Mathematica Pannonica
20/2 (2009), 159–166

$\boldsymbol{A}$-GROUPS AND CENTRALIZERS

Lucyna **Kabza**

Department of Mathematics, Southeastern Louisiana University, Hammond, LA 70401, USA

Received: January 2009

MSC 2000: 20 E 36

Keywords: Groups, commuting automorphisms, A-groups, ring of automorphisms, near-rings.

Abstract: A finite group $(G,+)$ is an A-*group* if the near-ring $A = \langle \mathrm{Aut}(G)\rangle$-generated by the automorphisms of G is a ring. If in a group G the centralizer of every element is characteristic in G, then G is an A-group. We investigate situations where the converse is true.

$\boldsymbol{I}$-, $\boldsymbol{E}$- and $\boldsymbol{A}$-groups

Let G be a finite, additive group, and $\mathrm{Inn}(G)$, $\mathrm{Aut}(G)$, $\mathrm{End}(G)$ be the semigroups of inner automorphisms, automorphisms and endomorphisms, respectively. Let $M_0(G) = \{f : G \to G \mid f(0) = 0\}$ be the near-ring of all functions on G that preserve zero with operations of point-wise addition and function composition (substitution). A *right (left) near-ring* $(N,+,\cdot)$ is an algebraic structure such that $(N,+)$ is a group (not necessarily abelian), $(N,\cdot)$ is a semigroup, and the multiplication distributes over addition on the right (left), but not necessarily on both sides. For any group G, $M_0(G)$ is a right near-ring, with $(g+h)\circ f = g\circ f + h\circ g$ for any f, g, and h in $M_0(G)$. From now on, for simplicity, we will write fg for $f\circ g$. Let $I = \langle \mathrm{Inn}(G)\rangle$, $A = \langle \mathrm{Aut}(G)\rangle$, and $E = \langle \mathrm{End}(G)\rangle$, be the subnear-rings of $M_0(G)$ generated by $\mathrm{Inn}(G)$, $\mathrm{Aut}(G)$, $\mathrm{End}(G)$, respectively. Consider $\mathrm{End}(G)$. It may not be a group

E-mail address: lkabza@selu.edu

with respect to addition, since only if G is abelian is the sum of any two endomorphisms an endomorphism. In particular, for a nonabelian group, $id + id$ is not an endomorphism. Yet, for any two endomorphisms α and β in $\mathrm{End}(G)$, $\alpha + \beta$ is an element of the near-ring $E = \langle \mathrm{End}(G) \rangle$. The question was posed to determine when I, A or E is a ring. For the near-ring E to be a ring, addition has to be abelian; hence for any α and β in $\mathrm{End}(G)$, we must have that $\alpha + \beta = \beta + \alpha$, or equivalently, for any $x \in G$, $\alpha(x) + \beta(x) = \beta(x) + \alpha(x)$, and we have to have the distributivity on the right and left as well, but these properties, in our case, follow from the definition of composition of functions (right distributivity) and from the fact that endomorphisms are left distributive and finite sums of endomorphisms are left distributive as well. A group G for which I, A, or E is a ring is called an I-, A-, or *E-group* respectively. An endomorphism α is called a *commuting endomorphism* if $\alpha(x) + x = x + \alpha(x)$, or equivalently, if the commutator $[\alpha(x), x] = -\alpha(x) - x + \alpha(x) + x = 0$ for every $x \in G$. The definition of an A-group is equivalent with every automorphism α being a commuting automorphism (Lemma 4). Throughout the paper, we use both of these definitions interchangeably. If all automorphisms (endomorphisms) are commuting, then the group G has to be nilpotent of class ≤ 3 [3]. Since a nilpotent group is a direct product of its Sylow subgroups, which are fully invariant, a number of results can be reduced to the p-group case. Even though the root of this problem lies in the near-ring theory, due to the characterization of I-, A-, and E-groups in terms of the commuting inner-automorphisms, automorphisms and endomorphisms, respectively, for odd order groups, it became a group theory problem. The question when is I or E a ring has been answered by Chandy in [1] for I and by Maxson and Pettet in [7] for E.

