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GENERALIZED POLYNOMIALS IN ONE AND IN SEVERAL VARIABLES

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Abstract: In earlier papers the authors considered relations between generalized polynomials p of degree $\leq n$ and functions P in n variables being Jensen in each variable such that p is the diagonalization of P. Jensen in each variable means that P is a generalized polynomial of degree ≤ 1 in each variable. Here we derive analogous results connecting functions of several variables which are generalized polynomials of degree $\leq \beta_i$ in the i-th variable and generalized polynomials (in one variable) of degree $\leq \sum \beta_i$.

We also discuss the question whether a function being a polynomial separately in each variable has to be a polynomial jointly in all variables.

1. Motivating results and questions

Let V, W be vector spaces over $\mathbb{Q}$ and denote by $\Delta_h\colon W^V \to W^V$ the difference operator with increment h, which for $f\colon V \to W$ is defined by $(\Delta_h f)(x) := f(x+h) - f(x)$.

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Definition 1. Let be $n \in \mathbb{N}_0$. A function $p\colon V \to W$ is called a *generalized polynomial of degree* $\leq n$ if $\Delta_h^{n+1} p = 0$ for all $h \in V$.

We denote the vector space of all generalized polynomials of degree $\leq n$ defined on V and taking values in W by
$\mathcal{P}^n(V,W) := \{p\colon V \to W \mid p \text{ is a generalized polynomial of degree } \leq n\}$.

There is a large literature on generalized polynomials, see for example [D] for a description of $\mathcal{P}^n(V,W)$ in an even more general situation. In [PS] the generalized polynomials $p \in \mathcal{P}^n(V,W)$ have been described by means of n-Jensen functions. The motivation was the *blossoming method* which is used for calculating values of spline functions (see [R]).

Definition 2. 1. A function $q\colon V \to W$ is *Jensen* if

$$q\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(q(x) + q(y)\right)$$

for all $x, y \in V$.

2. A function $P\colon V \to W$ is called *n-Jensen* if the partial mappings $x_i \mapsto P(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ are Jensen functions for all i.

As usual, we denote by S_n the symmetric group of all permutations of the set

$$\mathbf{n} := \{1, 2, \ldots, n\},$$

and call a function $Q\colon V^n \to W$ *symmetric* if $Q(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}) = Q(x_1, x_2, \ldots, x_n)$ for all $(x_1, x_2, \ldots, x_n) \in V^n$ and all $\pi \in \mathsf{S}_n$. The vector space of all n-Jensen functions defined on V^n with values in W is denoted by

$$\mathcal{J}^n(V,W) := \{P\colon V \to W \mid P \ n\text{-Jensen}\},$$

and the subspace of all symmetric n-Jensen functions is denoted by

$$\mathcal{J}^{n,\mathrm{sym}}(V,W) := \{P \in \mathcal{J}^n(V,W) \mid P \text{ symmetric}\}.$$

Definition 3. 1. For $n \in \mathbb{N}$ the *diagonalization mapping* $\delta_n : V \to V^n$ is defined by

$$\delta_n(x) := (x, \ldots, x),$$

with x in each of the n components on the right-hand side.

2. For $P \in \mathcal{P}^n(V,W)$ the *diagonalization D of P* is defined by $D(P) := P \circ \delta_n$.

It has been shown in [PS] that given $P \in \mathcal{J}^n(V,W)$ the diagonalization $D(P)$ is contained in $\mathcal{P}^n(V,W)$. So D maps $\mathcal{J}^n(V,W)$ into $\mathcal{P}^n(V,W)$. It has also been proved that the restriction $D' := D\big|_{\mathcal{J}^{n,\mathrm{sym}}(V,W)}$

gives a bijection between $\mathcal{J}^{n,\mathrm{sym}}(V,W)$ and $\mathcal{P}^n(V,W)$. For $n \geq 1$ and $p \in \mathcal{P}^n(V,W)$ the inverse $P := D'^{-1}(p)$ is given by

$$P(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{S \subseteq \mathbf{n}} (-1)^{n-|S|} (r+|S|)^n p\left(\frac{y + \sum_{i \in S} x_i}{r+|S|}\right), \tag{1}$$

where $|S|$ denotes the cardinality of S and $(y,r) \in V \times \mathbb{Q}$ either equals $(0,0)$ (with $0^n p(0/0) := 0$) or y is arbitrary and $r \in \mathbb{Q} \setminus \{0, -1, \ldots, -n\}$.

Note that $P \in \mathcal{J}^n(V,W)$ if and only if all partial functions mappings

$$x_i \mapsto P(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

are generalized polynomials of degree at most 1, i. e., are contained in $\mathcal{P}^1(V,W)$. Generalizing one may ask the following questions:

1) Let $m \in \mathbb{N}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m$ with $|\beta| = \sum_{i=1}^m \beta_i = n$ and $P \colon V^m \to W$ be given such that

$$(x_i \mapsto P(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)) \in \mathcal{P}^{\beta_i}(V,W) \tag{2}$$

for all $1 \leq i \leq m$ and all $x_1, x_2, \ldots, x_m \in V$. Is it then true that $P \circ \circ\, \delta_m \in \mathcal{P}^n(V,W)$?

2) Is it true that, given $p \in \mathcal{P}^n(V,W)$ and n, β as above, there is some P satisfying (2) such that $p = P \circ \delta_m$?

3) If 2) is true, is there some "canonical" P with $p = P \circ \delta_m$? (In the case $m = n$, $\beta = (1, 1, ..., 1)$, formula (1) gives a kind of canonical P.)

Before answering these questions we need some results on generalized polynomials in several variables.

2. Generalized polynomials in several vector variables, basic definitions and results

We will use some notions and results from [B, App., pp. 88–89], adopt these notions for our situation of rational vector spaces and put aside all topological aspects.

Throughout this paper let $m \in \mathbb{N}$ and let $V_1, V_2, \ldots, V_m, W$ be vector spaces over $\mathbb{Q}$. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m$ with $|\alpha| := \sum_{j=1}^m \alpha_j > 0$ the sequence $s(\alpha) \in \mathbb{N}^{|\alpha|}$ is defined by

$$s(\alpha) := (1^{\alpha_1}, 2^{\alpha_2}, \ldots, m^{\alpha_m}) := (\underbrace{1, \ldots, 1}_{\alpha_1\text{-times}}, \underbrace{2, \ldots, 2}_{\alpha_2\text{-times}}, \ldots, \underbrace{m, \ldots, m}_{\alpha_m\text{-times}}). \tag{3}$$

Let $V_\alpha := \bigtimes_{i=1}^{|\alpha|} V_{s(\alpha)_i}$ and let $\delta_\alpha\colon V_1 \times V_2 \times \ldots \times V_m \to V_\alpha$ be defined by

$$\delta_\alpha(x_1, x_2, \ldots, x_m) := (x_{s(\alpha)_i})_{1\le i\le|\alpha|} =: (x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_m^{\alpha_m}).$$

The rational vector space of all mappings from V_α to W which are $\mathbb{Q}$-linear in each variable is denoted by

$$\mathrm{Hom}_\alpha(V_1, V_2, \ldots, V_m, W) := \{f : V_\alpha \to W \mid f \text{ additive in each variable}\}.$$

We also define

$$\mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W) := \{f \circ \delta_\alpha \,\big|\, f \in \mathrm{Hom}_\alpha(V_1, V_2, \ldots, V_m, W)\}$$

and the elements $p \in \mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W)$ are called *α-homogeneous* polynomials because of $p(r_1x_1, r_2x_2, \ldots, r_mx_m) = r^\alpha p(x_1, x_2, \ldots, x_m)$ if $(x_1, x_2, \ldots, x_m) \in V := V_1 \times V_2 \times \ldots \times V_m$ and $r = (r_1, r_2, \ldots, r_m) \in \mathbb{Q}^m$, where $r^\alpha := \prod_{i=1}^m r_i^{\alpha_i}$.

