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GRAPHICAL RELATIONSHIPS BETWEEN THE INFIMUM AND INTERSECTION CONVOLUTIONS

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Abstract: After some preparations, we show that the hypograph of a generalized infimum convolution of extended real-valued functions coincides with the corresponding generalized intersection convolution of the hypographs of these functions.

This statement, together with a useful observation on the union convolution of relations, allows of a natural derivation of the fact that the strict epigraph of the infimum convolution of extended real-valued functions coincides with the global sum of the strict epigraphs of these functions.

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1. A few basic facts on relations and functions

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. Thus, a relation F on X to Y is also a relation on $X \cup Y$.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the domain and range of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse relation F^{-1} can be defined such that $F^{-1}(y) = \{x \in X : y \in \in F(x)\}$ for all $y \in Y$.

Moreover, if in addition G is a relation on Y to Z, then the composition relation $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

If $\leq$ is a relation on X and $+$ is a function of X^2 to X, then the ordered pairs $X(\leq) = (X, \leq)$ and $X(+) = (X, +)$ are called a goset and a groupoid, respectively. In this case, for any $x, y \in X$, we simply write $x \leq y$ and $x + y$ in place of $(x, y) \in \leq$ and $+((x, y))$, respectively. Moreover, we simply write X in place of $X(\leq)$ and $X(+)$.

The most basic order theoretic and algebraic notions can also be naturally defined in a goset and a groupoid, respectively. For instance, for any subset A of a goset X, we may naturally define

$$\operatorname{lb}(A) = \bigcap_{a \in A} \,]-\infty, a] \qquad \text{and} \qquad \operatorname{ub}(A) = \bigcap_{a \in A} [a, +\infty[\,,$$

where $]-\infty, a] = \{x \in X : x \leq a\}$ and $[a, +\infty[= \{x \in X : a \leq x\}$, and

$$\max(A) = A \cap \operatorname{ub}(A) \qquad \text{and} \qquad \inf(A) = \max\big(\operatorname{lb}(A)\big).$$

Moreover, for any subsets A and B of a groupoid X, we may naturally define

$$A + B = \{a + b : \ a \in A, \ b \in B\}.$$

Thus, a relation F on one groupoid X to another Y may be naturally called additive if

$$F(x + y) = F(x) + F(y)$$

for all $x, y \in X$.

In the sequel, $\mathbb{R}$ will stand for the conditionally complete ordered field of real numbers. Moreover, we shall also use the notation

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \ +\infty\}.$$

In $\overline{\mathbb{R}}$ we shall consider the natural extensions of the inequality and addition in $\mathbb{R}$. Moreover, having in mind the upper addition of Moreau [8], we also define

$$-\infty + (+\infty) = +\infty \qquad \text{and} \qquad +\infty + (-\infty) = +\infty.$$

Thus, we can at once state the following

Theorem 1.1. $\overline{\mathbb{R}}$*, with the extended inequality, is a complete ordered set.*

Remark 1.2. The completeness of $\overline{\mathbb{R}}$ means that $\inf(A) \neq \emptyset$ and $\sup(A) \neq \emptyset$ for all $A \subset \overline{\mathbb{R}}$. (See, for instance, [2].)

By the anti-symmetry of the inequality in $\overline{\mathbb{R}}$, $\inf(A)$ is actually singleton for all $A \subset \overline{\mathbb{R}}$ which will be identified with its element.

Moreover, we can also easily check the following

Theorem 1.3. $\overline{\mathbb{R}}$*, with the extended addition, is a commutative semigroup with zero element.*

Remark 1.4. To check the associativity of the addition in $\overline{\mathbb{R}}$, note that if at least one of the quantities $x, y, z \in \overline{\mathbb{R}}$ is equal to $+\infty$, then both $(x + y) + z$ and $x + (y + z)$ are also equal to $+\infty$.

2. Additivity properties of the relations $<$ and $\leq$

In this section, slightly improving the treatment of Moreau [8], we prove some basic facts concerning the compatibility of the inequality and the addition in $\overline{\mathbb{R}}$.