In the next theorem and for the rest of the paper $C_G(x) = \{g \in G \mid [g, x] = 0\}$ is the *centralizer of the element x in G.*

Theorem 1 (Chandy [1])**.** *I is a ring if and only if all conjugate elements commute, i.e., for every $x \in G$, $C_G(x) \trianglelefteq G$.*

Even more is known about I:

Theorem 2 (Chandy [1])**.** *I is a commutative ring if and only if G is nilpotent of class 2.*

Here is a characterization for E-groups.

Theorem 3 (Maxson–Pettet [7])**.** *Let G be a finite p-group, $p > 2$. Then*

G is an E-group if and only if for each $x \in G$, $C_G(x)$ is a fully invariant subgroup.

In view of the above theorems, one naturally wonders if a similar characterization exists for A-groups, namely "If G is a finite p-group, $p > 2$, then G is an A-group if and only if for each $x \in G$, $C_G(x)$ is a characteristic subgroup of G" [7].

However, as Pettet showed in [9] there is an infinite family of counterexamples to this conjecture. In this paper we investigate some types of groups for which this conjecture is true. As is often the case in the theory of p-groups, even though the result analogous to the E-groups characterization is not true in general, we find situations and classes of groups where the above characterization still holds.

First, we establish some properties of A-groups. The first result is analogous to the corresponding result in E-groups. The proof is similar to that of Lemma III.1 in [7].

Lemma 4. *Let G be a finite group. Then the following properties hold:*

(1) *G is an A-group if and only if $[\alpha(x), x] = 0$, for all $x \in G$, and all $\alpha \in \mathrm{Aut}(G)$.*

(2) *G is an A-group if and only if $[\alpha(x), y] = [x, \alpha(y)]$, for all x, $y \in G$, and all $\alpha \in \mathrm{Aut}(G)$.*

(3) *If G is an A-group then $[\alpha(x), \beta(y)] = [\beta(y), \alpha(x)]$, for all x, $y \in G$, and all α, $\beta \in \mathrm{Aut}(G)$. If G is a group of odd order then the converse also holds.*

Proof. (1) G is an A-group $\Leftrightarrow$ for all $\alpha, \beta \in \mathrm{Aut}(G)$, $\alpha + \beta = \beta + \alpha$ $\Leftrightarrow$ for all $\alpha, \beta \in \mathrm{Aut}(G)$, $id + \alpha^{-1}\beta = \alpha^{-1}\beta + id$ $\Leftrightarrow$ for all $\gamma \in \mathrm{Aut}(G)$, $id + \gamma = \gamma + id$ $\Leftrightarrow$ for all $\gamma \in \mathrm{Aut}(G)$, $x \in G$, $x + \gamma(x) = \gamma(x) + x$ $\Leftrightarrow$ for all $\gamma \in \mathrm{Aut}(G)$, $x \in G$, $[x, \gamma(x)] = 0$.

(2) Let G be an A-group. Using (1), since $\alpha \in \mathrm{Aut}(G)$, $\alpha(x - y)$ commutes with $x - y$, it follows that

$$\alpha(x) - \alpha(y) + x - y = x - y + \alpha(x) - \alpha(y).$$

Applying (1) again: $\alpha(x)$ commutes with x and $\alpha(y)$ commutes with y we obtain

$$-\alpha(x) - y + \alpha(x) + y = -x - \alpha(y) + x + \alpha(y),$$

which is exactly $[\alpha(x), y] = [x, \alpha(y)]$.

The converse follows by letting $y = x$ and applying (1).

(3) Let G be an A-group. Replacing y with $\beta(y)$ in (2) we obtain

$$[\alpha(x), \beta(y)] = [x, \alpha\beta(y)] = [\alpha\beta(x), y] = [\beta(x), \alpha(y)].$$

The converse follows by letting $x = y$ and $\beta = id$ in (2). Then $[\alpha(x), x] = [x, \alpha(x)] = [\alpha(x), x]^{-1}$ and since G is of odd order, $[\alpha(x), x]$ cannot be an element of order 2. Hence $[\alpha(x), x] = 0$ and applying (1) the result follows. $\Diamond$

Remark 5. It is easy to see that if G is a group such that the centralizer of every element in G is characteristic, then G is an A-group.

Hence the question reduces to the following: Characterize A-groups G for which the centralizer, $C_G(x)$, of every $x \in G$ is characteristic.