There is a subspace of $\mathrm{Hom}_\alpha(V_1, V_2, \ldots, V_m, W)$ which may be identified with the space $\mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W)$. Let S_α be that subgroup of the symmetric group $\mathsf{S}_{|\alpha|}$ which contains those permutations $\pi \in \mathsf{S}_{|\alpha|}$ which satisfy

$$\begin{aligned}\pi\left(\{\alpha_1 + \ldots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \ldots + \alpha_i\}\right) = \\ = \{\alpha_1 + \ldots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \ldots + \alpha_i\}\end{aligned}$$

for all $1 \le i \le m$. Here the number $\alpha_1 + \ldots + \alpha_{i-1}$ is equal to 0 if $i = 1$ and the set $\{\alpha_1 + \ldots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \ldots + \alpha_i\}$ is empty if $\alpha_i = 0$. Given $\pi \in \mathsf{S}_\alpha$ and $g\colon V_\alpha \to W$, we define $g^\pi\colon V_\alpha \to W$ by

$$g^\pi(x_1, x_2, \ldots, x_{|\alpha|}) := g(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(|\alpha|)}).$$

Then $g^\pi \in \mathrm{Hom}_\alpha(V_1, V_2, \ldots, V_m, W)$ if $g \in \mathrm{Hom}_\alpha(V_1, V_2, \ldots, V_m, W)$. Let

$$\begin{aligned}&\mathrm{Hom}_\alpha^{\mathrm{sym}}(V_1, V_2, \ldots, V_m, W) := \\ &:= \left\{g \in \mathrm{Hom}_\alpha(V_1, V_2, \ldots, V_m, W) \,\big|\, g^\pi = g \text{ for all } \pi \in \mathsf{S}_\alpha\right\},\end{aligned}$$

let $p \in \mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W)$ and let $P \in \mathrm{Hom}_\alpha(V_1, V_2, \ldots, V_m, W)$ be such that $p = P \circ \delta_\alpha$. Put $\alpha! := \prod_{i=1}^m \alpha_i!\,(= |\mathsf{S}_\alpha|)$ and define $\widehat{P} := \frac{1}{\alpha!}\sum_{\pi\in\mathsf{S}_\alpha} P^\pi$.

Then $\widehat{P} \in \mathrm{Hom}_\alpha^{\mathrm{sym}}(V_1, V_2, \ldots, V_m, W)$ and $P \circ \delta_\alpha = \widehat{P} \circ \delta_\alpha$. Thus $\mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W) = \left\{P \circ \delta_\alpha \,\big|\, P \in \mathrm{Hom}_\alpha^{\mathrm{sym}}(V_1, V_2, \ldots, V_m, W)\right\}$.

We want to show that even more is true, namely that the mapping given by $\mathrm{Hom}_\alpha^{\mathrm{sym}}(V_1, V_2, \ldots, V_m, W) \ni P \mapsto P \circ \delta_\alpha \in \mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W)$ is a (linear) isomorphism.

Definition 4. Let $V = V_1 \times \cdots \times V_m$, let $1 \le i \le m$ and $h \in$ $\in V_i$. Denote by $\sigma_i\colon V_i \to V$ the embedding of V_i into V, $\sigma_i(h) :=$ $:= (0, 0, \ldots, 0, h, 0, \ldots, 0)$, where h is in the i-th component. The *partial difference operator* $\Delta_{i,h} : W^V \to W^V$ is then defined by

$$(\Delta_{i,h}f)(x) := f(x + \sigma_i(h)) - f(x).$$

Theorem 1. *The rational vector spaces* $\mathrm{Hom}_\alpha^{\mathrm{sym}}(V_1,\ldots,V_m,W)$ *and* $\mathcal{P}_\alpha(V_1,\ldots,V_m,W)$ *are isomorphic. An isomorphism is given by* $P\mapsto$ $\mapsto P\circ\delta_\alpha$. *The inverse is given by* $p\mapsto\widehat{p}$,

$$\widehat{p}(x_{11},\ldots,x_{1\alpha_1},x_{21},\ldots,x_{2\alpha_2},\ldots,x_{m1},\ldots,x_{m\alpha_m}) := \tag{4}$$
$$:= \frac{1}{\alpha!}\left(\bigcirc_{i=1}^{m}\bigcirc_{j_i=1}^{\alpha_i}\Delta_{i,x_{ij_i}}\right)p(y_1,y_2,\ldots,y_m),$$

where $(y_1,y_2,\ldots,y_m)\in V$ *may be chosen arbitrarily.*

Proof. Clearly the mapping $P\mapsto P\circ\delta_\alpha$ is surjective and linear. If $m=1$ and $p\in\mathcal{P}_\alpha(V_1,W)$ the formula for $\widehat{p}$ is the classical *polarization formula* (see for example [K, Lemma 2, p. 394], [D]) which states that a given symmetric k-additive function $F:V^k\to W$ may be reconstructed from its diagonalization $f:=F\circ\delta_k$:

$$\bigcirc_{j=1}^{k}\Delta_{x_j}f(y)=k!F(x_1,x_2,\ldots,x_k). \tag{5}$$

Now let $m\geq 2$ and let $\mathcal{P}_\alpha(V_1,\ldots,V_m,W)\ni p=P\circ\delta_\alpha$ with $P\in$ $\in\mathrm{Hom}_\alpha^{\mathrm{sym}}(V_1,\ldots,V_m,W)$. Then, using the case $m=1$ and the properties of the difference operators $\Delta_{i,x_{ij_i}}$, it is easy to show by induction that for any k, $1\leq k\leq m$, for any $(y_1,y_2,\ldots,y_m)\in V$, and for any $(x_{11},\ldots,x_{1\alpha_1},\ldots,x_{m1},\ldots,x_{m\alpha_m})\in V_\alpha$

$$P(x_{11},\ldots,x_{1\alpha_1},\ldots,x_{k1},\ldots,x_{k\alpha_k},y_{k+1}^{\alpha_{k+1}},\ldots,y_m^{\alpha_m}) = \tag{6}$$
$$=\frac{1}{\prod_{i=1}^k\alpha_i!}\left(\bigcirc_{i=1}^{k}\bigcirc_{j_i=1}^{\alpha_i}\Delta_{i,x_{ij_i}}\right)p(y_1,y_2,\ldots,y_m).$$

The case $k=m$ gives the desired result. $\Diamond$

We identify W with the space of constant functions defined on $V=$ $=V_1\times V_2\times\ldots\times V_m$ and taking values in W and write

$$\mathcal{P}_0(V,W):=\mathcal{P}_{(0,0,\ldots,0)}(V_1,V_2,\ldots,V_m,W):=W.$$

Then homogeneous polynomials in one and in several variables are connected to each other in the following way.

Theorem 2. *For any* $k\in\mathbb{N}_0$ *we have*

$$\mathcal{P}_k(V,W)=\bigoplus_{\alpha\in\mathbb{N}_0^m,|\alpha|=k}\mathcal{P}_\alpha(V_1,V_2,\ldots,V_m,W).$$

Proof. We may suppose that $k\geq 1$. Let $\alpha\in\mathbb{N}_0^m$ with $|\alpha|=k$. Then $\mathcal{P}_\alpha(V_1,\ldots,V_m,W)\subseteq\mathcal{P}_k(V,W)$. In fact, let $p\in\mathcal{P}_\alpha(V_1,\ldots,V_m,W)$ and

$p = P \circ \delta_\alpha$ with $P \in \mathrm{Hom}_\alpha^{\mathrm{sym}}(V_1, \ldots, V_m, W)$ and let $\pi_i \colon V \to V_i$ be the projection to the i-th coordinate. Define

$$\widetilde{P} := P \circ (\underbrace{\pi_1, \ldots, \pi_1}_{\alpha_1\text{-times}}, \underbrace{\pi_2, \ldots, \pi_2}_{\alpha_2\text{-times}}, \ldots, \underbrace{\pi_m, \ldots, \pi_m}_{\alpha_m\text{-times}}).$$

Then $\widetilde{P} \in \mathrm{Hom}_k(V, W)$ and $\widetilde{P} \circ \delta_k = P \circ \delta_\alpha = p$. Thus

$$\mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W) \subseteq \mathcal{P}_k(V, W).$$

