Mathematica Pannonica
20/1 (2009), 1–9

AN IMPROVEMENT OF A CRITERION FOR STARLIKENESS

Róbert **Szász**

Department of Mathematics and Informatics, Sapientia University, Calea Sighişoarei 1c, Corunca, Romania

Received: September 2008??

MSC 2000: 30 C 45

Keywords: The operator of Alexander, starlike functions, close-to-convex functions.

Abstract: In this paper a result concerning the starlikeness of the image of the Alexander operator is improved. The technique of differential subordinations is used.

1. Introduction

We introduce the notations $U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ and $U(0, 1) = U$.

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disc U and having the form $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$.

The subclass of $\mathcal{A}$ consisting of functions for which the domain $f(U)$ is starlike with respect to 0, is denoted by S^*. An analytic description of S^* is

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

Another subclass of $\mathcal{A}$ which we deal with is defined by

$$C = \left\{ f \in \mathcal{A} \mid \exists\ g \in S^* : \operatorname{Re} \frac{zf'(z)}{g(z)} > 0,\ z \in U \right\}.$$

E-mail address: szasz_robert2001@yahoo.com
This work was supported by the Sapientia Research Foundation.

This is the class of close-to-convex functions.

We mention that C and S^* contain univalent functions.

The Alexander integral operator is defined by the equality:

$$A(f)(z) = \int_0^z \frac{f(t)}{t}\,dt.$$

Recall that if f and g are analytic in U and g is univalent, then the function f is said to be subordinate to g, written $f \prec g$ if $f(0) = g(0)$ and $f(U) \subset g(U)$.

In [2] it has been proved that $A(C) \not\subset S^*$.

In [1] (p. 310–311) the authors have proved the following result:

Theorem 1. *Let A be the operator of Alexander and let $g \in \mathcal{A}$ satisfy*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \geq \left|\operatorname{Im}\frac{z(zg'(z))'}{g(z)}\right|, \quad z \in U. \tag{1}$$

If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \quad z \in U$$

then $F = A(f) \in S^*$.

The aim of this paper is to prove an improvement of Th. 1.

2. Preliminaries

In order to prove the main result we need the following lemmas.

Lemma 1 [1] p. 22**.** *Let $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$ be analytic in U with $p(z) \not\equiv a$, $n \geq 1$ and let $q : U(0,1) \to \mathbb{C}$ be a univalent function with $q(0) = a$. If there exist two points $z_0 \in U(0,1)$ and $\zeta_0 \in \partial U(0,1)$ so that q is defined in ζ_0, $p(z_0) = q(\zeta_0)$ and $p(U(0,r_0)) \subset q(U)$, where $r_0 = |z_0|$, then there exists an $m \in [n, +\infty)$ so that*

(i) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

and

(ii) $\operatorname{Re}\left(1 + \dfrac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m\operatorname{Re}\left(1 + \dfrac{\zeta_0 q''(\zeta_0)_0}{q'(\zeta_0)}\right)$.

We mention that $z_0 p'(z_0)$ is the outward normal to the curve $p(\partial U(0,r_0))$ at the point $p(z_0)$. ($\partial U(0,r_0)$ denotes the border of the disc $U(0,r_0)$).

Lemma 2 [1] p. 26**.** *Let $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$, $p(z) \not\equiv a$ and $n \geq 1$. If $z_0 \in U$ and*

$$\operatorname{Re} p(z_0) = \min\{\operatorname{Re} p(z) : |z| \leq |z_0|\},$$

then

(i) $z_0 p'(z_0) \leq -\dfrac{n}{2} \dfrac{|p(z_0) - a|^2}{\operatorname{Re}(a - p(z_0))}$

and

(ii) $\operatorname{Re}[z_0^2 p''(z_0)] + z_0 p'(z_0) \leq 0.$

Lemma 3. *If p is an analytic function in U, $p(0) = 1$ and*

$$\operatorname{Re} p(z) \geq \frac{1}{2} |\operatorname{Im}(zp'(z) + p^2(z))|, \quad z \in U, \tag{2}$$

then $|\operatorname{Im}(p(z))| \leq 1, \quad z \in U.$

Proof. Note from (2) we know that $\operatorname{Re} p(z) \geq 0, \quad z \in U$. Let $\epsilon > 0$ be arbitrary and let $\mathcal{B}$ be the band defined by the equality

