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TRANSVERSAL PERTURBATIONS OF CONVEXITY

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Abstract: A generalization of Carahteodory's cutting theorem is proved for graphs of continuous functions over convex domains. This fact gives a chance to find an appropriate upper estimate of "nonconvexity" for such transversal perturbations of convex sets. As a corollary, a series of results on continuous selections of nonconvex-valued mappings are presented.

1. One-dimensional case

The famous Caratheodory's theorem [6,7] states that the convex hull $\operatorname{conv}(A)$ of a subset A of an N-dimensional real vector space E_N coincides with the set $\operatorname{conv}_{N+1}(A)$ which is defined as the union of all at most N-dimensional simplexes with vertexes from the set A. For a specific subsets $A \subset E$ some stronger results are true. In fact, [6, Th. 2.29] for a connected A it is sufficient to collect all at most $(N-1)$-dimensional simplexes with vertexes from the set.

Unlikely, sizes of such $(N-1)$-dimensional simplexes in general are principally greater than sizes of simplexes in the representation $\operatorname{conv}(A) = \operatorname{conv}_{N+1}(A)$. For example, if

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$$A = S \cup [B_0, C_0] \cup [B_1, C_1] \cup [B_2, C_2]$$

where

$$S = \{(\cos t, \sin t) : 0 \le t \le 2\pi\},$$

$$B_k = \left(\cos\frac{2\pi k}{3}, \sin\frac{2\pi k}{3}\right), \quad C_k = \left(\varepsilon\cdot\cos\frac{2\pi k}{3}, \varepsilon\cdot\sin\frac{2\pi k}{3}\right)$$

then the origin $(0,0)$ lies in some ε-small two-dimensional simplex from $\mathrm{conv}_3(A)$, but the length of an arbitrary segment from $\mathrm{conv}_2(A)$, passing through the point $(0,0)$, is more than 1.

The main aim of the present paper is to prove that for the *graphs* of continuous functions with convex domains such an effect is not possible. Roughly, for this type of a connected sets A the size of each simplex from $\mathrm{conv}_N(A)$ always is less than or equal to the size of some simplex from $\mathrm{conv}_{N+1}(A)$. First, let us consider the rather elementary case of functions from $\mathbb{R}$ into itself.

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a continuous function over the segment I. Let A, B, C be points on the graph Γ_f of f and let $D \in$ $\in \mathrm{conv}\{A, B, C\} = \Delta$. Then $D \in [E; F]$ for some points $E \in \Gamma_f, F \in \Gamma_f$ with $EF \le \max\{AB, BC, AC\}$. Moreover, one can assume that the segment $[E; F]$ is parallel to one of the sides of the triangle Δ.*

Sketch of proof. For definiteness let $x_A < x_B < x_C$ and let the point B be placed above the line AC. If the point D stands below the graph Γ_f then it suffices to draw the line through D parallel to AC and to apply the theorem on intermediate values for segments $[x_A; x_B]$ and $[x_B; x_C]$. If D is above the graph Γ_f and $x_A < x_D < x_B$ (or $x_B < x_D < x_C$) then it analogously suffices to draw the line throw D parallel to AB (or parallel to BC). $\Diamond$

Roughly speaking Th. 1.1 states that $\mathrm{conv}(\Gamma_f)$ coincides not only with $\mathrm{conv}_3(\Gamma_f)$ as in Caratheodory theorem but with $\mathrm{conv}_2(\Gamma_f)$ and, additionally, the sizes of one-dimensional simplexes from $\mathrm{conv}_2(\Gamma_f)$ are less than or equal to sizes of two-dimensional simplexes from $\mathrm{conv}_3(\Gamma_f)$.

Let us demonstrate an application in the theory of continuous selection. Using E. Michael's approach [2], for a nonempty closed subset $P \subset Y$ of a Banach space $(Y, \|\cdot\|)$ and for an open ball $D \subset Y$ of radius r, one can define

$$\delta(P, D) = \sup\{\mathrm{dist}(q, P)/r \mid q \in \mathrm{conv}(P \cap D)\},$$

where $\delta(P, D) = 0$ for the empty intersection $P \cap D$. Clearly, for a closed set P with nonempty intersection $P \cap D$, the equality $\delta(P, D) = 0$ means that the intersection $P \cap D$ is a convex subset of D.

