Mathematica Pannonica
20/1 (2009), 1–23

AFFINE POLAR SPACES, THEIR GRASSMANNIANS, AND ADJACENCIES

Małgorzata **Prażmowska**

Institute of Mathematics, University of Białystok, Akademicka 2, 15-267 Białystok, Poland

Krzysztof **Prażmowski**

Institute of Mathematics, University of Białystok, Akademicka 2, 15-267 Białystok, Poland

Mariusz **Żynel**

Institute of Mathematics, University of Białystok, Akademicka 2, 15-267 Białystok, Poland

Received: ??November 2008??

MSC 2000: 51 A 50, 51 A 15

Keywords: Affine polar spaces, Grassmann spaces, adjacency.

Abstract: The underlying affine space, as well as the affine polar space can be reconstructed in terms of each of the three adjacencies on k-subspaces of an affine polar space. Our approach to affine polar spaces differs from that of Cohen and Shult in [5] in that our affine polar space is a point-line incidence structure derived from a metric-affine space, i.e. an affine space equipped with an orthogonality determined by a symmetric bilinear form. This causes that our affine polar spaces include Minkowskian geometry and, in particular, our result generalizes Alexandrov–Zeeman theorems originally concerning adjacency of points of an affine polar space; on the other hand, we exclude symplectic geometry.

E-mail addresses: malgpraz@math.uwb.edu.pl, krzypraz@math.uwb.edu.pl, mariusz@math.uwb.edu.pl

1. Introduction

There is a great deal of papers which discuss adjacencies in Grassmann spaces and related geometries, just to mention [9], [10], [11], [18], [19]. They merely say that the geometry of the underlying structure can be expressed in the language of binary adjacency on subspaces of a given space. This can also be put in the following way: an adjacency preserving bijective transformation is given by an automorphism of the underlying structure. These are just analogues of the well known Chow theorem (cf. [4]) for null systems or a bit more general result by Dieudonné (cf. [7, Ch. III s. 2,3]).

What we would like to do here is to define the affine geometry in terms of adjacency on an affine polar Grassmannian. There are two approaches to affine polar Grassmannians possible. One can start with a polar space in its original sense given by Veldkamp in [27] and further developed in [3], [6] as a pure point-line geometry (a bit different approach to polar geometry is given in [26] which, as shown in [12], is equivalent to the aforementioned one). Then a hyperplane, i.e. a proper subspace meeting every line of the space, needs to be singled out in it as Cohen and Shult do in [5]. That way one gets a synthetic affine polar space. This construction resembles the construction of an affine space in a projective space and that was the intention of [5]. Now, provided that we have a dimension function, it is enough to put a Grassmannian structure on an affine polar space in a standard way to get an affine polar Grassmannian. We are however, following another idea. We start with an affine space over a vector space equipped with a reflexive form and put a Grassmannian structure on isotropic subspaces as in [20]. That way we are able to use relatively better known adjacencies in analytic, versus synthetic, affine Grassmannians.

In fact, our work can be also considered as a natural extension of investigations on Alexandrov–Zeeman-type theorems which state that a bijection of a metric affine space preserving a fixed distance d is actually an automorphisms of this space (cf. [2, Ch. 4], [13], [17]). In the particular case where $d = 0$ (cf. [1], [28]) the bijections in question are exactly those which preserve the adjacency (binary collinearity) of points of the corresponding affine polar space[1] and the Alexandrov–Zeeman-type theorems correspond well with the known fact that the geometry of a polar space can be expressed in terms of the adjacency of its points (cf. [6]).

The course of reasoning that leads to our main result can be divided into three steps. First we find cliques of all the three adjacencies investigated in the paper. They let us reconstruct the linear structure and parallelism in affine polar k-Grassmannians (Sec. 3). The next step is to get the dimension k down to 1. It is done by means of cliques of type star (Sec. 4). Finally, we get our principal result (Th. 4.1) that the underlying affine space as well as affine polar space both can be expressed in terms of each of the three adjacencies.