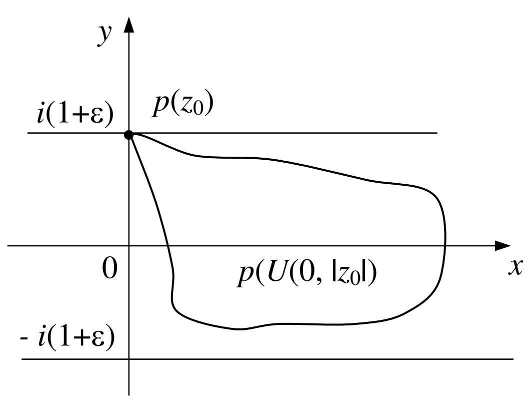
$$\mathcal{B} = \big\{z \in \mathbb{C} \mid |\operatorname{Im} z| \leq 1 + \epsilon, \operatorname{Re} z \geq 0\big\}.$$

We will prove that

$$\operatorname{Im} p(z) \leq 1 + \epsilon, \quad z \in U. \tag{3}$$

If (3) does not hold then according to Lemma 1 there exist a point $z_0 \in U$ and a real number $s \in [0, +\infty)$ so that

$p(U(0, |z_0|)) \subset \mathcal{B}$ and
$p(z_0) = s + i(1 + \epsilon)$, $s \geq 0$
$z_0 p'(z_0) = i\alpha$, $\alpha \geq 0$.



$z_0 p'(z_0)$ is the outward normal to the smooth curve $\gamma = \{p(z) : z \in \in \mathbb{C},\ |z| = |z_0|\}$.

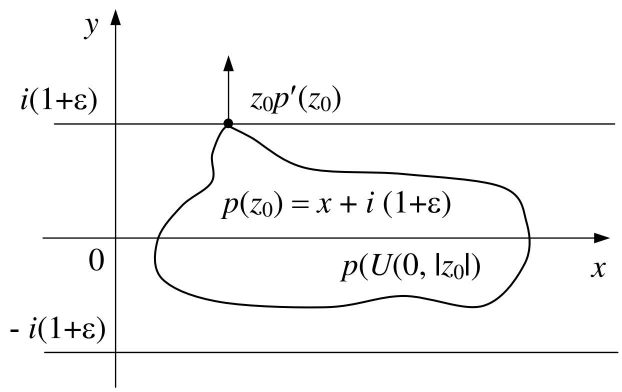
Condition (2) becomes

$$s \geq \frac{1}{2} \big|\operatorname{Im}(i\alpha + [s + i(1 + \epsilon)]^2)\big|$$

or equivalently

$$s \geq \frac{1}{2}|\alpha + 2(1+\epsilon)s|.$$

This inequality can be true only if $\alpha = 0$ and $s = 0$, but this means that $p(U(0,|z_0|)) \subset \mathcal{B}$ and $p(z_0) = i(1+\epsilon)$.



This contradicts the fact that γ is a smooth curve. The case $p(z_0) = = s - i(1+\epsilon)$, $z_0p'(z_0) = -i\alpha$ can be treated analogously. The obtained contradiction implies that

$$|\mathrm{Im}\,(p(z))| \leq 1+\epsilon, \quad z \in U$$

for every $\epsilon > 0$. Now if we put $\epsilon \to 0$ then results

$$|\mathrm{Im}\,(p(z))| \leq 1, \quad z \in U. \qquad \Diamond$$

Remark 1. If we put in Lemma 3 $p(z) = \frac{zg'(z)}{g(z)}$, then

$$zp'(z) + p^2(z) = \frac{z(zg'(z))'}{g(z)} \tag{4}$$

and we get that the condition

$$\mathrm{Re}\,\frac{zg'(z)}{g(z)} \geq \frac{1}{2}\left|\mathrm{Im}\,\frac{z(zg'(z))'}{g(z)}\right|, \quad z \in U$$

implies the inequality $\left|\mathrm{Im}\,\frac{zg'(z)}{g(z)}\right| \leq 1$, $z \in U$.

Lemma 4. *Let q be an analytic function in U and $q(0) = 1$. If $g \in \mathcal{A}$, $\left|\mathrm{Im}\,\frac{zg'(z)}{g(z)}\right| \leq 1$, $z \in U$, then the inequality*

$$\mathrm{Re}\left(zq'(z) + \frac{zg'(z)}{g(z)}q(z)\right) > 0, \quad z \in U \tag{5}$$

implies that $\operatorname{Re} q(z) > 0$, $z \in U$.