If $p = P \circ \delta_k \in \mathcal{P}_k(V, W)$ with $P \in \mathrm{Hom}_k(V, W)$ we put, for given i and $x_i \in V_i$, as before $\sigma_i(x_i) := (0, 0, \ldots, 0, x_i, 0, \ldots, 0) \in V$. Then $V \ni$ $\ni x = (x_1, x_2, \ldots, x_m) = \sum_{i=1}^m \sigma_i(x_i)$ and (by the multinomial theorem)

$$p(x) = P(x^k) = \sum_{\alpha \in \mathbb{N}_0^m, |\alpha| = k} \frac{k!}{\alpha!} P\big(\sigma_1(x_1)^{\alpha_1}, \sigma_2(x_2)^{\alpha_2}, \ldots, \sigma_m(x_m)^{\alpha_m}\big).$$

Put $p_\alpha(x_1, x_2, \ldots, x_m) = \frac{k!}{\alpha!} P\left(\sigma_1(x_1)^{\alpha_1}, \sigma_2(x_2)^{\alpha_2}, \ldots, \sigma_m(x_m)^{\alpha_m}\right)$. Then p_α is contained in $\mathcal{P}_\alpha^{\mathrm{sym}}(V_1, V_2, \ldots, V_m, W)$ since $p_\alpha = P_\alpha \circ \delta_\alpha$ where $P_\alpha \in$ $\in \mathrm{Hom}_\alpha^{\mathrm{sym}}(V_1, V_2, \ldots, V_m, W)$ is defined by

$$P_\alpha(x_{11}, \ldots, x_{1\alpha_1}, x_{21}, \ldots, x_{2\alpha_2}, \ldots, x_{m1}, \ldots, x_{m\alpha_m}) :=$$
$$:= \frac{k!}{\alpha!} P(\sigma_1(x_{11}), \ldots, \sigma_1(x_{1\alpha_1}), \ldots, \sigma_m(x_{m1}), \ldots, \sigma_m(x_{m\alpha_m})).$$

So $p = \sum_{|\alpha| = k} p_\alpha$ with $p_\alpha \in \mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W)$.

The sum is also direct since $\sum_{|\alpha| = k} p_\alpha = 0$ implies

$$0 = \sum_{|\alpha| = k} p_\alpha(r_1 x_1, \ldots, r_m x_m) = \sum_{|\alpha| = k} r^\alpha p_\alpha(x_1, \ldots, x_m)$$

for all $r = (r_1, \ldots, r_m) \in \mathbb{Q}^m$. This implies that $p_\alpha(x_1, \ldots, x_m) = 0$ for all $x = (x_1, \ldots, x_m) \in V$ and all $\alpha \in \mathbb{N}_0^m$ with $|\alpha| = k$. $\Diamond$

Remark 1. In the last part of the proof above we used the following (see [L, chap. V, p. 121]): Let $\beta \in \mathbb{N}_0^m$ and let

$$N_\beta := \times_{i=1}^m N_{\beta_i}, \quad N_k := \{0, 1, \ldots, k\}.$$

Then, given a family $(u_\alpha)_{\alpha \in N_\beta}$ of elements $u_\alpha \in W$, the relation $\sum_{\alpha \in N_\beta} r^\alpha u_\alpha = 0$ for all $r = (r_1, r_2, \ldots, r_m) \in \mathbb{Q}^m$ implies that all u_α vanish. In fact this is true even if $\sum_{\alpha \in N_\beta} r^\alpha u_\alpha = 0$ holds true (only) for all $r \in \times_{i=1}^m Q_i$, where each $Q_i \subset \mathbb{Q}$ contains at least $\beta_i + 1$ elements.

Definition 5. For $\beta = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m$ with $|\beta| \geq 1$ let

$$(7) \qquad \mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W) :=$$
$$:= \left\{ p \colon V \to W \;\middle|\; p = \sum_{\alpha \in N_\beta} p_\alpha \text{ for some } (p_\alpha)_{\alpha \in N_\beta} \in \times_{\alpha \in N_\beta} \mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W) \right\}.$$

The functions $p \in \mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ are called *generalized polynomials of multidegree* $\leq \beta$.

$\mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ is isomorphic to $\times_{\alpha \in N_\beta} \mathcal{P}_\alpha(V_1, V_2, \ldots, V_m, W)$ by Rem. 1 since for p as above and $r = (r_1, \ldots, r_m) \in \mathbb{Q}^m$ we may write $p(r_1x_1, \ldots, r_mx_m) = \sum_{\alpha \in N_\beta} r^\alpha p_\alpha(x_1, \ldots, x_m)$. We want to characterize the functions in $\mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ and the following is a first step in this direction.

Theorem 3. *Let $m \in \mathbb{N}$, $m \geq 2$, let $V_1, V_2, \ldots, V_m$, $V = V_1 \times \ldots \times V_m$, W as above. Assume that $p\colon V \to W$ has the property that for fixed $\beta' = (\beta_1, \beta_2, \ldots, \beta_{m-1}) \in \mathbb{N}_0^{m-1}$ and $\beta_m \in \mathbb{N}_0$ all partial functions $V_m \ni x_m \mapsto p(x_1, ..., x_{m-1}, x_m)$ are contained in $\mathcal{P}_{\beta_m}(V_m, W)$. Assume furthermore that all partial functions $V_1 \times \ldots \times V_{m-1} =: V' \ni (x_1, x_2, ..., x_{m-1}) \mapsto p(x_1, ..., x_{m-1}, x_m)$ are contained in $\mathcal{P}_{\beta'}(V_1, V_2, \ldots, V_{m-1}, W)$. Then $p \in \mathcal{P}_{(\beta_1, \ldots, \beta_{m-1}, \beta_m)}(V_1, \ldots, V_{m-1}, V_m, W)$.*

Proof. Fixing $x_m \in V_m$ we find by assumption some $P_{x_m} \in \mathrm{Hom}^{\mathrm{sym}}_{\beta'}$ such that

$$p(x_1, x_2, \ldots, x_{m-1}, x_m) = P_{x_m}\left(x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_{m-1}^{\beta_{m-1}}\right)$$

for all $(x_1, x_2, \ldots, x_{m-1}) \in V'$. By (4) we get

$$\begin{aligned} &P_{x_m}(x_{11}, \ldots, x_{1\beta_1}, \ldots, x_{m-1,1}, \ldots, x_{m-1,\beta_{m-1}}) = \\ &= \frac{1}{\beta'!}\left(\bigcirc_{i=1}^{m-1} \bigcirc_{j_i=1}^{\beta_i} \Delta_{i,x_{ij_i}}\right) p(y_1, y_2, \ldots, y_{m-1}, x_m) \end{aligned} \tag{8}$$

with arbitrary $(y_1, \ldots, y_{m-1}) \in V'$. By assumption the mappings $x_m \mapsto p(z_1, \ldots, z_{m-1}, x_m)$ belong to $\mathcal{P}_{\beta_m}(V_m, W)$ for all $(z_1, z_2, \ldots, z_{m-1}) \in V'$. The right-hand side of (8) considered as a function of x_m is a linear combination of functions of that type. So $\widehat{P}\colon V_m \to W$, defined by $\widehat{P}(x_m) := P_{x_m}\left(x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_{m-1}^{\beta_{m-1}}\right)$, is contained in $\mathcal{P}_{\beta_m}(V_m, W)$, too. Now let

$$\begin{aligned} &P\big(x_{11}, \ldots, x_{1\beta_1}, \ldots, x_{m-1,1}, \ldots, x_{m-1,\beta_{m-1}}, x_{m1}, \ldots, x_{m\beta_m}\big) := \\ &:= \frac{1}{\beta_m!} \bigcirc_{j_m=1}^{\beta_m} \Delta_{m,x_{mj_m}} \widehat{P}(y_m) \end{aligned}$$

with arbitrary $y_m \in V_m$. Then

$$P \in \mathrm{Hom}_{(\beta_1, \ldots, \beta_{m-1}, \beta_m)}(V_1, V_2, \ldots, V_{m-1}, V_m, W).$$

Moreover

$$P\big(x_{11},\dots,x_{1\beta_1},\dots,x_{m-1,1},\dots,x_{m-1,\beta_{m-1}},x_{m1},\dots,x_{m\beta_{m-1}},x_m^{\beta_m}\big)= \\ = P_{x_m}(x_{11},\dots,x_{1\beta_1},\dots,x_{m-1,1},\dots,x_{m-1,\beta_{m-1}}).$$