2. Basic notions and preliminary facts

Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ be a partial linear space. We write $\widetilde{\wp}(\mathfrak{M})$ for the class of all subspaces of $\mathfrak{M}$, and $\wp(\mathfrak{M})$ for the class of strong subspaces of $\mathfrak{M}$. If $\mathfrak{M}$ is an exchange space, then the dimension function is well defined for elements of $\wp(\mathfrak{M})$ and then we write $\wp_k(\mathfrak{M})$ for k-dimensional subspaces of $\mathfrak{M}$. For distinct $X_1, X_2 \in \wp_k(\mathfrak{M})$ we write (comp. general discussion in [8])

$$\begin{aligned} X_1 \sim_- X_2 \quad &\text{iff}\quad X_1 \cap X_2 \in \wp_{k-1}(\mathfrak{M}),\\ X_1 \sim^+ X_2 \quad &\text{iff}\quad X_1 \cup X_2 \subset X \text{ for some } X \in \wp_{k+1}(\mathfrak{M}),\\ X_1 \approx X_2 \quad &\text{iff}\quad X_1 \sim_- X_2 \text{ and } X_1 \sim^+ X_2. \end{aligned}$$

Assume that $\wp_{k+1}(\mathfrak{M}) \neq \emptyset$. Let $X', X'' \in \wp(\mathfrak{M})$ such that $X' \subset X''$. We define:

$$\begin{aligned} [X', X'']_k &:= \{X \in \wp_k(\mathfrak{M})\colon X' \subset X \subset X''\},\\ \mathrm{S}(X') &:= \{X \in \wp_k(\mathfrak{M})\colon X' \subset X\},\ \text{ for } \dim(X') = k-1 \text{ – \textit{a star}}\\ \mathrm{T}(X'') &:= \{X \in \wp_k(\mathfrak{M})\colon X \subset X''\},\ \text{ for } \dim(X'') = k+1 \text{ – \textit{a top}} \end{aligned}$$

$$\mathbf{p}(X', X'') := [X', X'']_k = \mathrm{S}(X') \cap \mathrm{T}(X'') \text{ for } {\scriptstyle \dim(X') = k-1 \atop \dim(X'') = k+1} \text{ – \textit{a pencil.}}$$

[1]More precisely, dealing with the problems related to Alexandrov–Zeeman-type theorems we must define in terms of the adjacency the metric structure of the underlying metric affine space (i.e. we must define *orthogonality* as well as the *affine structure of lines*). Here we are primarily interested in the incidence structure of an affine polar space and we pass over the problem how to define orthogonality in our approach.

Then we put:

$$\mathcal{P}_k(\mathfrak{M}) := \big\{\mathbf{p}(X', X'')\colon X' \in \wp_{k-1}(\mathfrak{M}),\ X'' \in \wp_{k+1}(\mathfrak{M}),\ X' \subset X''\big\},$$
$$\mathbf{P}_k(\mathfrak{M}) := \langle \wp_k(\mathfrak{M}), \mathcal{P}_k(\mathfrak{M})\rangle.$$

If $\mathfrak{M}$ is a projective or affine space, then $\mathbf{P}_k(\mathfrak{M})$ is a well known projective or affine Grassmannian respectively.

Let $\mathbb{V} = \langle V, +, \theta, \cdot\rangle$ be a vector space over a commutative field $\mathfrak{F}$ with characteristic $\neq 2$, and let ξ be a nondegenerate reflexive bilinear form on $\mathbb{V}$ with index $m > 0$ (cf. [14, Ch. I]). The form ξ determines the orthogonality relation $\perp_\xi = \perp$. We write $\mathrm{Sub}_k(\mathbb{V})$ for the set of k-dimensional vector subspaces of $\mathbb{V}$, and $\mathrm{Q}_k(\xi)$ for the set of k-dimensional isotropic subspaces; then $\mathrm{Q}_k(\xi) \neq \emptyset$ iff $k \leq m$. Next, let $\mathfrak{A}$ be the affine space over $\mathbb{V}$, i.e. the affine space $\mathfrak{A} = \mathbf{A}(\mathbb{V}) = \langle V, \mathcal{L}\rangle$, whose lines (elements of $\mathcal{L}$) are all the cosets $a + U$ with $U \in \mathrm{Sub}_1(\mathbb{V})$; we write shortly $\mathcal{L} = V + \mathrm{Sub}_1(\mathbb{V})$. It is a folklore that $\widetilde{\wp}(\mathfrak{A}) = \wp(\mathfrak{A}) = V + \mathrm{Sub}(\mathbb{V})$ and $\wp_k(\mathfrak{A}) = V + \mathrm{Sub}_k(\mathbb{V})$ for every $k \leq \dim(\mathbb{V})$. The substructure $\mathfrak{U}(\xi)$ of $\mathfrak{A}$ defined by