Proposition 2.1. *If $x, y, z, w \in \overline{\mathbb{R}}$ such that $x < z$ and $y < w$, then $x + y < z + w$.*

Proof. Because of $x < z$ and $y < w$, we necessarily have $-\infty < z$, $-\infty < w$ and $x < +\infty$, $y < +\infty$. Hence, it follows that $-\infty < z + w$ and $x + y < +\infty$.

Now, we can note that if either $z = +\infty$ or $w = +\infty$, then because of $x+y < +\infty$ and $z+w = +\infty$, the required inequality holds. Therefore, we may assume that $z < +\infty$ and $w < +\infty$.

In this case, because of $-\infty < z$ and $-\infty < w$, we necessarily have $z, w \in \mathbb{R}$. Therefore, if in particular $x, y \in \mathbb{R}$ also holds, then by the corresponding property of $\mathbb{R}$ the required inequality is true. While, if in particular either $x = -\infty$ or $y = -\infty$, then because of $x + y = -\infty$ and $z + w \in \mathbb{R}$, the required inequality again holds. $\Diamond$

Proposition 2.2. *If $x, y, \tau \in \overline{\mathbb{R}}$ such that $x + y < \tau$, then there exist $z, w \in \overline{\mathbb{R}}$ such that $x < z$, $y < w$ and $\tau = z + w$.*

Proof. Because of $x + y < \tau$, we necessarily have $-\infty < \tau$ and $x + y < < +\infty$. Hence, it follows that $x < +\infty$ and $y < +\infty$.

Now, we can note that if in particular $\tau = +\infty$, then the extended numbers $z = +\infty$ and $w = +\infty$ have the required properties. Therefore, we may assume that $\tau < +\infty$.

In this case, because of $-\infty < \tau$, we necessarily have $\tau \in \mathbb{R}$. Therefore, if in particular $x, y \in \mathbb{R}$ also holds, then by taking $r = \tau - (x + y)$ we can easily check that the numbers $z = x + r/2$ and $w = y + r/2$ have the required properties.

Therefore, we need only consider the case when either $x = -\infty$ or $y = -\infty$. For this, note that if for instance $x = -\infty$ and $y \in \mathbb{R}$, then we can take $w = y + 1$ and $z = \tau - (y + 1)$. While, if $x = -\infty$ and $y = -\infty$, then we can take $z = \tau/2$ and $w = \tau/2$. Now, by the commutativity of the addition in $\overline{\mathbb{R}}$, it is clear that the required assertion is always true. $\Diamond$

Now, as a common generalization of the above propositions, we can prove the following

Theorem 2.3. *For any $x, y \in \overline{\mathbb{R}}$, we have*

$$]x + y,\ +\infty] =]x,\ +\infty] +\ ]y,\ +\infty].$$

Proof. If $\tau \in\ ]x, +\infty] +\ ]y, +\infty]$, then there exist $z \in\ ]x, +\infty]$ and $w \in \in\ ]y, +\infty]$ such that $\tau = z + w$. The above inclusions imply that $x < z$ and $y < w$. Hence, by Prop. 2.1, it follows that $x+y < z+w$. Therefore, $\tau = z + w \in\ ]x + y, +\infty]$. This shows that $]x, +\infty] +\ ]y, +\infty] \subset\ ]x + y, + +\infty]$.

Conversely, if $\tau \in]x + y, +\infty]$, then $x + y < \tau$. Thus, by Prop. 2.2, there exist $z, w \in \overline{\mathbb{R}}$ such that $x < z$, $y < w$ and $\tau = z + w$. Hence, it is clear that $\tau = z + w \in\ ]x, +\infty] +\ ]y, +\infty]$. Therefore, $]x + y, +\infty] \subset \subset\ ]x, +\infty] +\ ]y, +\infty]$ is also true. $\Diamond$

Analogously to Prop. 2.1, we can also easily prove the following

Proposition 2.4. *If* $x, y, z, w \in \overline{\mathbb{R}}$ *such that* $x \leq z$ *and* $y \leq w$*, then* $x + y \leq z + w$*.*

Proof. Because of the corresponding property of $\mathbb{R}$, we need only consider the case when at least one of the above numbers is equal to either $-\infty$ or $+\infty$.