For any nonempty closed subset $P \subset Y$ of a Banach space $(Y, \|\cdot\|)$ the value of its *function of nonconvexity* α_P at a point $r > 0$ is defined as $\alpha_P(r) = \sup\{\delta(P, D)\}$, where supremum is taken over the family of all open balls of radius r. Next a subset of a Banach space is said to be *α-paraconvex* if its function of nonconvexity majorates by the preassigned constant $\alpha \in [0, 1)$.

Corollary 1.2 [3]. *Let $f : I \to \mathbb{R}$ be a Lipshitz with constant k function or be a continuous monotone function over the closed convex domain I. Then the graph Γ_f of f is α-paraconvex subset of Euclidean plane $\mathbb{R}^2$ with $\alpha = \sin(\arctan(k))$, or with $\alpha = \sin(\arctan(1)) = 2^{-0{,}5}$, respectively.*

Sketch of proof. For an open ball $D \subset \mathbb{R}^2$ of radius r pick a point $Q \in \operatorname{conv}(\Gamma_f \cap D)$. By Caratheodori's theorem $Q \in \operatorname{conv}\{A, B, C\} = \Delta$ for some $A, B, C \in \Gamma_f \cap D$. Clearly, $\max\{AB, BC, AC\} < 2r$. Due to Th. 1.1 one can assume that $Q \in \operatorname{conv}\{E, F\} = [E; F]$ for some $E, F \in \Gamma_f \cap D$. Moreover, $EF \le \max\{AB, BC, AC\} < 2r$.

Hence it suffices to estimate the distances $\operatorname{dist}(Q, \Gamma_f)$ between points Q of segments $[E; F]$ with endpoints in graph and the graph itself. Moreover, it is easy to check that it suffices to estimate these distances only for the middle points Q of segments $[E; F]$.

For definiteness let $f : I \to \mathbb{R}$ be a continuous and increasing function over the closed convex domain I and $x_E < x_F$. Draw the lines $y = y_E, x = x_F$ and denote $K = (x_F; y_E)$ its intersection. Draw the lines $x = x_Q, y = y_Q$ and let $L = (x_F, y_Q), M = (x_Q, y_E)$. Triangles ΔEMQ and ΔQLF are equal and the graph Γ_f intersects both the sides QM and QL. But $QM^2 + QL^2 = QF^2 < r^2$. Therefore $QM < 2^{-0{,}5}r$ or $QL < 2^{-0{,}5}r$. That is why $\operatorname{dist}(Q, \Gamma_f) < 2^{-0{,}5}r$ and $\alpha_{\Gamma_f}(\cdot) \le 2^{-0{,}5}$. $\Diamond$

Denote $G(\mathbb{R}^2)$ the family of all closed subsets of the plane which are graphs of continuous functions defined on a convex domain with respect to some (not fixed!) orthogonal coordinate system. Denote respectively $G\operatorname{Lip}_k(\mathbb{R}^2)$ and $G\operatorname{Mon}(\mathbb{R}^2)$ the subfamilies of $G(\mathbb{R}^2)$ consisting of graphs of Lipshitz with constant k functions and graphs of monotone functions. Observe that in fact $G\operatorname{Mon}(\mathbb{R}^2) \subset G\operatorname{Lip}_1(\mathbb{R}^2)$. To demonstrate the inclusion it suffice for increasing function (for decreasing function) to rotate the coordinate system in the clockwise direction (in the counterclockwise direction).

Cor. 1.2 together with E. Michael's selection theorem for paraconvex-valued mappings [2] show that the following statement is true. For basic facts on the selection theory see for example [1, 5].

Corollary 1.3. *Any lower semicontinuous mapping $F : X \to G\,\mathrm{Lip}_k(\mathbb{R}^2)$ from a paracompact space X admits a continuous singlevalued selection.*

2. Two-dimensional case

The analog of Th. 1.1 is true for any dimensions, but the proof is more complicated and based in fact on the proof for the case of two variables. We formulate such a statement in purely affine terms. For the shortness we denote $[X_1, \ldots, X_k]$ the convex hull $\mathrm{conv}\,\{X_1, \ldots, X_k\}$ and $(X_1, \ldots, X_k)$ the set of all inner (in convex sense) points of $[X_1, \ldots, X_k]$.