Maxson and Pettet in [7] give an example of an A-group that is not an E-group for any p with $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ ($\mathrm{Aut}_c(G) = \{\alpha \in \mathrm{Aut}(G) \mid -x + \alpha(x) \in Z(G)\}$ – the normal subgroup of central automorphisms), hence the centralizers are characteristic (below Th. 8(1)).

Similarly to the situation of E-groups we have the next property.

Theorem 6. *If G is an A-group then $\mathrm{Aut}(G)/\mathrm{Aut}_c G$ is abelian.*

Proof. Let α, β be in $\mathrm{Aut}(G)$. Let $x, y \in G$. Then since G is an A-group by Th. 4(2)

$$[\beta\alpha(x), y] = [\alpha(x), \beta(y)] = [x, \alpha\beta(y)] = [x, \beta\alpha(y)].$$

Hence

$$[x, \alpha\beta(y)] = [x, \beta\alpha(y)],$$

and

$$-x - \alpha\beta(y) + x + \alpha\beta(y) = -x - \beta\alpha(y) + x + \beta\alpha(y)$$
$$-\alpha\beta(y) + x + \alpha\beta(y) = -\beta\alpha(y) + x + \beta\alpha(y)$$
$$x = (\alpha\beta - \beta\alpha)(y) + x + (\beta\alpha - \alpha\beta)(y)$$
$$x = (\alpha\beta - \beta\alpha)(y) + x - (\alpha\beta - \beta\alpha)(y).$$

Since x is arbitrary, for every $\alpha, \beta \in \mathrm{Aut}(G)$, and for every $y \in G$, $(\alpha\beta - \beta\alpha)(y) \in Z(G)$. This is equivalent to $\alpha\beta(y) = \beta\alpha(y) + z$, where $z \in Z(G)$. Since $Z(G)$ is always characteristic, application of $\alpha^{-1}\beta^{-1}$ to the previous equation results in $-y + [\alpha, \beta](y) \in Z(G)$. Thus $[\mathrm{Aut}(G), \mathrm{Aut}(G)] \leq \mathrm{Aut}_c G$ and the result follows. $\Diamond$

Here are some further properties of commutators related to centralizers. We use the notation $[x, \alpha] = -x + \alpha(x)$ and $x^y = -y + x + y$.

Lemma 7. *Let G be an arbitrary group. Let $x, y \in G$, $y \in C_G(x)$, and $\alpha \in \mathrm{Aut}(G)$. The following are equivalent:*

(1) $[x, \alpha(y)] = 0$; (2) $[x, [y, \alpha]] = 0$;

(3) $[[x, \alpha], y] = 0$; (4) $[[x, \alpha], [y, \alpha]] = 0$.

Proof. The equivalency of (1), (2) and (3) follows from the commutator properties, see [4, p. 18] and assumption that $y \in C_G(x)$. Namely, we have that

$$0 = [x, \alpha(y)] = [x, y + [y, \alpha]] = [x, [y, \alpha]] + [x, y]^{[y,\alpha]} = [x, [y, \alpha]].$$

Now we show that these properties imply (4):

$$\begin{aligned} 0 = \alpha[x, y] &= [\alpha(x), \alpha(y)] = [x + [x, \alpha], y + [y, \alpha]] \\ &= [x + [x, \alpha], [y, \alpha]] + [x + [x, \alpha], y]^{[y,\alpha]}. \end{aligned}$$

Since y commutes with x by assumption, and by (3) y commutes with $[x, \alpha]$, the above is equivalent to

$$0 = [x + [x, \alpha], [y, \alpha]] = [x, [y, \alpha]]^{[x,\alpha]} + [[x, \alpha], [y, \alpha]].$$

Applying (2), (4) follows. The converse is proved by similar steps in the other direction. ♢

If $\mathrm{Aut}(G)$ is of even order, then there is an automorphism $\alpha \in$ $\in \mathrm{Aut}(G)$ of order 2. For $x \in G$, $\alpha(x) = x + a$ for some $a \in G$. Since α has order 2, $\alpha(\alpha(x)) = x$. This implies that

$$x = \alpha(\alpha(x)) = \alpha(x + a) = \alpha(x) + \alpha(a) = x + a + \alpha(a).$$

Hence $x = x + a + \alpha(a)$, which in turn implies that $\alpha(a) = -a$. It follows that if there is an automorphism α of G that has order 2, then there is an element a of G such that $\alpha(a) = -a$. A group G for which there is an automorphism mapping an element of G to its inverse is called an *s.i.* (some inversion) group, as opposed to *n.i.* (no inversion) group.