For this, note that if for instance $x = +\infty$, then because of $x \leq z$ we also have $z = +\infty$. Therefore, $z + w = +\infty$, and thus the required inequality trivially holds.

While, if $z = -\infty$, then because of $x \leq z$ we also have $x = -\infty$. Therefore, if $y \neq +\infty$, then $x+y = -\infty$, and thus the required inequality trivially holds. While, if $y = +\infty$, then because of $y \leq w$ we also have $w = +\infty$. Therefore, $z + w = +\infty$, and thus the required inequality again trivially holds. $\Diamond$

By using Prop. 2.2, we can also easily prove the following

Proposition 2.5. *If* $x, y, \tau \in \overline{\mathbb{R}}$ *such that* $x + y \leq \tau$*, then there exist* $z, w \in \overline{\mathbb{R}}$ *such that* $x \leq z$*,* $y \leq w$ *and* $\tau = z + w$*.*

Proof. Because of $x + y \leq \tau$, we have either $x + y = \tau$ or $x + y < \tau$. Note that, in the first case, we can simply take $z = x$ and $w = y$. While, in the second case, Prop. 2.2 can be applied. $\Diamond$

Now, as a common generalization of Props. 2.4 and 2.5, we can also state

Theorem 2.6. *For any* $x, y \in \overline{\mathbb{R}}$*, we have*

$$[x + y,\ +\infty] = [x,\ +\infty] + [y,\ +\infty].$$

Remark 2.7. The above two theorems show that $<$ and $\leq$ are additive relations on $\overline{\mathbb{R}}$.

In the sequel, we shall also need the following partial dual of Prop. 2.2.

Proposition 2.8. *If* $x, y, \tau \in \overline{\mathbb{R}}$ *such that* $\tau < x + y$*,* $x \neq -\infty$ *and* $y \neq -\infty$*, then there exist* $z, w \in \overline{\mathbb{R}}$ *such that* $z < x$*,* $w < y$ *and* $\tau = z + w$*.*

Proof. Because of $\tau < x + y$, we necessarily have $-\infty < x + y$ and $\tau < +\infty$. Therefore, either $x + y \in \mathbb{R}$ or $x + y = +\infty$. Moreover, either $\tau = -\infty$ or $\tau \in \mathbb{R}$.

If in particular $x + y \in \mathbb{R}$, then $x, y \in \mathbb{R}$. Now, if in particular $\tau \in \mathbb{R}$ also holds, then by taking $r = x + y - \tau$, we can easily check that the numbers $z = x - r/2$ and $w = y - r/2$ have the required properties. While, if $\tau = -\infty$, then we can simply take $z = -\infty$ and $w = -\infty$.

While, if in particular $x + y = +\infty$, then either $x = +\infty$ or $y = = +\infty$. Now, if in particular $x = +\infty$ and $y \in \mathbb{R}$, then we can easily see that the numbers $w = y - 1$ and $z = \tau - (y - 1)$ have the required properties. While, if $x = +\infty$ and $y = +\infty$, then we can simply take $z = \tau/2$ and $w = \tau/2$ with the usual convention that $\tau/2 = -\infty$ if $\tau = = -\infty$. Now, by the commutativity of the addition in $\overline{\mathbb{R}}$, it is clear that the required assertion is always true. $\Diamond$

Remark 2.9. Note that if in particular $x, y, \tau \in \overline{\mathbb{R}}$ such that $\tau < x + y$, $x \neq +\infty$ and $y \neq +\infty$, then we necessarily have $x \neq -\infty$ and $y \neq -\infty$. Therefore, the conclusion of the above proposition is again true.

Now, as a common generalization of Props. 2.1 and 2.8 and Rem. 2.9, we can also state

Theorem 2.10. *For any $x, y \in \mathbb{R} \cup \{-\infty\}$ or $x, y \in \mathbb{R} \cup \{+\infty\}$, we have*

$$[-\infty,\ x + y[= [-\infty,\ x[+ [-\infty,\ y[.$$

By using Prop. 2.8, we can also easily prove the following partial dual of Prop. 2.5.