Theorem 2.1. *Let E be a three-dimensional real vector space and $P \subset E$ be a subset of E such that $P = \Gamma_f$ for some basis $\{e_1, e_2, e_3\}$ of E, for some convex two-dimensional subset $V \subset \mathrm{span}\,(e_1, e_2)$ and for some continuous function $f : V \to \mathrm{span}\,(e_3)$. Then for any points A_0, A_1, A_2, A_3 of the set P and for each point $Q \in [A_0, A_1, A_2, A_3]$ there are points Q_1, Q_2, Q_3 of the set P such that $Q \in [Q_1, Q_2, Q_3] = \Delta$ and the triangle Δ can be moved into some of the faces of the tetrahedron $[A_0, A_1, A_2, A_3]$ by means of some parallel shift.*

Proof. For a point $A \in E$ denote $l(A)$ the line through A parallel to $\mathrm{span}\,(e_3)$. Due to the affine invariance of the assumptions and statement of Th. 2.1 one can assume that:

– the point A_0 is placed above the plane $\mathrm{span}\,\{A_1, A_2, A_3\}$;

– the line $l(A_0)$ meets the triangle $[A_1, A_2, A_3]$ in its inner point, say H; and

– the point $Q \in [A_0, A_1, A_2, A_3]$ belongs to the tetrahedron $[A_0, A_1, A_2, H]$.

Draw two planes through the point Q. The first – the plane α parallel to the face $[A_1, A_2, A_3]$ and the second – the plane β parallel to the face $[A_1, A_2, A_0]$. Denote $H' = \alpha \cap l(A_0)$ and $H'' = \beta \cap l(A_0)$. Consider two parallel shifts. First, the shift of the face $[A_1, A_2, A_3]$ into the plane α defined by the vector $\overrightarrow{HH'}$. Let $B_i = A_i + \overrightarrow{HH'}, i = 1, 2, 3$. Second, the shift of the face $[A_1, A_2, A_0]$ into the plane β defined by the vector $\overrightarrow{A_0H''}$. Let $C_j = A_j + \overrightarrow{A_0H''}, j = 0, 1, 2$.

Triangles $[B_1, B_2, B_3]$ and $[C_1, C_2, C_0]$ are intersected by the segment, say $[D_1; D_2]$. More precisely, $D_k = (H'; B_k) \cap (C_0; C_k)$, $k = 1, 2$. Clearly $D \in (D_1; D_2)$ and $[D_1; D_2] = \alpha \cap \beta \cap [A_0, A_1, A_2, H]$. The lines $l(D_k)$, $k = 1, 2$ meet the graph $P = \Gamma_f$ at the points P_k, $k = 1, 2$. Rest of the proof depends on the position of the points P_1 and P_2 relatively the plane α.

So, let both two points P_1 and P_2 be placed above the plane α (or belong to α). Applying the proof of Th. 1.1 to three segments $[A_1; H]$, $[A_2; H]$ and $[A_3; H]$ we find three points $Q_1 \in \Gamma_f \cap [B_1; D_1]$, $Q_2 \in \Gamma_f \cap$ $\cap [B_2; D_2]$, $Q_3 \in \Gamma_f \cap [B_3; H']$. Then $Q \in [Q_1, Q_2, Q_3] \subset [B_1, B_2, B_3]$ and $[B_1, B_2, B_3] = [A_1, A_2, A_3] + \overrightarrow{HH'}$.

Next, let both two points P_1 and P_2 be placed below the plane α. Then points P_k and $A_0, k = 1, 2$ are in the different half-spaces with respect to the plane β. Hence there are $G_k \in \Gamma_f \cap (D_k, C_0)$, $k = 1, 2$. Similarly, there are $F_k \in \Gamma_f \cap (D_k, C_k), k = 1, 2$. Clearly $Q \in [F_1, F_2, G_1, G_2] \subset$ $\subset [C_0, C_1, C_2]$ and $[C_0, C_1, C_2] = [A_0, A_1, A_2] + \overrightarrow{A_0 H''}$.