$$\mathbf{U}_1(\mathbb{V}, \xi) := \mathfrak{U}(\xi) := \langle V, \mathcal{G}\rangle, \text{ with } \mathcal{G} = V + \mathrm{Q}_1(\xi)$$

will be referred to as *the affine polar space determined by* ξ. Clearly, $\mathfrak{U}(\xi)$ is a partial linear space. The following is known.

Fact 2.1. *$\mathfrak{U}(\xi)$ is an exchange space. If ξ is antisymmetric then* $\mathrm{Q}_1 = \mathrm{Sub}_1(\mathbb{V})$ *and then simply* $\mathfrak{U}(\xi) = \mathfrak{A}$.

Let ξ be symmetric. Then $\wp(\mathfrak{U}(\xi)) = V + \mathrm{Q}(\xi)$ so, $\wp_k(\mathfrak{U}(\xi)) = V + \mathrm{Q}_k(\xi) \subset \wp_k(\mathfrak{A})$ for every k with $1 \leq k \leq m$.

From now on we assume that

the form ξ is symmetric, $\dim(\mathbb{V}) = \dim(\mathfrak{A}) \geq 3$, $\mathfrak{U} := \mathfrak{U}(\xi)$, and $k \leq m$.

Consequently, the structure $\mathbf{U}_k(\mathbb{V}, \xi) = \mathbf{P}_k(\mathfrak{U}(\xi))$ is a substructure of the affine Grassmannian $\mathbf{P}_k(\mathfrak{A})$. An important, though evident is the following corollary to the known properties of quadratic Grassmannians (cf. [29]).

Fact 2.2. *If a pencil $q = \mathbf{p}(A, B) \in \mathcal{P}_k(\mathfrak{A})$ contains at least three elements of $\wp(\mathfrak{U})$, then $B \in \wp(\mathfrak{U})$ and thus $q \subset \wp(\mathfrak{U})$ and $q \in \mathcal{P}_k(\mathfrak{U})$.*

Let us recall that, when the affine Grassmannian $\mathbf{P}_k(\mathfrak{A})$ is considered, then quite frequently it is assumed that some other type of lines (parallel pencils) is also admitted. Namely, for $U \in \wp_k(\mathfrak{A})$ and $B \in \wp_{k+1}(\mathfrak{A})$ such that $U \subset B$ we write

$$\mathbf{p}^*(U, B) := [U]_\parallel \cap \mathrm{T}(B), \text{ where } [U]_\parallel = \{U'\colon U \parallel U\},$$
$$\mathcal{P}^*_k(\mathfrak{A}) := \big\{\mathbf{p}^*(U, B)\colon U \in \wp_k(\mathfrak{A}),\ B \in \wp_{k+1}(\mathfrak{A}),\ U \subset B\big\}.$$