Proposition 2.11. *If $x, y, \tau \in \overline{\mathbb{R}}$ such that $\tau \leq x + y$, $x \neq -\infty$ and $y \neq -\infty$, then there exist $z, w \in \overline{\mathbb{R}}$ such that $z \leq x$, $w \leq y$ and $\tau = z + w$.*

Proof. Because of $\tau \leq x + y$, we have either $\tau = x + y$ or $\tau < x + y$. Note that, in the first case, we can simply take $z = x$ and $w = y$. While, in the second case Prop. 2.8 can be applied. $\Diamond$

Remark 2.12. Note that if $x, y, \tau \in \overline{\mathbb{R}}$ such that $\tau \leq x + y$, $x \neq +\infty$ and $y \neq +\infty$, then by Rem. 2.9 the conclusion of the above proposition is also true.

Thus, in addition to Prop. 2.11, we can also state that if for instance $x, y, \tau \in \overline{\mathbb{R}}$ such that $\tau \leq x + y$, $x = -\infty$ and $y \neq +\infty$, then the conclusion of Prop. 2.11 is also true.

Now, as a common generalization of Props. 2.4 and 2.11 and Rem. 2.12, we can also state

Theorem 2.13. *For any $x, y \in \mathbb{R} \cup \{-\infty\}$ or $x, y \in \mathbb{R} \cup \{+\infty\}$, we have*

$$[-\infty,\ x + y] = [-\infty,\ x] + [-\infty,\ y].$$

Remark 2.14. The latter two theorems show that $>$ and $\geq$ are additive relations on both $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R} \cup \{+\infty\}$ to $\overline{\mathbb{R}}$.

3. Further results on the inverse relation of $\leq$

For our subsequent purposes, it seems convenient to introduce a particular notation for the inverse relation $\geq$ of $\leq$ in $\overline{\mathbb{R}}$.

Definition 3.1. We define a relation Φ on $\overline{\mathbb{R}}$ such that $\Phi(x) = [-\infty, x]$ for all $x \in \overline{\mathbb{R}}$.

Thus, we can at once state the following proposition which shows that Φ is just the inverse relation of $\leq$ in $\overline{\mathbb{R}}$.

Proposition 3.2. *For any $x, y \in \overline{\mathbb{R}}$, we have*

$$x \Phi y \iff y \leq x.$$

Proof. By the corresponding definitions, it is clear that

$$x \Phi y \iff y \in \Phi(x) \iff y \in [-\infty, x] \iff y \leq x. \qquad \Diamond$$

Remark 3.3. From this proposition, by using Th. 1.1, we can easily derive that Φ is also a complete order relation on $\overline{\mathbb{R}}$.

In connection with the relations $\leq$ and Φ, we can also easily establish the following proposition which needs only the reflexivity and transitivity of the inequality in $\overline{\mathbb{R}}$.

Proposition 3.4. *For any $x, y \in \overline{\mathbb{R}}$, we have*

$$x \leq y \iff \Phi(x) \subset \Phi(y).$$

Proof. If $z \in \Phi(x)$, then $z \leq x$. Hence, if $x \leq y$ holds, we can infer that $z \leq y$, and thus $z \in \Phi(y)$. Therefore, $\Phi(x) \subset \Phi(y)$ also holds.

Conversely, if the latter inclusion holds, then because of $x \in \Phi(x)$, we also have $x \in \Phi(y)$. Therefore, $x \leq y$ also holds. $\Diamond$

Remark 3.5. From this proposition, we can already see that the set-valued mapping $x \mapsto \Phi(x)$, where $x \in \overline{\mathbb{R}}$, is an order monomorphism of $\overline{\mathbb{R}}$ to the family $\mathcal{P}(\overline{\mathbb{R}})$ of all subsets of $\overline{\mathbb{R}}$.

Moreover, by using Prop. 3.4, we can easily prove the following

Theorem 3.6. *For any family $(x_i)_{i \in I}$ in $\overline{\mathbb{R}}$, we have*

$$\Phi\left(\inf_{i \in I} x_i\right) = \bigcap_{i \in I} \Phi(x_i).$$

Proof. If $x = \inf_{i \in I} x_i$, then $x \leq x_i$ for all $i \in I$. Hence, by Prop. 3.4, it follows that $\Phi(x) \subset \Phi(x_i)$ for all $i \in I$, and thus $\Phi(x) \subset \bigcap_{i \in I} \Phi(x_i)$.