Finally, considering the third possibility, let P_1 lies below the plane α whereas P_2 lies above α. Then there exists $D_3 \in \Gamma_f \cap (D_1, D_2)$. Considering analogously to the previous steps the cases $Q \in (D_1; D_3)$ or $Q \in (D_2; D_3)$ one can find the desired triangle $[Q_1, Q_2, Q_3]$ in $[C_0, C_1, C_2]$ or respectively, in $[B_1, B_2, B_3]$. $\Diamond$

Remarks. 1. Clearly in Th. 2.1 there are no chances to replace triangles $[Q_1, Q_2, Q_3]$ by a segments.

2. In the proof we never use the fact that the set P exactly is the graph Γ_f of continuous function f. Really it suffices to consider an arbitrary subset $P \subset E$ which is "continuous" with respect to the third basic vector e_3. Last sentence means that Th. 2.1 holds for each subset $P \subset E$ with the following property:

If $(a_1, a_2, a_3) \in P$ *and* $(b_1, b_2, b_3) \in P$ *then* $[(a_1, a_2, c); (b_1, b_2, c)] \cap$ $\cap P \neq \emptyset$ *for every* c *between* a_3 *and* b_3.

Obviously this property holds for curvilinear trapeziums, not only for graphs of continuous functions.

3. The question about possible analog of Ths. 1.1 and 2.1 for continuous mappings from $\mathbb{R}$ to $\mathbb{R}^2$ is still open.

3. General case

Theorem 3.1. *Let* E *be a* $(n+1)$*-dimensional real vector space and* $P \subset E$ *be a subset of* E *such that* $P = \Gamma_f$ *for some basis* $\{e_1, e_2, \ldots, e_{n+1}\}$ *of* E*, for some convex* n*-dimensional subset* $V \subset \operatorname{span}(e_1, \ldots, e_n)$ *and for some continuous function* $f : V \to \operatorname{span}(e_{n+1})$*. Then for any points*

$A_0, A_1, A_2, \dots, A_{n+1}$ of the set P and for each point

$$Q \in [A_0, A_1, A_2, \dots, A_{n+1}]$$

there are points $Q_1, Q_2, \dots, Q_{n+1}$ of the set P such that

$$Q \in [Q_1, Q_2, \dots, Q_{n+1}] = \Delta$$

and the n-simplex Δ can be moved into some of the faces of the $(n+1)$-simplex $[A_0, A_1, A_2, \dots, A_{n+1}]$ with respect to some parallel shift.

Proof. Repeating notations from the proof of Th. 2.1 we denote $l(A)$ the line through $A \in E$ parallel to $\operatorname{span}(e_{n+1})$ and restrict ourselves to the case when the point A_0 is placed above the plane $\operatorname{span}\{A_1, A_2, \dots, A_n\}$ and

$$H = l(A_0) \cap [A_1, A_2, \dots, A_{n+1}] \in (A_1, A_2, \dots, A_{n+1}),$$
$$Q \in (A_0, A_1, \dots, A_n, H).$$

Let the hyperplane α be parallel to the face $[A_1, A_2, \dots, A_{n+1}]$, $Q \in \alpha$ and the hyperplane β be parallel to the face $[A_0, A_1, \dots, A_n]$, $Q \in \beta$. Denote:

a) $H' = \alpha \cap l(A_0)$, $H'' = \beta \cap l(A_0)$;
b) $B_i = A_i + \overrightarrow{HH'}$, $i = 1, 2, \dots, n+1$;
c) $C_j = A_j + \overrightarrow{A_0H''}$, $j = 0, 1, \dots, n$;
d) $D_k = (H'; B_k) \cap (C_0; C_k)$, $k = 1, 2, \dots, n$;
e) $P_k = l(D_k) \cap P$, $k = 1, 2, \dots, n$.

As in the proof of Th. 2.1 the simplex $D = \operatorname{conv}\{D_1, \dots, D_n\}$ coincides with the intersection $\alpha \cap \beta \cap [A_0, A_1, \dots, A_n, H]$ and contains the point Q.