It is known that the structure $\mathbf{P}^*_k(\mathfrak{A}) = \langle \wp_k(\mathfrak{A}), \mathcal{P}^*_k(\mathfrak{A})\rangle$ is a (not connected) partial linear space, and also the structure $\mathbf{P}^\dagger_k(\mathfrak{A}) :=$ $:= \langle \wp_k(\mathfrak{A}), \mathcal{P}_k(\mathfrak{A}) \cup \mathcal{P}^*_k(\mathfrak{A})\rangle$ is a partial linear space, which is sometimes also called an affine Grassmann space like $\mathbf{P}_k(\mathfrak{A})$ is. Now we can extend the family of pencils of the Grassmann space $\mathbf{U}_k(\mathbb{V}, \xi)$ by the set $\mathcal{P}^*_k(\mathfrak{U})$ of all parallel pencils $\mathbf{p}^*_k(U, B)$ with $B \in \wp_{k+1}(\mathfrak{U}(\xi))$, and this extension, which will be denoted by $\mathbf{U}^\dagger_k(\mathbb{V}, \xi)$ (or shortly by $\mathfrak{U}^\dagger_k$) is a partial linear space as well. Note that 2.2 does not remain valid for elements of $\mathcal{P}^*(\mathfrak{A})$.

Consequently, it may be also convenient to define the (auxiliary) relation $\sim^{\shortparallel}$ for $X_1, X_2 \in \wp_k(\mathfrak{U})$:

$X_1 \sim^{\shortparallel} X_2$ iff $X_1 \cup X_2 \subset X$ for some $X \in \wp_{k+1}(\mathfrak{U})$ and $X_1 \parallel X_2$;

After that we see that $\approx$ is simply the (binary) collinearity of points of $\mathfrak{U}_k$, $\sim^+$ is the collinearity in $\mathfrak{U}^\dagger_k$ (and $\sim^{\shortparallel}$ is the collinearity in $\mathfrak{U}^*_k$).

In the considerations presented below the symbol $\mathrm{S}(H)$ with $H \in$ $\in \wp_{k-1}(\mathfrak{U})$ may be somehow misleading, since it may denote either the set of all k-subspaces of $\mathfrak{A}$ which contain H, or its proper subset consisting of all k-subspaces of $\mathfrak{U}$, which contain H. To avoid this trouble we use the symbol $\mathrm{S}(H)$ for the first set, and $\mathrm{S}_0(H)$ for the second one. Happily, if $B \in \wp_{k+1}(\mathfrak{U})$ then the top $\mathrm{T}(B)$ defined in $\mathfrak{A}$ is also a top in $\mathfrak{U}$.

In the sequel we shall study the three adjacencies $\sim_-$, $\sim^+$, and $\approx$ in the family $\wp_k(\mathfrak{U}(\xi))$. As an auxiliary tool we shall also consider corresponding adjacencies in $\mathfrak{A}$ and in the projective completion $\mathfrak{P}$ of $\mathfrak{A}$.

Let $X_i = a_i + U_i \in \wp_k(\mathfrak{U})$ be distinct for $i = 1, 2$; i.e. let $U_1, U_2 \in \mathrm{Q}_k$. We consider several particular cases. We set $X' = X_1 \cap X_2$, $U' = U_1 \cap U_2$, $U'' = U_1 + U_2$, and let $X'' = X_1 \sqcup X_2$ be the affine subspace spanned by $X_1 \cup X_2$. Recall that

– either $X' = \emptyset$, or $X' = a + U'$, where $a \in X'$ is arbitrary, and

– $X'' = a_i + (U'' + \langle a_2 - a_1\rangle)$ both for $i = 1$ and $i = 2$.

$*$ Let $X' \in \wp_{k-1}(\mathfrak{A})$**:** Then $U' \in \mathrm{Q}_{k-1}$. We can write $X_i = a + U_i$ and $X' = a + U'$ for any $a \in X'$. In particular, $X_1 \sim_- X_2$.

In such a case we have $U'' \in \mathrm{Sub}_{k+1}(\mathbb{V})$ and $X'' = a + U'' \in \wp_{k+1}(\mathfrak{A})$. Consequently, $X_1 \sim^+ X_2$ iff $U'' \in \mathrm{Q}_{k+1}$ which, on the other hand, is equivalent to $U_1 \perp U_2$.

$*$ Let $X'' \in \wp_{k+1}(\mathfrak{A})$**:** Then one of the following two possibilities hold:

a) either $U' = U_1 = U_2$ and then $X_1 \parallel X_2$, or

b) $U'' \in \mathrm{Sub}_{k+1}(\mathbb{V})$.