On the other hand, if $y \in \bigcap_{i \in I} \Phi(x_i)$, then $y \in \Phi(x_i)$ for all $i \in I$. Hence, by the transitivity of Φ, it follows that

$$\Phi(y) \subset \Phi[\Phi(x_i)] = \Phi^2(x_i) \subset \Phi(x_i)$$

for all $i \in I$. Moreover, by Prop. 3.4, $\Phi(y) \subset \Phi(x_i)$ implies $y \leq x_i$ for all $i \in I$. Hence, by the definition of x, it is clear that $y \leq x$, and thus $y \in \Phi(x)$. Therefore, $\bigcap_{i \in I} \Phi(x_i) \subset \Phi(x)$ is also true. $\Diamond$

In general, the inverse of an additive relation need not be additive. However, by Th. 2.13, we can once state the following

Theorem 3.7. *For any $x, y \in \mathbb{R} \cup \{-\infty\}$ or $x, y \in \mathbb{R} \cup \{+\infty\}$, we have*
$$\Phi(x+y) = \Phi(x) + \Phi(y).$$

Remark 3.8. Because of this theorem, we can note that the restriction of the set-valued mapping considered in Rem. 3.5 to either $\mathbb{R} \cup \{-\infty\}$ or $\mathbb{R} \cup \{+\infty\}$ is not only an order, but also an algebraic monomorphism to $\mathcal{P}(\overline{\mathbb{R}})$.

In general, the complement of an additive relation need not also be additive. However, by using Th. 2.3, we can easily prove the following

Theorem 3.9. *Under the notation $\Phi^c = \overline{\mathbb{R}}^2 \setminus \Phi$, for any $x, y \in \overline{\mathbb{R}}$, we have*
$$\Phi^c(x+y) = \Phi^c(x) + \Phi^c(y).$$

Proof. By the corresponding definitions, it is clear that
$$\Phi^c(x) = \overline{\mathbb{R}} \setminus \Phi(x) = [-\infty, +\infty] \setminus \; [-\infty, x] = \,]x, +\infty]$$
for all $x \in \overline{\mathbb{R}}$. Therefore, Th. 2.3 can be applied. $\Diamond$

4. Generalized infimum and intersection convolutions

To briefly formulate the definitions of the above mentioned convolutions, it seems convenient to introduce the following

Definition 4.1. Let X be a set and Γ be a relation on X to X^2. Moreover, for any $U, V \subset X$, define
$$\Delta(x, U, V) = \Gamma(x) \cap (U \times V).$$

Remark 4.2. An important particular case is when X is groupoid and for some relation $\leq$ on X we have
$$\Gamma(x) = \big\{ (u, v) \in X^2 : \; x \leq u + v \big\},$$
for all $x \in X$. Here, $\leq$ may, in particular, be the equality relation on X which is only a partial order on X.

Definition 4.3. Under the notation of Def. 4.1, for any two functions f and g on X to $\overline{\mathbb{R}}$, we define a function $f * g$ on X to $\overline{\mathbb{R}}$ such that
$$(f * g)(x) = \inf \big\{ f(u) + g(v) : \; (u, v) \in \Delta(x, D_f, D_g) \big\}$$

for all $x \in X$. The function $f * g$ will be called the infimum convolution of f and g corresponding to the relation Γ.

Remark 4.4. Note that if f and g do not take on the value $+\infty$, then $(f * g)(x) = +\infty$ if and only if $\Delta(x, D_f, D_g) = \emptyset$. That is, $\Gamma(x) \cap (D_f \times \times D_g) = \emptyset$, or equivalently $x \notin \Gamma^{-1}[D_f \times D_g]$.

The infimum convolution corresponding to the equality relation on a semigroup was already applied by several mathematicians in minimization problems and regularization processes. (See Moreau [8] and Strömberg [9].)

However, the infimum convolution corresponding to an inequality relation on a groupoid has only been explicitly applied by the second author in [14] to put the derivation of an increasing Hahn–Banach type extension theorem of Fuchssteiner and Lusky [6, p. 13] into a proper perspective.