Therefore

$$P\big(x_1^{\beta_1},x_2^{\beta_2},\dots,x_{m-1}^{\beta_{m-1}},x_m^{\beta_m}\big)=P_{x_m}\big(x_1^{\beta_1},x_2^{\beta_2},\dots,x_{m-1}^{\beta_{m-1}}\big)= \\ = p(x_1,x_2,\dots,x_{m-1},x_m).$$

This means that $p=P\circ\delta_{(\beta_1,\beta_2,\dots,\beta_{m-1},\beta_m)}$ with

$$P\in \mathrm{Hom}_{(\beta_1,\dots,\beta_{m-1},\beta_m)}(V_1,\dots,V_{m-1},V_m,W)$$

and thus that $p\in\mathcal{P}_{(\beta_1,\dots,\beta_{m-1},\beta_m)}(V_1,\dots,V_{m-1},V_m,W)$. $\Diamond$

The following theorem gives a characterization of generalized polynomials of multi-degree $\le\beta$. The one-dimensional case of this theorem may be found in [D] and [K, chap. XV, p. 378, pp. 393–397].

Theorem 4. *Let $m\in\mathbb{N}$ and $\beta=(\beta_1,\beta_2,\dots,\beta_m)\in\mathbb{N}_0^m$. Then the following conditions on $p\colon V_1\times\ldots\times V_m\to W$ are equivalent to each other.*

1) $p\in\mathcal{P}^\beta(V_1,V_2,\dots,V_m,W)$;

2) $\bigcirc_{j_i=1}^{\beta_i+1}\Delta_{i,h_{i,i_j}}p=0$ *for all* $1\le i\le m$ *and all* $h_{i1},h_{i2},\dots,h_{i,\beta_i+1}\in V_i$;

3) $\Delta_{i,h_i}^{\beta_i+1}p=0$ *for all* $1\le i\le m$ *and all* $h_i\in V_i$.

Proof. Let $p\in\mathcal{P}^\beta(V_1,\dots,V_m,W)$. Then $p=\sum_{\alpha\in N_\beta}p_\alpha$ with $p_\alpha\in$ $\in\mathcal{P}_\alpha(V_1,\dots,V_m,W)$. For fixed i, α the mapping

$$x_i\mapsto p_\alpha(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_m),\qquad x_j\in V_j$$

fixed when $j\ne i$, is contained in $\mathcal{P}_{\alpha_i}(V_i,W)$. Thus (using the one-dimensional case) $\bigcirc_{j_i=1}^{\alpha_i+1}\Delta_{i,h_{i,j_i}}p_\alpha=0$. Since $\alpha_i\le\beta_i$ this implies $\bigcirc_{j_i=1}^{\beta_i+1}\Delta_{i,h_{i,j_i}}p_\alpha=0$. And this holds true for all $\alpha\in N_\beta$. So condition 1) implies condition 2). Condition 2) obviously implies condition 3). Finally we prove that condition 3) implies condition 1) by induction on m. The case $m=1$ is the "classical" one-dimensional case. Suppose now that the implication 3) $\Rightarrow$ 1) holds true for $m-1$ where $m\ge 2$.

For fixed $x_m\in V_m$ we define $p_{x_m}:V_1\times\ldots\times V_{m-1}\to W$ by $p_{x_m}(x_1,x_2,\dots,x_{m-1},x_m):=p(x_1,x_2,\dots,x_m)$. By assumption $\Delta_{i,h_i}^{\beta_i+1}p_{x_m}=0$ for $1\le i\le m-1$ and $h_i\in V_i$. Thus by the induction hypothesis $p_{x_m}\in$ $\in\mathcal{P}^{\beta'}(V_1,V_2,\dots,V_{m-1},W)$ where $\beta':=(\beta_1,\beta_2,\dots,\beta_{m-1})$. This means that there are $q_{\alpha'}=q_{\alpha',x_m}\in\mathcal{P}_{\alpha'}(V_1,V_2,\dots,V_{m-1},W)$ such that $p_{x_m}=$ $=\sum_{\alpha'\in N_{\beta'}}q_{\alpha'}$.

We also know that $\Delta_{x_m,h_m}^{\beta_m+1}p=0$. Writing $\widehat{q}_{\alpha'}(x_1,\dots,x_{m-1},x_m):=$ $:=q_{\alpha',x_m}(x_1,\dots,x_{m-1})$ and observing $\widehat{q}_{\alpha'}(s_1x_1,s_2x_2,\dots,s_{m-1}x_{m-1},x_m)=$ $=s^{\alpha'}\widehat{q}_{\alpha'}(x_1,x_2,\dots,x_{m-1},x_m)$ for all $s=(s_1,s_2,\dots,s_{m-1})\in\mathbb{Q}^{m-1}$ we get

$$0 = \Delta_{m,h_m}^{\beta_m+1} p(s_1x_1, s_2x_2, \ldots, s_{m-1}x_{m-1}, x_m) =$$
$$= \sum_{\alpha' \in N_{\beta'}} s^{\alpha'} \Delta_{m,h_m}^{\beta_m+1} \widehat{q}_{\alpha'}(x_1, x_2, \ldots, x_{m-1}, x_m)$$

for all $s \in \mathbb{Q}^{m-1}$ and all $(x_1, x_2, \ldots, x_{m-1}, x_m) \in V_1 \times \ldots \times V_{m-1} \times V_m$. Therefore $\Delta_{m,h_m}^{\beta_m+1} \widehat{q}_{\alpha'} = 0$ for all $\alpha' \in N_{\beta'}$ (and all $h_m \in V_m$).

This implies that there exist mappings $\widehat{q}_{j,\alpha'} \colon V_1 \times V_2 \times \ldots \times V_m \to \to W$ such that $\widehat{q}_{\alpha'} = \sum_{j=0}^{\beta_m} \widehat{q}_{j,\alpha'}$ and $(x_m \mapsto \widehat{q}_{j,\alpha'}(x_1, x_2, \ldots, x_{m-1}, x_m)) \in \in \mathcal{P}_j(V_m, W)$, $0 \le j \le \beta_m$.

So $\widehat{q}_{j,\alpha'}(x_1, x_2, \ldots, x_{m-1}, s_m x_m) = s_m^j \widehat{q}_{j,\alpha'}(x_1, x_2, \ldots, x_{m-1}, x_m)$ with $s_m \in \mathbb{Q}$ and also

$$\widehat{q}_{\alpha'}(x_1, x_2, \ldots, x_{m-1}, \ell x_m) = \sum_{j=0}^{\beta_m} \ell^j \widehat{q}_{j,\alpha'}(x_1, x_2, \ldots, x_{m-1}, x_m).$$

Using the inverse of the Vandermonde matrix $(\ell^j)_{0 \le \ell, j \le \beta_m}$ we may find rational numbers $b_{j\ell}$ such that

$$\widehat{q}_{j,\alpha'}(x_1, x_2, \ldots, x_{m-1}, x_m) = \sum_{\ell=0}^{\beta_m} b_{j\ell} \widehat{q}_{\alpha'}(x_1, x_2, \ldots, x_{m-1}, \ell x_m).$$

Thus $\widehat{q}_{j,\alpha'}$ as a function of the first $m-1$ variables is a generalized polynomial of multidegree $\le \beta'$. By the previous theorem this implies that

$$\widehat{q}_{j,\alpha'} \in \mathcal{P}_{(\alpha'_1, \alpha'_2, \ldots, \alpha'_{m-1}, j)}(V_1, V_2, \ldots, V_{m-1}, V_m, W).$$

Thus $p \in \mathcal{P}^{\beta}(V_1, V_2, \ldots, V_m, W)$, as desired. $\Diamond$

This theorem immediately implies the following result.