In case b) we directly get $X_1 \sim_- X_2$. In case a) we have, evidently, $X_1 \not\sim_- X_2$, and also $\dim(U_1 + \langle a_2 - a_1\rangle) = k+1$ so, to obtain $X_1 \sim^+ X_2$ we need $U_1 + \langle a_2 - a_1\rangle \in \mathrm{Q}$, which means that $U_1 \perp (a_2 - a_1)$ and $(a_2 - a_1) \perp (a_2 - a_1)$.

$*$ Let $X_1 \parallel X_2$**:** Then $U_1 = U_2$ and $X'' \in \wp_{k+1}(\mathfrak{A})$ so, we come to the case a) above and then the above investigations explain also when $X_1 \sim^+ X_2$ holds.

Now, we are going to determine maximal cliques of the introduced adjacencies.

Fact 2.3. *Let $H \in \wp_{k-1}(\mathfrak{U})$, $B \in \wp_{k+1}(\mathfrak{U})$, and $Y \in \wp_m(\mathfrak{U})$. Assume that $H \subset Y$. Finally, let $A \in \wp_k(\mathfrak{U})$ and Z be the direction of A, i.e. a suitable subspace of $\mathfrak{P}$; assume that $A \subset Y$.*

(i) *The set $\mathrm{S}_0(H)$ is a maximal $\sim_-$-clique, but it is not a $\sim^+$-clique. It is a subspace of $\mathfrak{U}_k$ but it is not strong.*

(ii) *If $k < m-1$, then the set $[H, Y]_k$ is a maximal $\sim^+$-clique so, it is also a maximal $\approx$-clique. Moreover, it is a subspace of $\mathfrak{U}_k$, isomorphic to the projective space $\mathbf{P}_1(Y/H)$. It is also a subspace of $\mathfrak{U}_k^\dagger$.*

(iii) *If $k < m$, then the set $\mathrm{T}(B)$ is a maximal $\sim^+$-clique, but it is not a $\sim_-$-clique. Clearly, it is a subspace of $\mathfrak{U}_k^\dagger$; in fact it is the affine Grassmann space $\mathbf{P}_k^\dagger(B)$ of hyperplanes of the affine space B.*

(iv) *Let us write $\eth(Z) = [A]_\parallel$ for the parallel class of A, and $[A, Y]_k^* = [A^\infty, Y]_k^* := \eth(Z) \cap [\emptyset, Y]_k$. If $k < m-1$, then the set $[Z, Y]_k^*$ is a maximal $\sim^+$-clique. Clearly, no two distinct elements of $[Z, Y]_k^*$ are in the relation $\sim_-$. Nevertheless, this set is a subspace of $\mathfrak{U}_k^\dagger$, isomorphic to the affine space $\mathbf{A}(Y/A)$.*

(v) *Let $k < m$ and $\mathcal{K} \subset \wp_k(B)$. Note that $\mathcal{K}$ consists of hyperplanes of the affine space B. Assume that $\mathcal{K}$ does not contain a pair of distinct parallel hyperplanes, and for every hyperplane direction in B (i.e. through every hyperplane of the projective horizon of B) there is a hyperplane in $\mathcal{K}$ with this direction. Then $\mathcal{K}$ is a maximal $\approx$-clique (and, consequently, a maximal $\sim_-$-clique as well).*

(vi) *In case $k = m-1$ both the sets $[H, Y]_k$ and $[Z, Y]_k^*$ are lines of $\mathfrak{U}_k$ and $\mathfrak{U}_k^\dagger$ respectively, properly contained in the $\sim^+$-clique $\mathrm{T}(Y)$.*

Consequently, every line of $\mathbf{P}_k(\mathfrak{U}) =: \mathfrak{U}_k$ can be properly extended to at least one maximal $\sim^+$-clique, and at least one maximal $\sim_-$-clique.