Definition 4.5. Under the notation of Def. 4.1, for any two relations F and G on X to a groupoid Y, we define a relation $F * G$ on X to Y such that

$$(F * G)(x) = \bigcap \big\{ F(u) + G(v) : \ (u, v) \in \Delta(x, D_F, D_G) \big\}$$

for all $x \in X$. The relation $F * G$ will be called the intersection convolution of F and G corresponding to the relation Γ.

Remark 4.6. The intersection convolution corresponding to the equality relation on a group and a groupoid was first studied by the second author in [10] and [11], respectively.

Some of the results of the former paper [10] have later been extended to fuzzy multifunctions by Beg [1]. Moreover, motivated by the results of [10], the intersection convolution has recently been also intensively investigated in [3, 4, 5, 11, 12, 13].

Analogously to the infimum and intersection convolution, the supremum and union convolutions can also be naturally introduced.

Definition 4.7. Under the notation of Def. 4.1, for any two relations F and G on X to a groupoid Y, we define a relation $F \circledast G$ on X to Y such that

$$(F \circledast G)(x) = \bigcup \big\{ F(u) + G(v) : \ (u, v) \in \Delta(x, D_F, D_G) \big\}$$

for all $x \in X$. The relation $F \circledast G$ will be called the union convolution of F and G corresponding to the relation Γ.

Remark 4.8. Now, in contrast to Defs. 4.3 and 4.5, we may simply write $\Gamma(x)$ in place $\Delta(x, D_F, D_G)$.

Namely, if either $u \notin D_F$ or $v \notin D_G$, then we necessarily have $F(u)+G(v)=\emptyset$ which does not influence the union.

However, in contrast to the intersection convolution, the union convolution corresponding to the equality relation need not, in general, be introduced since by a particular case of [7, Th. 3.1] we have the following

Theorem 4.9. *If in particular X is a groupoid and*

$$\Gamma(x)=\big\{(u,v)\in X^2:\ x=u+v\big\}$$

for all $x\in X$, then for any two relations F and G on X to another groupoid Y, we have

$$F \circledast G = F \oplus G.$$

Remark 4.10. In this theorem,

$$F\oplus G=\big\{(x+z,y+w):\ (x,y)\in F,\quad (z,w)\in G\big\}.$$

is the global sum of the relations F and G.

This greatly differs from the more usual pointwise sum $F+G$ which is defined such that $(F+G)(x)=F(x)+G(x)$ for all $x\in X$.

5. Graphical relationships between the various convolutions

The intersection convolution is actually a particular case of a straightforward generalization of the infimum convolution.

However, our purpose is here to investigate only the graphical relationships between the above two particular kinds of convolutions.

For this, we shall also need the following definition of the hypograph which slightly differs from the usual one. (See, for instance, [8, p. 140].)

Definition 5.1. For any function f on a set X to $\overline{\mathbb{R}}$, we define a relation H_f on D_f to $\overline{\mathbb{R}}$ such that

$$H_f(x)=[-\infty, f(x)]$$

for all $x\in D_f$. The relation H_f will be called the hypograph of f.

Remark 5.2. Now, the complement $E_f=(D_f\times\overline{\mathbb{R}})\setminus H_f$ may be called the strict epigraph of f.

Namely, thus E_f is a relation on D_f to $\overline{\mathbb{R}}$ such that

$$E_f(x)=\overline{\mathbb{R}}\setminus H_f(x)=\ [-\infty,+\infty]\ \setminus\ [-\infty, f(x)]=\,]f(x),+\infty]$$

for all $x\in D_f$.

Note that, in contrast to $D_{H_f}=D_f$, we now only have $D_{E_f}=D_f\setminus \setminus f^{-1}(+\infty)$. Moreover, by [9, p. 7], the set $E_f\cap(X\times\mathbb{R})$ should be called the strict epigraph of f.