Corollary 1. *Let $m \in \mathbb{N}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{N}_0^m$. Then $p \colon \times_{i=1}^m V_i \to W$ is a generalized polynomial of multidegree $\le \beta$ if and only if all partial functions $x_i \mapsto p(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$ are generalized polynomials of (simple) degree $\le \beta_i$.*

3. Polynomials in several variables and multi-Jensen functions

The characterization of polynomials in $\mathcal{P}^{\beta}(V_1, V_2, \ldots, V_m, W)$ by Th. 4 is done by a system of functional equations. In [PS, Thm. 6] the connection between polynomials in one variable of degree $\le n$ and n-Jensen functions has been used to show that given rational vector spaces

U, W, a function $q\colon U \to W$ is in $\mathcal{P}^n(U, W)$ if and only if the functional equation

$$q(x) = \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(1+j)^n q\left(\frac{y+jx}{1+j}\right), \quad x, y \in U \tag{9}$$

is satisfied. This may be generalized and as a result we get a characterization of the polynomials of multi-degree $\leq \beta$.

Theorem 5. *Let $V_1, V_2, \ldots, V_, W$ be rational vector spaces and let $\beta \in \in \mathbb{N}_0^m$. Then a function $p\colon V_1 \times V_2 \times \ldots \times V_m \to W$ is a polynomial of multi-degree $\leq \beta$ if and only if the functional equation*

$$\begin{aligned} p(x_1, x_2, \ldots, x_m) = \frac{1}{\beta!}\sum_{\alpha \in N_\beta}\left(\prod_{j=1}^{m}(-1)^{\beta_j - \alpha_j}\binom{\beta_j}{\alpha_j}(1+\alpha_j)^{\beta_j}\right) \times \\ \times p\left(\frac{y_1+\alpha_1 x_1}{1+\alpha_1}, \frac{y_2+\alpha_2 x_2}{1+\alpha_2}, \ldots, \frac{y_m+\alpha_m x_m}{1+\alpha_m}\right) \end{aligned} \tag{10}$$

is satisfied for all $x_j, y_j \in V_j$ and all $1 \leq j \leq m$.

Proof. Let $p \in \mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$. By Cor. 1 this implies that p as a function of the j-th variable is a polynomial of degree $\leq \beta_j$. Thus by (9) we get

$$\begin{aligned} & p(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) = \\ & = \frac{1}{\beta_j!}\sum_{\alpha_j=0}^{\beta_j}(-1)^{\beta_j-\alpha_j}\binom{\beta_j}{\alpha_j}(1+\alpha_j)^{\beta_j} p\left(x_1, \ldots, x_{j-1}, \frac{y_j+\alpha_j x_j}{1+\alpha_j}, x_{j+1}, \ldots, x_m\right) \end{aligned} \tag{11}$$

for all $1 \leq j \leq m$ and all $x_j, y_j \in V_j$. This implies (10).

Conversely, let (10) be satisfied. (9) for the polynomial $q = 1 \in \in \mathbb{Q} = W$ implies that

$$\frac{1}{\beta_l!}\sum_{\alpha_l=0}^{\beta_l}(-1)^{\beta_l-\alpha_l}\binom{\beta_l}{\alpha_l}(1+\alpha_l)^{\beta_l} = 1.$$

Fixing j, putting $y_l = x_l$ for $l \neq j$ and using the above identity we derive equation (11) from (10). Thus p is a polynomial of degree $\leq \beta_j$ in the j-th variable for all j. By Cor. 1 this implies the desired result $p \in \mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$. $\Diamond$

For $\beta = (1^m)$ the space $\mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ is the space of all functions $q\colon V_1 \times V_2 \times \ldots \times V_m \to W$ which are Jensen in each variable.

Now let $\beta \in \mathbb{N}_0^m$ be arbitrary. For convenience and generalizing the case of a single variable we denote the space of all functions $q\colon V_\beta \to W$ which are Jensen in each variable by

$$\mathcal{J}^\beta(V_1, V_2, \ldots, V_m, W) := \mathcal{P}^{(1^{|\beta|})}(\underbrace{V_1, \ldots, V_1}_{\beta_1\text{-times}}, \underbrace{V_2, \ldots, V_2}_{\beta_2\text{-times}}, \ldots, \underbrace{V_m, \ldots, V_m}_{\beta_m\text{-times}}),$$

and by

$$\mathcal{J}^{\beta,\mathrm{sym}}(V_1, V_2, \ldots, V_m, W) := \\ := \{q \in \mathcal{J}^\beta(V_1, V_2, \ldots, V_m, W) \mid q^\pi = q \text{ for all } \pi \in \mathsf{S}_\beta\}$$

the subspace of all *symmetric* q from $\mathcal{J}^\beta(V_1, V_2, \ldots, V_m, W)$.

Theorem 6. *Let $\beta \in \mathbb{N}_0^m$ be given. Then*

1) *$q \circ \delta_\beta \in \mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ for all $q \in \mathcal{J}^\beta(V_1, V_2, \ldots, V_m, W)$.*

2) *For any $p \in \mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ there is some*
$$q \in \mathcal{J}^{\beta,\mathrm{sym}}(V_1, V_2, \ldots, V_m, W) \quad \text{such that } p = q \circ \delta_\beta.$$

Such a q may be written as

(12)
$$q(x_{11}, \ldots, x_{1\beta_1}, x_{21}, \ldots, x_{2\beta_2}, \ldots, x_{m1}, \ldots, x_{m\beta_m}) = \\ = \frac{1}{\beta!} \sum_{\substack{S_1 \subseteq \boldsymbol{\beta_1} \\ \vdots \\ S_m \subseteq \boldsymbol{\beta_m}}} \prod_{j=1}^m \big((-1)^{\beta_j - \alpha_j}(1 + |S_j|)^{\beta_j}\big) \times \\ \times p\left(\frac{y_1 + \sum_{i_1 \in S_1} x_{1i_1}}{1 + |S_1|}, \ldots, \frac{y_m + \sum_{i_m \in S_m} x_{mi_m}}{1 + |S_m|}\right),$$

where the choice of the $y_j \in V_j$ does not affect the values of q.

Proof. Let $q \in \mathcal{J}^\beta(V_1, V_2, \ldots, V_m, W)$. Then

$$\big((x_{j1}, \ldots, x_{j\beta_j}) \mapsto q(\ldots, x_{j1}, \ldots, x_{j\beta_j}, \ldots)\big) \in \mathcal{J}^{\beta_j}(V_j, W).$$

By [PS, Cor. 1] $(x_j \mapsto q(\ldots, \underbrace{x_j, \ldots, x_j}_{\beta_j\text{-times}}, \ldots)) \in \mathcal{P}^{\beta_j}(V_j, W)$. Therefore $q \circ \delta_\beta \in \mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ by Cor. 1.

Now let $p \in \mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ be given. Take some $y_l \in V_l$, $l = 1, 2, \ldots, m$, and define q by (12). For fixed j the right-hand side of (12) may be written as

$$\sum_{S_l \subseteq \boldsymbol{\beta_l}, l \neq j} \prod_{l \neq j} \left(\frac{(-1)^{\beta_l - |S_l|}(1 + |S_l|)^{\beta_l}}{\beta_l!}\right) \times \\ \times \frac{1}{\beta_j!} \sum_{S_j \subseteq \boldsymbol{\beta_j}} (-1)^{\beta_j - |S_j|}(1 + |S_j|)^{\beta_j} p\left(\ldots, \frac{y_j + \sum_{i_j \in S_j} x_{ji_j}}{1 + |S_j|}, \ldots\right).$$

Using (1) with $r = 1$ we may conclude that the inner sum does not depend on y_j and that it is Jensen in each of the variables x_{ji_j}, $1 \le i_j \le$ $\le \beta_j$. Since this holds true for all j the definition of q does not depend on the y_j, $1 \le j \le m$. Moreover q is Jensen in each variable. Obviously $q^\pi = q$ for all $\pi \in \mathsf{S}_\beta$. Finally, putting $x_{j1} = x_{j2} = \ldots = x_{j\beta_j} = x_j$ for all j results in

$$(q \circ \delta_\beta)(x_1, x_2, \ldots, x_m) =$$

$$= \frac{1}{\beta!} \sum_{\substack{S_1 \subseteq \boldsymbol{\beta_1} \\ \vdots \\ S_m \subseteq \boldsymbol{\beta_m}}} \prod_{j=1}^{m} \left((-1)^{\beta_j - \alpha_j} (1+|S_j|)^{\beta_j}\right) p \left(\frac{y_1 + |S_1|\, x_1}{1 + |S_1|}, \ldots, \frac{y_m + |S_m|\, x_m}{1 + |S_m|}\right) =$$

$$= \frac{1}{\beta!} \sum_{\alpha \in N_\beta} \prod_{j=1}^{m} \left((-1)^{\beta_j - \alpha_j} \binom{\beta_j}{\alpha_j} (1+\alpha_j)^{\beta_j}\right) p \left(\frac{y_1 + \alpha_1 x_1}{1 + \alpha_1}, \ldots, \frac{y_1 + \alpha_m x_m}{1 + \alpha_m}\right).$$

This and Th. 5 implies $q \circ \delta_\beta = p$. $\Diamond$

Now we may generalize the result from the first section concerning the relation between $\mathcal{P}^n(V, W)$ and $\mathcal{J}^{n,\mathrm{sym}}(V, W)$.