Proof. Possibly, (v) needs a small justification. Note, first, that if $\mathcal{K}$ is a $\approx$-clique, then every two $A_1, A_2 \in \mathcal{K}$ must intersect so, they cannot be parallel. Two nonparallel hyperplanes of B are in the relation $\approx$ so, if there would be $A \in \wp_k(B)$ which is not parallel to any one in $\mathcal{K}$, then $\mathcal{K} \cup \{A\}$ would be also a $\approx$-clique and then $\mathcal{K}$ is not maximal. $\Diamond$

It turns out that these are all the possible maximal cliques.

Proposition 2.4. *The cliques listed in 2.3 are all the maximal cliques of corresponding adjacencies.*

(i) *Every maximal $\sim^+$-clique of $\mathfrak{U}_k$ either is contained in an affine star and has form 2.3*(ii)*, or is contained in an affine direction and has form 2.3*(iv)*, or is contained in a top and has form 2.3*(iii).

(ii) *Every maximal $\sim_-$-clique of $\mathfrak{U}_k$ either is contained in a star and has form 2.3*(i) *or is contained in a top and has form 2.3*(v).

(iii) *Every maximal $\approx$-clique of $\mathfrak{U}_k$ either is contained in a star and has form 2.3*(ii) *or is contained in a top and has form 2.3*(v).

Proof. Let us begin a more detailed analysis. Let $\mathcal{K}$ be a maximal $\sim^+$-clique of $\mathfrak{U}$, (or a maximal $\sim_-$-clique). Then $\mathcal{K}$ is contained in a maximal $\sim^+$-clique (a maximal $\sim_-$-clique resp.) $\widetilde{\mathcal{K}}$ of $\mathfrak{A}$, which on the other hand can be extended to a maximal clique $\overline{\mathcal{K}}$ of $\mathfrak{P}$. From 2.3, $\overline{\mathcal{K}}$ is not a line of $\mathbf{P}_k(\mathfrak{P})$. From the known properties of Grassmannians of projective spaces, either

c) $\overline{\mathcal{K}} = \mathrm{S}(Z)$ for some $Z \in \wp_{k-1}(\mathfrak{P})$, or

d) $\overline{\mathcal{K}} = \mathrm{T}(D)$ for some $D \in \wp_{k+1}(\mathfrak{P})$.

Let H^∞ be the hyperplane of $\mathfrak{P}$ which completes $\mathfrak{A}$ to $\mathfrak{P}$.

Consider the case d). If $D \subset H^\infty$ we have $\mathcal{K} = \emptyset$ so, $B := D \setminus H^\infty \in \wp_{k+1}(\mathfrak{A})$. Therefore, $\widetilde{\mathcal{K}} = \mathrm{T}(B)$ holds in $\mathfrak{A}$.

Next, we consider the case c). Suppose that $Z \subset H^\infty$. Then $\widetilde{\mathcal{K}}$ consists of all the k-subspaces of $\mathfrak{A}$ which have Z as its direction so, it is a parallel pencil $[A]_\parallel =: \eth(Z)$ ($A \in \wp_k(\mathfrak{A})$ with direction Z is arbitrary) of subspaces of $\mathfrak{A}$.

Next, let $H := Z \setminus H^\infty \in \wp_{k-1}(\mathfrak{A})$. In this case $\widetilde{\mathcal{K}} = \mathrm{S}(H)$ holds in $\mathfrak{A}$. In every of the above cases we have to determine what are maximal cliques of adjacencies of $\mathfrak{U}$ contained in corresponding affine cliques. In any case we assume that $\mathcal{K}$ contains at least two distinct points A, A', which are then adjacent in the currently investigated sense.

$*$ Let $\mathcal{K} \subset \mathrm{T}(B)$**:** Then $B = A \sqcup A' \in \wp_{k+1}(\mathfrak{U})$. If $\mathcal{K}$ is maximal $\sim^+$-clique, this gives simply that $\mathcal{K} = \mathrm{T}(B)$ holds in $\mathfrak{U}$.

Next, assume that $\mathcal{K}$ is a maximal $\sim_-$-clique. Then, clearly, $\mathcal{K}$ has form described in 2.3(v).