Proposition 5.3. *For any function f on a set X to $\overline{\mathbb{R}}$, we have*
$$H_f = \Phi \circ f.$$

Proof. By the corresponding definitions, it is clear that
$$H_f(x) = [-\infty, f(x)] = \Phi(f(x)) = (\Phi \circ f)(x)$$
for all $x \in D_f$. Thus, in particular the required equality is also true. $\Diamond$

Remark 5.4. Now, we can also easily see that
$$E_f(x) = \overline{\mathbb{R}} \setminus H_f(x) = \overline{\mathbb{R}} \setminus \Phi\big(f(x)\big) = \Phi^c\big(f(x)\big) = (\Phi^c \circ f)(x)$$
for all $x \in D_f$. Therefore, $E_f = \Phi^c \circ f$ is also true.

Moreover, by using Prop. 5.3, we can also prove the following

Theorem 5.5. *Under our former notation, for any two functions f and g on X to either $\mathbb{R} \cup \{-\infty\}$ or $\mathbb{R} \cup \{+\infty\}$, we have*
$$H_{f*g} = H_f * H_g.$$

Proof. By using Prop. 5.3 and Ths. 3.6 and 3.7, we can see that

$$\begin{aligned} H_{f*g}(x) &= \big(\Phi \circ (f * g)\big)(x) = \Phi\big((f * g)(x)\big) = \\ &= \Phi\left(\inf\big\{f(u) + g(v) : \ (u, v) \in \Delta(x, D_f, D_g)\big\}\right) = \\ &= \bigcap\big\{\Phi\big(f(u) + g(v)\big) : \ (u, v) \in \Delta(x, D_f, D_g)\big\} = \\ &= \bigcap\big\{\Phi\big(f(u)\big) + \Phi\big(g(v)\big) : \ (u, v) \in \Delta(x, D_f, D_g)\big\} = \\ &= \bigcap\big\{(\Phi \circ f)(u) + (\Phi \circ g)(v) : \ (u, v) \in \Delta(x, D_f, D_g)\big\} = \\ &= \bigcap\big\{H_f(u) + H_g(v) : \ (u, v) \in \Delta(x, D_{H_f}, D_{H_g})\big\} = (H_f * H_g)(x) \end{aligned}$$

for all $x \in X$. Therefore, the required equality is also true. $\Diamond$

Now, as a dual of the latter theorem, we can also prove the following

Theorem 5.6. *Under our former notation, for any two functions f and g on X to $\overline{\mathbb{R}}$, we have*
$$E_{f*g} = E_f \circledast E_g.$$

Proof. By the proof of Th. 5.5, Rem. 5.4 and Th. 3.9, it is clear that

$$\begin{aligned}E_{f*g}(x) &= \overline{\mathbb{R}} \setminus H_{f*g}(x) = \\ &= \overline{\mathbb{R}} \setminus \bigcap \big\{ \Phi\big(f(u)+g(v)\big) : \ (u,v) \in \Delta(x, D_f, D_g) \big\} = \\ &= \bigcup \big\{ \overline{\mathbb{R}} \setminus \Phi\big(f(u)+g(v)\big) : \ (u,v) \in \Delta(x, D_f, D_g) \big\} = \\ &= \bigcup \big\{ \Phi^c\big(f(u)+g(v)\big) : \ (u,v) \in \Delta(x, D_f, D_g) \big\} = \\ &= \bigcup \big\{ \Phi^c\big(f(u)\big) + \Phi^c\big(g(v)\big) : \ (u,v) \in \Delta(x, D_f, D_g) \big\} = \\ &= \bigcup \big\{ E_f(u) + E_g(v) : \ (u,v) \in \Delta(x, D_f, D_g) \big\} = \\ &= \bigcup \big\{ E_f(u) + E_g(v) : \ (u,v) \in \Delta(x, D_{E_f}, D_{E_g}) \big\} = (E_f \circledast E_g)(x)\end{aligned}$$

for all $x \in X$. Therefore, the required equality is also true. $\Diamond$

From this theorem, by using Th. 4.9, we can immediately get the following counterpart of [8, Prop. 7.b] and [9, Th. 2.2(b)].

Corollary 5.7. *If in particular X is a groupoid and*
$$\Gamma(x) = \big\{ (u,v) \in X^2 : \ x = u+v \big\}$$
for all $x \in X$, then for any two functions f and g on X to $\overline{\mathbb{R}}$, we have
$$E_{f*g} = E_f \oplus E_g .$$

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