Proof. If $\operatorname{Re} q(z) > 0$, $z \in U$ does not hold true, then Lemma 2 implies that there are two real numbers $s, t \in \mathbb{R}$ and a complex number $z_0 \in U$ so that $q(z_0) = is$, $zq'(z_0) = t \le -\frac{1}{2}(s^2+1)$.

Thus

$$\operatorname{Re}\left(z_0 q'(z_0) + \frac{z_0 g'(z_0)}{g(z_0)} q(z_0)\right) = \operatorname{Re}\left(t + \frac{z_0 g'(z_0)}{g(z_0)} is\right) \le$$
$$\le -\frac{1}{2}s^2 - s \operatorname{Im} \frac{z_0 g'(z_0)}{g(z_0)} - \frac{1}{2}.$$

According to the conditions of the lemma we have

$$\Delta = \left(\operatorname{Im} \frac{z_0 g'(z_0)}{g(z_0)}\right)^2 - 1 \le 0$$

and so

$$-\frac{1}{2}s^2 - s \operatorname{Im} \frac{z_0 g'(z_0)}{g(z_0)} - \frac{1}{2} \le 0$$

for all $s \in \mathrm{R}$.

This contradicts condition (5) and yields $\operatorname{Re} q(z) > 0$, $z \in U$. $\Diamond$

3. The main result

Theorem 2. *Let $g \in \mathcal{A}$ be a function which satisfies the condition*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \ge \frac{1}{2}\left|\operatorname{Im}\left(\frac{z(zg'(z))'}{g(z)}\right)\right|, \quad z \in U. \tag{6}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U,$$

then $F = A(f) \in S^*$, where A denotes the Alexander operator.

Proof. The first part of the proof follows the idea of the authors of Th. 1.

From $F = A(f)$ we get that

$$F'(z) + zF''(z) = f'(z).$$

This can be rewritten in the form

$$P(z)(zp'(z) + p^2(z)) = \frac{zf'(z)}{g(z)}, \quad z \in U$$

where $p(z)=\frac{zF'(z)}{F(z)}$ and $P(z)=\frac{F(z)}{g(z)}$.

The conditions of the theorem imply that:

$$\operatorname{Re}\left[P(z)(zp'(z)+p^2(z))\right]>0,\quad z\in U. \tag{7}$$

In the first step we will prove that $\operatorname{Re}P(z)>0$, $z\in U$. A differentiation of the equality $g(z)\cdot P(z)=F(z)$ leads to $g(z)\cdot zP'(z)+zg'(z)P(z)=f(z)$. Differentiating again, we get that

$$z^2P''(z)+zP'(z)+2zP'(z)\cdot\frac{zg'(z)}{g(z)}+P(z)\cdot\frac{z(zg'(z))'}{g(z)}=\frac{zf'(z)}{g(z)}.$$

If $\operatorname{Re}P(z)>0$ does not hold for every $z\in U$, then according to Lemma 2 there are two real numbers $s,t\in\mathbb{R}$ and a point $z_0\in U$ so that

$$\begin{aligned} P(z_0)&=is \\ z_0P'(z_0)&=t\le-\frac{1}{2}(s^2+1) \\ \operatorname{Re}\left[z_0^2P''(z_0)+z_0P'(z_0)\right]&\le 0. \end{aligned} \tag{8}$$

The conditions of the theorem imply $\operatorname{Re}\frac{z_0g'(z_0)}{g(z_0)}\ge 0$ and

$$\Delta=\left(\operatorname{Im}\left(\frac{z_0(z_0g'(z_0))'}{g(z_0)}\right)\right)^2-4\left(\operatorname{Re}\frac{z_0g'(z_0)}{g(z_0)}\right)^2\le 0.$$

From this and (8) results

$$\begin{aligned} \operatorname{Re}\frac{z_0f'(z_0)}{g(z_0)} &= \operatorname{Re}\left[z_0^2P''(z_0)+z_0P'(z_0)\right]+2z_0P'(z_0)\operatorname{Re}\frac{z_0g'(z_0)}{g(z_0)}+ \\ &+\operatorname{Re}\left(P(z_0)\frac{z_0(z_0g'(z_0))'}{g(z_0)}\right)\le 2t\operatorname{Re}\frac{z_0g'(z_0)}{g(z_0)}+ \\ &-s\operatorname{Im}\left(\frac{z_0(z_0g'(z_0))'}{g(z_0)}\right)\le -s^2\operatorname{Re}\frac{z_0g'(z_0)}{g(z_0)}- \\ &-s\operatorname{Im}\left(\frac{z_0(z_0g'(z_0))'}{g(z_0)}\right)-\operatorname{Re}\frac{z_0g'(z_0)}{g(z_0)}\le 0. \end{aligned}$$

This means that $\operatorname{Re}\frac{z_0f'(z_0)}{g(z_0)}\le 0$ is in contradiction with the hypothesis of the theorem and so $\operatorname{Re}P(z)>0$ for all $z\in U$.