Renumbering the indexes one can assume that all points $P_1, \dots, P_m$ are placed below the hyperplane α whereas each of the points $P_{m+1}, \dots, P_n$ is placed above the hyperplane α or belongs to α. For every pair $(s;t)$ of indexes with $1 \le s \le m < t \le n$ pick a point $D_{st} \in P \cap (D_s, D_t)$ and define the simplexes D_- and D_+ by settings

$$D_- = \operatorname{conv}\{D_s, D_{st} : 1 \le s \le m < t \le n\},$$
$$D_+ = \operatorname{conv}\{D_t, D_{st} : 1 \le s \le m < t \le n\},$$

Clearly $Q \in D = D_- \cup D_+$. Let $Q \in D_-$. Then $Q = \sum \lambda_p S_p$, $0 \le \lambda_p \le 1$, $\sum \lambda_p = 1$ for some points $S_p \in \{D_s, D_{st} : 1 \le s \le m < t \le n\}$.

Applying Th. 2.1 to every point from the set $\{D_s, D_{st} : 1 \le s \le$ $\le m < t \le n\}$ we see that it belongs to some subtriangle of the simplex $[C_0, C_1, \dots, C_n]$. Hence, the point Q (as a convex combination of such points) belongs to some subsimplex of the simplex $[C_0, C_1, \dots, C_n]$.

Applying the Caratheodory's cutting theorem to this subsimplex we can replace it by some n-dimensional subsimplex of $[C_0, C_1, \ldots, C_n] = [A_0, A_1, \ldots, A_n] + \overrightarrow{A_0H''}$. Similarly, for the case $Q \in D_+$. So, Th. 3.1 is proved. $\Diamond$

4. Applications

In this section we shortly list some applications of Th. 3.1.

Corollary 4.1. *Let $P = \Gamma_f$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous real-valued function with convex closed n-dimensional domain. Suppose that all restrictions $f|_l$ over one-dimensional lines l are monotone functions. Then P is α-paraconvex subset of the Euclidean space $\mathbb{R}^{n+1}$ for some constant $\alpha = \alpha(n) \in [0; 1)$.*

Corollary 4.2 [4]. *Graph of any Lipschitz function with constant k function over convex closed finite-dimensional domain is α-paraconvex subset of the Hilbert space, $\alpha = \sin(\arctan(k))$.*

Corollary 4.3 [4]. *For every $n \in \mathbb{N}$ and $C > 0$ there exists an increasing continuous function $\alpha : (0; +\infty) \to [0; 1)$ such that for every polynomial $f = f(x_1, \ldots, x_n) = \sum a_{ij}x_ix_j + \sum a_kx_k + a_0$ of degree two with $|a_{ij}| \leq C$ the function of nonconvexity of its graph Γ_f is pointwise less or equal than α.*

Each of Cors. 4.1–4.3 implies some selection theorem for graph-valued lower semicontinuous mappings as it has been pointed out in the Sec. 1 above.

Note that without uniform restrictions on coefficients the statement of Cor. 4.3 is false and a corresponding selection theorem in general is false, too.

Unfortunately the degree two is maximal for positive results in this direction. Even for polynomials $f_t(x, y) = x^3 + txy, 0 < t < 1$ of third degree on two variables the supremum of their functions of nonconvexity identically equals to 1. For more higher degree's the worse possibility occurs even for a single polynomial.

Corollary 4.4 [4]. *Let $f(x, y) = x^9 + x^3y$. Then the function of nonconvexity of the graph Γ_f identically equals to 1.*

Finally let us state an interesting purely geometrical problem.

Question 4.5. Let $f : \mathbb{R} \to \mathbb{R}^2$ be a continuous mapping, $A, B, C, D \in \Gamma_f$ and $E \in \text{conv}\,\{A, B, C, D\}$. Is it true that $E \in \text{conv}\,\{F, G, H\}$ for some points $F, G, H \in \Gamma_f$ with the property that the triangle conv $\{F, G, H\}$ can be moved into one of the faces of tetrahedron conv $\{A, B, C, D\}$ with respect to some parallel shift?

As a more "selection" version we have the following question.

Question 4.6. *Let $f : \mathbb{R} \to \mathbb{R}^2$ be a Lipschitz mapping. Is it true that the graph Γ_f is α-paraconvex subset of three-dimensional Euclidean space for some appropriate $\alpha \in [0; 1)$?*

Even for very special case of the helix curve Quest. 4.6 is open.

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