G)$ generated by $\mathrm{Inn}(G) \cup \{f\}$. Then $|K| = |G|$ so $\mathrm{Pol}_0(G) \subsetneqq K \subsetneqq M_0(G)$, i.e., $\mathrm{Pol}_0(G)$ is not maximal in $M_0(G)$.

Finally, suppose G is simple and finite. If G is non-abelian, then by Fröhlich, [5], $\mathrm{Pol}_0(G) = I(G) = M_0(G)$ so, again, $\mathrm{Pol}_0(G)$ is not maximal in $M_0(G)$. If G is abelian then $G \cong \mathbb{Z}_p$ for some prime p which in turn implies $\mathrm{Pol}_0(G) = I(G) = E(G)$ (the near-ring in $M_0(G)$ generated by the endomorphisms of G). From [8], since G is finite, $E(G)$ is maximal as a subnear-ring of $M_0(G)$ if and only if $G \cong \mathbb{Z}_3$ or $G \cong (\mathbb{Z}_2)^n$, $n \geq 2$. Since $(\mathbb{Z}_2)^n$, $n \geq 2$ is not simple, we must have $G \cong \mathbb{Z}_3$.

Theorem 4. *Let G be a group. Then* $\mathrm{Pol}_0(G)$ *is maximal as a subnear-ring of $M_0(G)$ and hence* $\mathrm{Pol}(G)$ *is a maximal subnear-ring of $M(G)$ if and only if $G \cong \mathbb{Z}_3$.*

III. Units in POL(G) and POL$_0$(G)

In this short section we present recent work on the group of units of $\mathrm{Pol}_0(G)$. We then indicate possible directions for future research.

Let G be a group and let $u(\mathrm{Pol}_0(G))$ denote the group of "multiplicative" units of the near-ring $\mathrm{Pol}_0(G)$. Suppose $u(\mathrm{Pol}_0(G))$ is an abelian group. Since $\mathrm{Inn}(G) \subseteq \mathrm{Pol}_0(G)$ and since $G/Z(G) \cong \mathrm{Inn}(G)$, we get $G/Z(G)$ is abelian which means G is nilpotent of class 2. From

Chandy, [4], we get G is a 2-Engel group and $I(G) = \mathrm{Pol}_0(G)$ is a commutative ring.

Theorem 5. *Let G be a group. The following are equivalent:*

i) $u(\mathrm{Pol}_0(G))$ *is an abelian group;*

ii) G *is nilpotent of class* 2*;*

iii) $\mathrm{Pol}_0(G)$ *is a commutative ring;*

iv) G *has a cover by normal abelian subgroups and* G *is central by abelian.*

Proof. From the above discussion we get i) implies ii) implies iii). Since iii) clearly implies i) we have the equivalence of i), ii) and iii). From Th. 1, iii) implies that G has an abelian normal cover. Moreover, as we have seen, $u(\mathrm{Pol}_0(G))$ abelian implies $G/Z(G)$ is abelian, i.e., G is central by abelian. On the other hand, from iv) we have that $G/Z(G)$ is abelian so G is nilpotent of class 2. $\Diamond$

We close with two questions for future investigation.

(Q1) What can be said about $\mathrm{Pol}(G)$ when $u(\mathrm{Pol}(G))$ is an abelian group?

(Q2) Which abelian groups arise as unit groups of $\mathrm{Pol}_0(G)$?

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