Now we return to the relation (7). If $\operatorname{Re}p(z)>0$ does not hold for every $z\in U$, then we apply Lemma 2 for the second time and we get that there are two real numbers $s_1,t_1\in\mathbb{R}$ and a point $z_1\in U$ so that

$$\begin{aligned} p(z_1)&=is_1 \\ z_1p'(z_1)&=t_1\le-\frac{1}{2}(s_1^2+1). \end{aligned}$$

This leads us to a contradiction with the inequality (7) as follows:

$$\operatorname{Re}\left[P(z_1)(z_1p'(z_1)+p^2(z_1))\right] = \operatorname{Re}\left[P(z_1)(t_1-s_1^2)\right] \le 0.$$

The obtained contradiction implies that

$$\operatorname{Re} p(z) = \operatorname{Re}\frac{zF'(z)}{F(z)} > 0, \quad z \in U$$

and so $F \in S^*$. $\Diamond$

We will prove that the condition (1) in Th. 1 can be replaced by the condition $\left|\operatorname{Im}\frac{zg'(z)}{g(z)}\right| \le 1, \quad z \in U$, namely by the inequality from the conclusion of Rem. 1.

Theorem 3. *Let $g \in \mathcal{A}$ be a function, which satisfies the condition*

$$\left|\operatorname{Im}\frac{zg'(z)}{g(z)}\right| \le 1, \quad z \in U. \tag{9}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \quad z \in U,$$

then $F = A(f) \in S^*$ where A denotes the Alexander operator.

Proof. From $F = A(f)$ we obtain that

$$F'(z) + zF''(z) = f'(z).$$

This can be rewritten using the notations $p(z) = \frac{zF'(z)}{F(z)}$ and $P(z) = \frac{F(z)}{g(z)}$ in the following way

$$P(z)(zp'(z)+p^2(z)) = \frac{zf'(z)}{g(z),}\, z \in U.$$

The conditions of Th. 3 imply that

$$\operatorname{Re} P(z)(zp(z)+p^2(z)) > 0, \quad z \in U. \tag{10}$$

First we prove that $\operatorname{Re} P(z) > 0$, $z \in U$.

If we let $Q(z) = \frac{f(z)}{g(z)}$ a simple differentiation of the equalities $g(z)\cdot Q(z) = f(z)$ and $g(z)P(z) = F(z)$ leads to

$$zQ'(z) + \frac{zg'(z)}{g(z)}Q(z) = \frac{zf'(z)}{g(z)} \tag{11}$$

and

$$zP'(z) + \frac{zg'(z)}{g(z)}P(z) = \frac{f(z)}{g(z)}, \quad z \in U. \tag{12}$$

The condition $\operatorname{Re}\frac{zf'(z)}{g(z)} > 0$, equality (11) and Lemma 4 imply that $\operatorname{Re} Q(z) > 0$, $z \in U$, namely $\operatorname{Re}\frac{f(z)}{g(z)} > 0$, $z \in U$.

Now equality (12) and Lemma 4 imply that $\operatorname{Re} P(z) > 0$, $z \in U$.

If $\operatorname{Re} p(z) > 0$, $z \in U$ would not be true, then according to Lemma 2 there are two real numbers $s, t \in \mathbb{R}$ and a point $z_0 \in U$ so that $p(z_0) = is$ and $z_0 p'(z_0) = t \leq -\frac{1}{2}(s^2+1)$. Thus

$$P(z_0)(z_0 p'(z_0) + p^2(z_0)) = P(z_0)(t - s^2)$$

and $\operatorname{Re} P(z_0) > 0$ implies that

$$\operatorname{Re}\left[P(z_0)(z_0 p'(z_0) + p^2(z_0))\right] \leq 0.$$

This inequality contradicts (10), hence we deduce $\operatorname{Re} p(z) = \operatorname{Re} \frac{zF'(z)}{F(z)} > 0$, $z \in U$. ♢