$*$ Let $\mathcal{K} \subset \eth(Z)$**:** Evidently, no $\sim_-$-clique may satisfy this condition so, let $\mathcal{K}$ be a maximal $\sim^{+}$-clique.

Set $Y = \bigcup\mathcal{K}$. We can write $A = a + U$ for some $a \in V$ and $U \in \mathrm{Q}_k$; clearly, $A \parallel U$. Then $\mathcal{M} = -a + \mathcal{K} = \{-a + C\colon C \in \mathcal{K}\}$ is a maximal $\sim^{+}$-clique, $\mathcal{K} = a + \mathcal{M}$, and $Y = a + M$, where $M = \bigcup\mathcal{M}$. Let $A_1, A_2 \in \mathcal{M}$, and let $y_i \in A_i$; then $A_i = U + y_i$. Since $\theta \in U \sqcup A_i = U + \langle y_i\rangle \in \wp_{k+1}(\mathfrak{U})$ we get $U \perp y_i$ and $y_i \perp y_i$. Moreover, $A_1 \sqcup A_2 = y_1 + (U + \langle y_2 - y_1\rangle) \in \wp_{k+1}(\mathfrak{U})$, which gives $(y_2 - y_1) \perp (y_2 - y_1)$. This yields that the subspace $\langle M\rangle$ of $\mathbb{V}$ spanned by M is isotropic, from the maximality of $\mathcal{M}$ we conclude with $\mathcal{M} = \langle M\rangle \in \mathrm{Q}_m$ and thus $Y \in \wp_m(\mathfrak{U})$. Finally, $\mathcal{K} = [Z, Y]^{*}_{k}$.

$*$ Let $\mathcal{K} \subset \mathrm{S}(H)$**:** We obtain $H = A \cap A' \in \wp_{k-1}(\mathfrak{U})$ so, if $\mathcal{K}$ is a maximal $\sim_{-}$-clique we conclude with $\mathcal{K} = \mathrm{S}_0(H)$.

Assume that $\mathcal{K}$ is a maximal $\sim^{+}$-clique and set, as above, $Y = \bigcup\mathcal{K}$. With analogous reasoning we come to $Y \in \wp_m(\mathfrak{U})$ and $\mathcal{K} = [H, Y]_k$.

This closes our analysis and completes the proof. $\Diamond$

As an immediate consequence and a by-product of 2.3 and 2.4 we obtain

Corollary 2.5. *The class of maximal strong subspaces of $\mathfrak{U}_k$ consists of sets of the form 2.3*(ii)*. The class of maximal strong subspaces of $\mathfrak{U}^{\dagger}_{k}$ consists of sets of the form 2.3*(ii)*, 2.3*(iv)*, and 2.3*(iii) *provided $k < m-1$; if $k = m-1$ then the class of maximal strong subspaces of $\mathfrak{U}^{\dagger}_{k}$ consists of sets of the form 2.3*(iii)*.*

In the next step we shall try to reconstruct the underlying geometries $\mathfrak{U}$ and $\mathfrak{A}$ in terms of the adjacencies $\sim_{-}$ and $\sim^{+}$. Dealing with $\sim_{-}$ we shall follow mostly some affine ideas (comp. [22]), while dealing with $\sim^{+}$ mainly ideas related to polar spaces (cf. [19]) will be used. In any case the crucial role will be played by the notion of a generating triangle:

> Let ρ be a binary symmetric relation on a set X. A triple x_1, x_2, x_3 of elements of X will be called *a ρ-generating triangle* iff $x_i \;\rho\; x_j$ for $1 \le i < j \le 3$, and there is the unique maximal ρ-clique that contains x_1, x_2, x_3. If x_1, x_2, x_3 is a generating triangle we write $\mathbf{\Delta}^{\rho}(x_1, x_2, x_3)$.