Theorem 7. *For given $\beta \in \mathbb{N}_0^m$ the mapping $q \mapsto q \circ \delta_\beta$ from $\mathcal{J}^\beta(V_1, V_2, \ldots, V_m, W)$ to $\mathcal{P}^\beta(V_1, V_2, \ldots, V_m, W)$ is a linear isomorphism. The inverse is given by* (12)*.*

Proof. By the considerations above we only must show that the mapping $q \mapsto q \circ \delta_\beta$ is injective. This is done by induction on m. For convenience we also give the argument for $m = 1$. So let $p = q \circ \delta_\beta$ with $q \in$ $\in \mathcal{J}^{\beta,\mathrm{sym}}(V_1, W)$ and assume $p = 0$. We have to show that $q = 0$. By [PS, Th. 3] we know that there are $M_i \in \mathrm{Hom}_i^{\mathrm{sym}}(V_1, W)$ such that $q(x_1, x_2, \ldots, x_\beta) = \sum_{i=0}^{\beta} \sum_{1 \le j_1 < j_2 < \ldots < j_i \le \beta} M_i(x_{j_1}, x_{j_2}, \ldots, x_{j_i})$. Thus $0 =$ $= p(x) = \sum_{i=0}^{\beta} \binom{\beta}{i} M_i(x^i)$. Therefore $M_i \circ \delta_i = 0$ for all i. Thus by Th. 1 $M_i = 0$ for all i. This means that $q = 0$.

Now let $m \ge 2$ and assume that the assertion holds true for $m-1$. Since $0 = p(x_1, x_2, \ldots, x_m) = q(x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_m^{\beta_m})$ and since q as a function of the last β_m variables is multi-Jensen and symmetric by using the case $m = 1$ we get that

$$q(x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_{m1}, \ldots, x_{m\beta_m}) = 0$$

for all $x_i \in V_i$, $1 \le i \le m-1$, and all $x_{mj} \in V_m$. Using then the induction hypothesis we may conclude that $q = 0$. $\Diamond$

4. Diagonalizations of multivariate polynomials

In this section we give answers to the questions posed in the introduction. Let $V_1 = V_2 = \ldots = V_m =: U$

Theorem 8. *Let $n \in \mathbb{N}$, let $\beta \in \mathbb{N}_0^m$, assume $|\beta| = n$. Then for any $P \in \mathcal{P}^\beta(\underbrace{U, \ldots, U}_{m\text{-times}}, W)$ the diagonalization $p = P \circ \delta_m$ is contained in $\mathcal{P}^n(U, W)$*

Proof. Choose $q \in \mathcal{J}^\beta(\underbrace{U, U, \ldots, U}_{m\text{-times}}, W) = \mathcal{J}^{|\beta|}(U, W)$ such that $P = q \circ \delta_\beta$. Then $p = q \circ \delta_n \in \mathcal{P}^n(U, W)$. $\Diamond$

Theorem 9. *Let $p \in \mathcal{P}^n(U, W)$, let $\beta \in \mathbb{N}_0^m$, and assume $|\beta| = n$. Then there is some $P \in \mathcal{P}^\beta(\underbrace{U, U, \ldots, U}_{m\text{-times}}, W)$ such that $P \circ \delta_m = p$.*

Proof. Let $q \in \mathcal{J}^n(U, W)$ with $p = q \circ \delta_n$. Define $P \colon U^m \to W$ by $P := q \circ \delta_\beta$, i.e., $P(x_1, x_2, \ldots, x_m) = q(x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_m^{\beta_m})$. Then the partial functions

$$x_i \mapsto P(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$$

are generalized polynomials of degree $\leq \beta_i$. Thus P is a generalized polynomial in m variables of multi-degree $\leq \beta$. Obviously $P \circ \delta_m = q \circ \delta_n = p$. $\Diamond$

Theorem 10. *Let us denote the P constructed in Th. 9 by p_β. Then $p_n = p$ and p_β may be constructed from p by*

$$(13) \quad p_\beta(x_1, x_2, \ldots, x_m) = \\ = \frac{1}{n!} \sum_{\alpha \in N_\beta} (-1)^{n-|\alpha|} \left(\prod_{j=1}^m \binom{\beta_j}{\alpha_j} \right) (r + |\alpha|)^n p \left(\frac{y + \sum_{j=1}^m \alpha_j x_j}{r + \sum_{j=1}^m \alpha_j} \right),$$

where $(y, r) \in U \times \mathbb{Q}$ either equals $(0, 0)$ (with $0^n p(0/0) := 0$) or y is arbitrary and $r \in \mathbb{Q} \setminus \{0, -1, \ldots, -n\}$.

Proof. Note that $p_\beta = q \circ \delta_\beta$ with $q = p_{(1^n)}$. (1) thus reads as

$$q(w_1, w_2, \ldots, w_n) = \frac{1}{n!} \sum_{S \subseteq \mathbf{n}} (-1)^{n-|S|} (r + |S|)^n p \left(\frac{y + \sum_{i \in S} w_i}{r + |S|} \right).$$

Let $M_j = \{\beta_1 + \ldots + \beta_{j-1} + 1, \ldots, \beta_1 + \ldots + \beta_j\}$. Then $\mathbf{n}$ is the disjoint union of the M_j. Moreover $p_\beta(x_1, x_2, \ldots, x_m) = q(w_1, w_2, \ldots, w_n)$ where $w_i = x_j$ for all $i \in M_j$. The sum over $S \subseteq \mathbf{n}$ may be written as a sum over $S_1 \subseteq M_1, S_2 \subseteq M_2, \ldots, S_m \subseteq M_m$. Doing so we get $y + \sum_{i \in S} w_i = y + \sum_{j=1}^m |S_j|\, x_j$ which only depends on the cardinality of the S_j. This finally gives (13). $\Diamond$

Definition 6. For given $p \in \mathcal{P}^n(U, W)$ and $\beta \in \mathbb{N}_0^m$ with $|\beta| = n$ the mapping $p_\beta \colon U^m \to V$ defined by (13) is called the *β-blossom* of p.

Thus additionally to the original blossom $p_{(1,1,\ldots,1)}$ we have a whole bunch of such blossoms.

Theorem 11. *Let $\beta \in \mathbb{N}_0^m$, $\gamma \in \mathbb{N}_0^l$ satisfy $|\beta| = |\gamma| = n$. Then given $p \in \mathcal{P}^n(U, W)$ the blossoms p_β and p_γ are related by*

$$\begin{aligned}(14)\qquad & p_\beta(x_1, x_2, \ldots, x_m) = \\ &= \frac{1}{n!} \sum_{\alpha \in N_\beta} (-1)^{n-|\alpha|} \left(\prod_{j=1}^m \binom{\beta_j}{\alpha_j} \right) (r + |\alpha|)^n (p_\gamma \circ \delta_l) \left(\frac{y + \sum_{j=1}^m \alpha_j x_j}{r + \sum_{j=1}^m \alpha_j} \right),\end{aligned}$$

where r and y are as in Th. 10.