Theorem 4. *If p is an analytic function in U, $p(0) = 1$ and*

(13) $$\operatorname{Re} p(z) > |\operatorname{Im}(zp'(z) + p^2(z))|, \quad z \in U,$$

then $\operatorname{Re} p(z) \geq |\operatorname{Im} p(z)|, \quad z \in U.$

Proof. To prove the assertion we introduce the notation

$$\mathcal{D} = \left\{ z \in \mathbb{C} : |\arg(z)| \leq \frac{\pi}{4} \right\}.$$

We observe that the assertion $\operatorname{Re} p(z) \geq |\operatorname{Im} p(z)|, \quad z \in U$ is equivalent to

(14) $$p \prec q,$$

where

$$q(z) = \sqrt{\frac{1+z}{1-z}}$$

is the Riemann mapping from U to $\mathcal{D}$. (The branch of $\sqrt{z}$ is chosen such that $\operatorname{Im}\sqrt{z} \geq 0$.)

If (14) does not hold true, then Lemma 1 implies that there are two points $z_0 \in U$ and $\zeta_0 \in \mathbb{C}$, $|\zeta_0| = 1$ so that $p(U(0, |z_0|)) \subset q(U)$,

$$p(z_0) = q(\zeta_0)$$

and

$$z_0 p'(z_0) = m \zeta_0 q(\zeta_0)$$

where $m \in \mathbb{R}$, $m \geq 1$.

If $\arg \zeta_0 = \beta$ then $q(\zeta_0) = \sqrt{\operatorname{ctg} \frac{\beta}{2}} \left(\frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2} \right)$, $\operatorname{ctg} \frac{\beta}{2} \geq 0$ and

$$\zeta_0 q'(\zeta_0) = \frac{-1}{4\sqrt{\operatorname{ctg} \frac{\beta}{2}} \sin^2 \frac{\beta}{2} \left(\frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2} \right)}.$$

We discuss the case

$$q(\zeta_0) = \sqrt{\operatorname{ctg} \frac{\beta}{2}} \left(\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right).$$

The other case is similar.

In this case condition (13) becomes

$$\frac{\sqrt{2}}{2}\sqrt{\operatorname{ctg}\frac{\beta}{2}} \geq \left|\frac{m}{4\sqrt{2}\sqrt{\operatorname{ctg}\frac{\beta}{2}}\sin^2\frac{\beta}{2}} + \operatorname{ctg}\frac{\beta}{2}\right|,$$

and using the notation $t = \sqrt{\operatorname{ctg}\frac{\beta}{2}}$, it can be rewritten as follows

$$mt^4 + 4\sqrt{2}t^3 - 4t^2 + m \leq 0. \tag{15}$$

The condition $m \geq 1$ implies that

$$t^4 + 4\sqrt{2}t^3 - 4t^2 + 1 \leq mt^4 + 4\sqrt{2}t^3 - 4t^2 + m.$$

An elementary analysis of the behaviour of the function

$$\varphi : [0, +\infty) \to \mathbb{R},\ \varphi(t) = t^4 + 4\sqrt{2}t^3 - 4t^2 + 1$$

shows that $\varphi(t) > 0, t \in [0, \infty)$ and this contradicts (15). The contradiction implies that $p \prec q$. $\Diamond$

Conclusions

1. The result of Th. 2 is stronger than Th. 1.

2. Th. 1 says that a subclass of the class of close-to-convex functions is mapped by the Alexander operator in the class of starlike functions.

3. Rem. 1 shows that the condition (6) of Th. 2 implies condition (9) of Th. 3 and so Th. 2 is a consequence of Th. 3. Th. 3 asserts that a larger class (as in the case of Th. 2) of analytic functions is mapped by the Alexander operator in S^*, but this larger class contains functions which are not necessary close-to-convex.

4. It would be interesting to study the validity of Th. 1 if we replace condition (1) by the weaker condition $\operatorname{Re}\operatorname{Re}\frac{zg'(z)}{g(z)} \geq |\operatorname{Im}\frac{zg'(z)}{g(z)}|, \quad z \in U$ (which is the consequence of Th. 4 and equality (4)).

References

[1] MILLER, S. S. and MOCANU, P. T.: Differential Subordinations Theory and Applications, Marcel Dekker, 2000.

[2] SZÁSZ, R.: A Counter-Example Concerning Starlike Functions, *Studia Univ. Babeş-Bolyai Mathematica* **LII**, No. 3 (2007), 171–172.