The main result of the paper is the following:

Theorem 8. *Let G be an A-group. Then each of the following conditions is sufficient for the centralizer of every element of G to be characteristic in G:*

(1) $\mathrm{Aut}(G)/\mathrm{Aut}_c(G)$ *is of odd order.*

(2) $C_G(G') = Z(G)$. *(This implies that* $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$.*)*

(3) $\Phi(G) \cap Z(G') = 1$. *(This implies that* $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$.*)*

(4) *If G is a p-group for a prime $p > 2$ and* $\mathrm{Aut}(G)$ *is nilpotent. (This implies that* $|\mathrm{Aut}(G)|$ *is odd.)*

(5) *For every $x \in G$, $C_G(x)$ is abelian.*

(6) *If G is n.i. group. (This implies that* $|\mathrm{Aut}(G)|$ *is odd.)*

(7) *If p is odd and G is a 2-generator p-group of class 2 with no normal subgroup having a complement in G. (This implies G is n.i.)*

Proof. (1) Let $|\mathrm{Aut}(G)/\mathrm{Aut}_c(G)| = 2k + 1$. Let $y \in C_G(x)$ and $\alpha \in$ $\in \mathrm{Aut}(G)$. Then using (2) in Lemma 4, one gets

$$\alpha^k([x,\alpha(y)]) = [\alpha^k(x),\alpha^{k+1}(y)] = [x,\alpha^{2k+1}(y)] = [x, y+y_Z],$$

where $y_Z \in Z(G)$. Hence $[x, y+y_Z] = [x,y] + [x,y_Z] = [x,y] = 0$. Thus $[x,\alpha(y)] = 0$ which implies that $\alpha(y) \in C_G(x)$ and we get that $C_G(x)$ is characteristic.

(2) $C_G(G') = Z(G)$ implies by Lemma 2.2 [2] that in an A-group G, for any x and any $\alpha \in \mathrm{Aut}(G)$, $[x,\alpha] = -x+\alpha(x) \in C_G(G') = Z(G)$ thus $\alpha \in \mathrm{Aut}_c(G)$and we have that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$, which is a special case of (1).

(3) $\Phi(G) \cap Z(G') = 1$ implies by Remark 4.1 in [2] that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$, which is a special case of (1).

(4) From [6] we know that if $\mathrm{Aut}(G)$ is nilpotent then it is a p-group. Thus $\mathrm{Aut}(G)$ is odd, which is a special case of (1).

(5) Let $y \in C_G(x)$. Since G is an A-group, $\alpha(x) \in C_G(x)$. Since $C_G(x)$ is abelian then $[x,\alpha(y)] = [\alpha(x), y] = 0$. Thus $C_G(x)$ is characteristic.

(6) By the remark before Th. 8, if G is n.i. then $|\mathrm{Aut}(G)|$ is odd, which is a special case of (1).

(7) Heneken and Liebeck in [5, p. 467, Cor. 1] proved that if p is odd then a 2-generator p-group of class 2 is an n.i. group if and only if it does not have a normal subgroup having a complement in G. Hence by (6) we are done. $\Diamond$

Situation (5) occurs, for example, when $C_G(x)/Z(G)$ is cyclic, in particular in A-groups where $[G : Z(G)] = p^2$.

Remark 9. If α is an element of $\mathrm{Aut}(G)$ of order 2 then for any $y \in G$, $\alpha([x,\alpha(y)]) = [\alpha(x),\alpha^2(y)] = [\alpha(x), y] = [x,\alpha(y)]$ or $[x,\alpha(y)]$ is a fixed point of α. So automorphisms of even order can not be f.p.f.

The s.i. and n.i. groups have been investigated and G. A. Miller gave examples of p-groups, for every p, class 2, orders p^9 and exponent p^4 that are n.i. groups [8]. Miller also showed that the automorphism group is itself a p-group. These groups are examples where the above theorem can be applied.