Proof. This is obvious since $p = p_\gamma \circ \delta_l$. $\Diamond$

Of course (13) is the special case of (14) with $l = 1$ and $\gamma_1 = n$. In the case $m = l$ and $\beta_i = \gamma_i$, $i = 1, 2, \ldots, m$, (14) renders a functional equation for the β-blossom p_β of $p \in \mathcal{P}^n(U, W)$. In particular, the functional equation (23) of [PS] may be read as the special case $m = l = 1$ of (14).

Remark 2. Given p, β as above one cannot expect uniqueness of $P \in \mathcal{P}^\beta(\underbrace{U, U, \ldots, U}_{m\text{-times}}, W)$ with $p = P \circ \delta_m$. Let, for example, $U = W = \mathbb{Q}$, $n = 4$, $\beta = (2, 2)$, and $m = 2$. Then $p \in \mathcal{P}^4(U, W)$ iff $p(x) = \sum_{i=0}^4 a_i x^i$ and $P \in \mathcal{P}^{(2,2)}(U, U, W)$ iff $P(x, y) = \sum_{i=0}^2 \sum_{j=0}^2 a_{ij} x^i y^j$ with some $a_i, a_{ij} \in \mathbb{Q}$. But $p = P \circ \delta_\beta$ is equivalent to

$$a_{00} = a_0,\ a_{10} + a_{01} = a_1,\ a_{20} + a_{11} + a_{02} = a_2,\ a_{21} + a_{12} = a_3,\ a_{22} = a_4$$

showing that there are four coefficients a_{ij} which may be chosen arbitrarily.

Even the consideration of symmetric functions P, which makes sense here since $\beta_1 = \beta_2$, still leaves room for one free parameter. Assuming $p(x) = P(x, x)$ *and* $P(x, y) = P(y, x)$ for all x, y is equivalent to

$$\begin{aligned} &a_{00} = a_0,\ a_{01} = a_{10} = \frac{a_1}{2},\ a_{11} \text{ arbitrary}, \\ &a_{20} = a_{02} = \frac{a_2 - a_{11}}{2},\ a_{12} = a_{21} = \frac{a_3}{2},\ a_{22} = a_4. \end{aligned}$$

The following theorem will give some final answer to all three questions from the introduction.

Theorem 12. *Let $n \in \mathbb{N}$, let $\beta \in \mathbb{N}_0^m$ and assume $|\beta| = n$. Then for any $P \in \mathcal{P}^\beta(\underbrace{U, U, \dots, U}_{m\text{-times}}, W)$ the diagonalization $p = P \circ \delta_m$ is contained in $\mathcal{P}^n(U, W)$. If $p \in \mathcal{P}^n(U, W)$ is given, there is exactly one $P \in \mathcal{P}^\beta(\underbrace{U, U, \dots, U}_{m\text{-times}}, W)$ such that $P \circ \delta_\beta = p$ and such that*

$$\text{(15)}\quad P(x_1, x_2, \dots, x_m) = \\ = \frac{1}{n!} \sum_{\alpha \in N_\beta} (-1)^{n-|\alpha|} \left(\prod_{j=1}^m \binom{\beta_j}{\alpha_j} \right) (r + |\alpha|)^n (P \circ \delta_m) \left(\frac{y + \sum_{j=1}^m \alpha_j x_j}{r + \sum_{j=1}^m \alpha_j} \right),$$

for all $x_1, x_2, \dots, x_m \in U$ with y and r as in the previous theorems.

Proof. This follows from the previous results by taking into consideration that (15) is the same as (13) but formulated in terms of P only. $\Diamond$

The question if functional equation (15) also has other solutions than that of the previous theorem can be answered negatively:

Theorem 13. *Let $\beta \in \mathbb{N}_0^m$, let $P \colon U^m \to W$ be an arbitrary function which satisfies* (15) *with some $r \in \mathbb{Q} \setminus \{0, -1, \dots, -n\}$ for all $y \in U$. Then $P \in \mathcal{P}^\beta(\underbrace{U, U, \dots, U}_{m\text{-times}}, W)$.*

Proof. The function $p = P \circ \delta_m$ satisfies

$$p(x) = \frac{1}{n!} \sum_{\alpha \in N_\beta} (-1)^{n-|\alpha|} \left(\prod_{j=1}^m \binom{\beta_j}{\alpha_j} \right) (r + |\alpha|)^n p\left(\frac{y + |\alpha|\, x}{r + |\alpha|} \right).$$

Note that $|\alpha| \le |\beta| = n$ for $\alpha \in N_\beta$. Thus we have $p(x) = \sum_{k=0}^n a_k p\left(\frac{y+kx}{r+k}\right)$ with $a_k = \frac{(-1)^{n-k}(r+k)^n}{n!} \sum_{\alpha \in N_\beta, |\alpha|=k} \prod_{j=1}^m \binom{\beta_j}{\alpha_j}$. So by [S, Th. 9.5, p. 73], which has also been used in to prove Th. 6 of [PS] we conclude that $p \in \mathcal{P}^n(U, W)$. So (15) becomes (13) with P instead of p_β. Therefore $P = p_\beta$ with $p \in \mathcal{P}^n(U, W)$. This implies the desired result since $p_\beta \in \in \mathcal{P}^\beta(\underbrace{U, U, \dots, U}_{m\text{-times}}, W)$. $\Diamond$

Remark 3. It is obvious that any P satisfying (15) is symmetric provided that all β_i are equal to each other. Continuing the example from the previous remark we see that in fact this condition is stronger than symmetry: $P \in \mathcal{P}^{(2,2)}(U, U, W)$ satisfies $P(x, x) = p(x)$ for all x and (15) if and only if

$$a_{00} = a_0,\ a_{01} = a_{10} = \frac{a_1}{2},\ a_{11} = \frac{2a_2}{3},$$
$$a_{20} = a_{02} = \frac{a_2}{6},\ a_{12} = a_{21} = \frac{a_3}{2},\ a_{22} = a_4,$$

which also demonstrates that P is determined uniquely by p if (15) is satisfied.

5. Functions being polynomials separately in each variable

In this section we consider arbitrary fields K and (classical) polynomial functions $f\colon K^n \to K$ which for $K = \mathbb{Q}$ are just generalized polynomials of some multi-degree β.

In [K, Lemma 4, p. 397] one finds the following assertion:

> *If a function $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(x) = f(\xi_1, \ldots, \xi_n)$ is a polynomial separately in each variable ξ_i, $i = 1, \ldots, n$, then f is a polynomial jointly in all variables.*

In the proof it is (implicitly) assumed that the partial degrees of f are bounded by p independently of the concrete variable and independently of the values of the other variables. This makes the proof rather easy; in fact what is used in the theorem following this Lemma 4 is this result under the mentioned stronger assumption.

In [C] the author proves what the title of the paper states in the case of functions from $\mathbb{R}^2$ to $\mathbb{R}$. Problem E 2940 of the Amer. Math. Monthly asks the question whether, given a function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is a polynomial separately in each variable is a polynomial jointly in both variables. The solution to this problem (Amer. Math. Monthly 91 (1984), p. 142) was a reference to [C].

But it turns out that the situation is quite interesting if the question is asked for an arbitrary field K instead of $\mathbb{R}$. The answer to this generalized Problem E 2940 depends on the cardinality of the field K.

Theorem 14. *Let K be a field and $n \in \mathbb{N}$. Suppose that $f\colon K^n \to K$ is a polynomial function separately in each variable. Then*

1. *f is a polynomial function in n variables provided that K is finite or uncountable.*

2. *For every countable infinite field K there exists a function $f: K^2 \to K$ which is not a polynomial in both variables jointly.*

Proof. If K is finite any function $f\colon K^n \to K$ is a polynomial function. (For completeness we give the arguments: Given $a = (a_1, a_2, \dots, a_n) \in K^n$ we consider the Lagrange polynomial $f_a(x) = f_a(x_1, x_2, \dots, x_n) := \prod_{i=1}^n \prod_{b_i \in K, b_i \neq a_i} \frac{x_i - b_i}{a_i - b_i}$. Then $f = \sum_{a \in K^n} f(a) f_a$.)

For uncountable K we use induction on n and the ideas from [C]. The case $n = 1$ is trivial. So suppose $n \geq 2$. Fixing x_n we get by the induction hypothesis that f is a polynomial in $x_1, x_2, \dots, x_{n-1}$. Thus there are functions $A_\alpha\colon K \to K$, $\alpha \in \mathbb{N}_0^{n-1}$ such that $I(\xi) := \{\alpha \in \mathbb{N}_0^{n-1} \,\big|\, A_\alpha(\xi) \neq 0\}$ is finite for all $\xi \in K$ and such that

$$f(x_1, \dots, x_{n-1}, x_n) = \sum_{\alpha \in \mathbb{N}_0^{n-1}} A_\alpha(x_n) x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{n-1}^{\alpha_{n-1}}$$

for all $(x_1, \dots, x_{n-1}, x_n) \in K^n$. Given $p \in \mathbb{N}$ we define $F_p := \{\xi \in K \,\big|\, I(\xi) \subseteq \{0, 1, \dots, p\}^{n-1}\}$. Since K is the union of the countably many sets F_p and since K is uncountable there is some m such that F_m is uncountable and thus infinite. Therefore

$$f(x_1, \dots, x_{n-1}, x_n) = \sum_{\alpha \in \{0,1,\dots,m\}^{n-1}} A_\alpha(x_n) x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{n-1}^{\alpha_{n-1}} \tag{16}$$

for all $(x_1, x_2, \dots, x_{n-1}) \in K^{n-1}$ and all $x_n \in F_m$. Let us choose subsets Q_i of K with $|Q_i| = m + 1$. By Rem. 1 for K instead of $\mathbb{Q}$ the linear system

$$0 = \sum_{\alpha \in \{0,1,\dots,m\}^{n-1}} u_\alpha y_1^{\alpha_1} y_2^{\alpha_2} \dots y_{n-1}^{\alpha_{n-1}}, \quad (y_1, y_2, \dots, y_{n-1}) \in Q := \bigtimes_{i=1}^{n-1} Q_i$$

with $(m+1)^{n-1}$ equations for the $(m+1)^{n-1}$ variables u_α has only the trivial solution $u_\alpha = 0$, $\alpha \in \{0, 1, \dots, m\}^{n-1}$. Thus (16) may be solved for the $A_\alpha(x_n)$, $x_n \in F_m$:

$$A_\alpha(x_n) = \sum_{(y_1, y_2, \dots, y_{n-1}) \in Q} c_\alpha(y_1, y_2, \dots, y_{n-1}) f(y_1, y_2, \dots, y_{n-1}, x_n).$$

The mapping $x_n \mapsto (y_1, y_2, \dots, y_{n-1}, x_n)$ is a polynomial in x_n. Thus a_α defined by

$$a_\alpha(x) := \sum_{(y_1, y_2, \dots, y_{n-1}) \in Q} c_\alpha(y_1, y_2, \dots, y_{n-1}) f(y_1, y_2, \dots, y_{n-1}, x)$$

is a polynomial in x such that $a_\alpha(x) = A_\alpha(x)$ for all $x \in F_m$ and all $\alpha \in \{0, 1, \dots, m\}^{n-1}$. So $g\colon K^n \to K$,

$$g(x_1,\dots,x_{n-1},x_n) := \sum_{\alpha\in\{0,1,\dots,m\}^{n-1}} a_\alpha(x_n)x_1^{\alpha_1}x_2^{\alpha_2}\dots x_{n-1}^{\alpha_{n-1}}$$

is a polynomial in n variables and thus of the form $g(x_1,\dots,x_{n-1},x_n) = = \sum_{l=0}^{k} g_l(x_1,\dots,x_{n-1})x_n^l$ for some positive integer k and certain polynomials g_l in $n-1$ variables. But

$$f(x_1,\dots,x_{n-1},x_n) = \sum_{l\in\mathbb{N}_0} f_l(x_1,\dots,x_{n-1})x_n^l$$

where for each fixed $(x_1,\dots,x_{n-1})$ only finitely many $f_l(x_1,\dots,x_{n-1})$ are different from 0. Since $f(x_1,\dots,x_{n-1},x_n) = g(x_1,\dots,x_{n-1},x_n)$ for $x_n \in \in F_m$ and F_m is infinite we may conclude that $f_l = g_l$ for $l \le k$ and that $f_l = 0$ for $l > k$. This means that $f = g$. So f is a polynomial in n variables since this is the case for g.

Finally suppose that K is countably infinite, $K = \{x_0, x_1, \dots\}$ with mutually distinct x_i. We define $f\colon K^2 \to K$ by $f(x_i,x_j) := \sum_{k=0}^{i} a_{ik}x_j^k$ such that with certain $a_{ik}, b_{jk} \in K$ we also have $f(x_i,x_j) = \sum_{k=0}^{j} b_{jk}x_i^k$ and $a_{ii} = 1$ for all i.

These coefficients may be constructed by induction: $a_{00} := b_{00} := 1$. If, for $n \ge 0$ we have already found a_{ij}, b_{ij}, $0 \le i,j \le n$ such that

$$\sum_{k=0}^{i} a_{ik}x_j^k = \sum_{k=0}^{j} b_{jk}x_i^k, \quad 0 \le i,j \le n,$$

we put $a_{n+1,n+1} := 1$ and determine the $a_{n+1,k}$ as the unique solution of the interpolation problem

$$\sum_{k=0}^{n} a_{n+1,k}x_j^k = \sum_{k=0}^{j} b_{jk}x_{n+1}^k - a_{n+1,n+1}x_j^{n+1}, \quad 0 \le j \le n.$$

Similarly the $b_{n+1,k}$, $0 \le k \le n+1$ are constructed as the unique solution of

$$\sum_{k=0}^{n+1} b_{n+1,k}x_i^k = \sum_{k=0}^{i} a_{ik}x_{n+1}^k, \quad 0 \le i \le n+1.$$

Then by construction f is a polynomial in the first or second variable if the value of the other variable is kept fixed. If f were a polynomial in both variables jointly we would have $f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{m} c_{ij}x^iy^j$ for all $x, y \in K$ where m is some positive integer and where c_{ij} are certain elements of K. But this would imply

$$\sum_{k=0}^{m+1} a_{m+1,k}y^k = f(x_{m+1},y) = \sum_{k=0}^{m}\left(\sum_{i=0}^{m} c_{ik}x_{m+1}^i\right)y^k, \quad y \in K,$$

a contradiction since $a_{m+1,m+1} = 1$. $\lozenge$

Remark 4. It is a kind of mathematical folklore that the finite fields are exactly those finite commutative rings R with unit for which all functions $f\colon R \to R$ are polynomial functions. Even something more is true:

1. *For any finite field F and any positive integer all functions from F^n to F are polynomial functions.*

2. *Let R be a commutative ring with unit, not necessarily finite. Assume that for some $n \in \mathbb{N}$ all functions from R^n to R are polynomial functions. Then R is a finite field.*

The proof of the first part is contained in the proof of the previous theorem. If R satisfies the hypotheses we get immediately that we may assume $n = 1$. Consider any $R \ni a \neq 0$ and take $f\colon R \to R$ with $f(0) = 0$ and $f(a) = 1$. Since f is a polynomial function there are $c_0, c_1, \dots, c_m \in R$ such that $f(x) = \sum_{j=0}^{m} c_j x^j$ for all $x \in R$. Accordingly $c_0 = f(0) = 0$ and $1 = f(a) = a(c_1 + c_2 a + \ldots + c_m a^{m-1})$. Thus a is a unit and therefore R is a field. Assume that R is infinite. Then we consider $f\colon R \to R$ with $f(1) = 1$ and $f(a) = 0$ for all $a \neq 1$. Thus there is a polynomial $F \in R[X]$ with $f(x) = F(x)$ for all $x \in R$. Since $F(a) = 0$ for $a \neq 1$, F is divisible by $\prod_{j=1}^{m}(X - a_j)$ where $m \in \mathbb{N}$ is arbitrary and $a_1, a_2, \dots, a_m$ are m mutually distinct elements of $R \setminus \{1\}$. So F is divisible by polynomials of arbitrary high degree which means that $F = 0$. But this contradicts $F(1) = 1 \neq 0$.

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Note added in proof. Recently the authors became aware of the fact that Th. 1 from the paper [FH] is closely related to Th. 4 presented here.

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