Generalization

The previous section and the results in Chandy, Malone, Pettet–Maxson and Pettet investigate when a group G is an F-group for F generated by a specific subset S of $(\mathrm{End}(G), \circ)$; for example $F = \langle \mathrm{Inn}(G) \rangle$,

$\langle \mathrm{End}(G) \rangle$ and $\langle \mathrm{Aut}(G) \rangle$, meaning that the near-ring $F = \langle S \rangle$ is a ring. Th. 4 above, and similar results for I-groups and E-groups can be generalized if $\mathrm{Inn}(G)$, $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$ is replaced with any subset S containing id of $\mathrm{End}(G)$ and $F = \langle S \rangle$. This generalization was suggested by Gary L. Walls, for which the author is very grateful.

Proposition 10. *Let G be a group with no elements of order 2 and S a subset of $\mathrm{End}(G)$, with $id \in S$. Let $F = \langle S \rangle$. Then G is an F-group if and only if for every $x \in G$ and for every $\alpha \in F$, $[x, \alpha(x)] = 0$.*

Proof. Since $\langle S \rangle$ is a ring then every element of S commutes with id, thus $[\alpha(x), x] = [x, \alpha(x)]$, for every $\alpha \in S$. The converse follows from the proof of Lemma 4. $\Diamond$

Remark 11. Let G be a group with no elements of order 2 and let S be any subset of $\mathrm{End}(G)$, $id \in S$. Let $F = \langle S \rangle$. Then the following are equivalent:

(1) G is an F-group.

(2) For every $x \in G$ and every $\alpha \in S$, $\alpha(Z(C_G(x))) \subseteq C_G(x)$.

(3) For every $x \in G$ $\cup_{\alpha \in S}(\alpha(Z(C_G(x))) \subseteq C_G(x)$.

(4) For every $x \in G$, $\cup_{\alpha \in S}\alpha(\langle x \rangle) \subseteq C_G(x)$.

Proof. It is enough to prove (1) $\Leftrightarrow$ (2).

$:\Rightarrow:$ Let $\alpha \in S$, $x \in G$ and $y \in Z(C_G(x))$. Then $\alpha(x) = x + a$ for some $a \in C_G(x)$ since G is an F-group. Consider

$$[x, \alpha(y)] = [\alpha(x), y] = [x + a, y] = 0$$

since y commutes with both x and a.

$:\Leftarrow:$ Follows immediately, since x is in $Z(C_G(x))$ and $\alpha(x) \in C_G(x)$ imply that x commutes with its image, which is equivalent to the definition of an F-group. $\Diamond$

This leads to a slightly different characterization, but one that unifies the three cases: I-, E- and A-groups.

Corollary 12. *Let G be a finite p-group with $p > 2$. Let $F = \langle S \rangle$ where $S \subseteq \mathrm{End}(G)$ with $id \in S$. Then G is an F-group if and only if for every $\alpha \in F$, and for every $x \in G$, $\alpha(Z(C_G(x))) \subseteq C_G(x)$.*

The author would like to thank C. J. Maxson, for introduction to this problem, Gary Walls, for very helpful comments, and the referees for a thorough evaluation and helpful suggestions that resulted in a better presentation.

References

[1] CHANDY, A. J.: Rings generated by the inner-automorphisms of non-abelian groups, *Proc. Amer. Math. Soc.* **30** (1971), 59–60.

[2] DEACONESCU, M., SILBERG, G. and WALLS, G. L.: On commuting automorphisms of groups, *Arch. Math. (Basel)* **79** (2002), no. 6, 423–429.

[3] FAUDREE, R.: Groups in which each element commutes with its endomorphic images, *Proc. Amer. Math. Soc.* **27** (1971), 236–240.

[4] GORENSTEIN, D.: Finite groups. Second edition, Chelsea Publishing Co., New York, 1980.

[5] HEINEKEN, H., LIEBECK, H.: On groups with odd order automorphism groups, *Arch. Math. (Basel)* **24** (1973), 464–471.

[6] MALINOWSKA, I.: p-automorphisms of finite p-groups: problems and questions, *Advances in group theory* **2002,** Aracne, Rome (2003), 111–127.

[7] MAXSON, C. J. and PETTET, M. R.: Maximal subrings and E-groups, *Arch. Math. (Basel)* **88** (2007), no. 5, 392–402.

[8] MILLER, G. A.: A group of order p^m whose group of isomorphisms is of order p^α, *Messenger of Math.* **43** (1913), 126–128.

[9] PETTET, M. R.: On automorphisms of A-groups, *Arch. Math. (Basel)* **91** (2008), no. 4, 289–299.