In an elementary language we can define even more generally:

$$\mathbf{\Delta}^{\rho}_{n}(x_1, \ldots, x_n) \;:\Longleftrightarrow\; \rho(x_1, \ldots, x_n) \text{ and } \forall a, b \left[\; a, b \;\rho\; x_1, \ldots, x_n \Longrightarrow a \;\rho\; b \;\right] \tag{1}$$

and then $\mathbf{\Delta}^{\rho} = \mathbf{\Delta}^{\rho}_{3}$. As an immediate consequence of 2.4(ii) we obtain

Lemma 2.6. *Let $B \in \wp_{k+1}(\mathfrak{U})$ and $\mathcal{Z} = \{A_1, \dots, A_n\}$ be any n-element subset of $\mathrm{T}(B)$ ($n < \infty$). Assume that $A_i \sim_- A_j$ for every $1 \le i < j \le n$ and n is less than the cardinality of the projective horizon of B. Then $\mathcal{Z}$ has at least two distinct extensions to a $\sim_-$-clique and therefore the relation $\mathbf{\Delta}_n^{\sim_-}(A_1, \dots, A_n)$ fails.*

3. Definability of geometry in terms of adjacencies

3.1. Minkowskian geometry and the Alexandrov–Zeeman theorem

Let $m = 1$ i.e. let the underlying metric affine space $\mathfrak{R} = (\mathfrak{A}, \perp)$ be a Minkowskian space. Then only two adjacencies are sensible: $\sim^+ = \approx$ on $\wp_0(\mathfrak{U})$ and $\sim_-$ on $\wp_1(\mathfrak{U})$. Here relevant results are known.

Theorem 3.1. *Let* $\dim(\mathfrak{A}) \ge 3$*. The structures $\mathfrak{U}$, $\mathfrak{A}$, and $\mathfrak{R}$ can be defined in terms of the relation $\sim^+$ on $\wp_k(\mathfrak{U})$ with $k = 0$. If $m = 1$ then the structures $\mathfrak{U}$, $\mathfrak{A}$, and $\mathfrak{R}$ can be defined in terms of the relation $\sim_-$ on the set $\wp_k(\mathfrak{U})$ with $k = 1$.*

Proof. The first claim is simply a reformulation of the Alexandrov–Zeeman theorem mentioned in the Introduction, as it is formulated in [17]. The second claim is a direct consequence of the first one, as a point of $\mathfrak{A}$ can be identified with the (equivalence class of) pair of lines of $\mathfrak{U}$ which pass through this point. $\Diamond$

In what follows, we generalize 3.1 to an arbitrary sensible k. Clearly, the relation $\sim_-$ on $\wp_0(\mathfrak{U})$ is total and therefore useless. Consequently, in view of 3.1 in the subsequent Subs. 3.2–3.5 we assume that $k > 0$.

3.2. The $\sim^+$-adjacency

Here, in this subsection, following 2.3(vi) we need to assume that $k < m - 1$ to have all three types of maximal $\sim^+$-cliques. We write $\mathcal{T}$ for the family of maximal $\sim^+$-cliques defined in (iii) of 2.3, $\mathcal{S}$ for those defined in (ii), and $\mathcal{S}^*$ for those defined in (iv). Clearly, if $\mathcal{K}, \mathcal{K}' \in \mathcal{T}$ then $|\mathcal{K} \cap \mathcal{K}'| \le 1$. We write $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{K}'$. In the projective or lattice-theoretic notation we have $[Z_1, Y_1]_k \cap [Z_2, Y_2]_k = [Z_1 \sqcup Z_2, Y_1 \cap Y_2]_k$. Therefore, to have $\mathcal{K}_0 \ne \emptyset$ we need, first, $Z_1 \subset Y_2$, $Z_2 \subset Y_1$, $\dim(Z_1 \sqcup Z_2) \le k$, and $\dim(Y_1 \cap Y_2) \ge k$. Similarly, to get that $|\mathcal{K}_0| \ge 2$ we need $\dim(Z_1 \sqcup Z_2) < k$ and $\dim(Y_1 \cap Y_2) > k$.

Let us consider the following cases:

$*$ $\mathcal{K} = \mathrm{T}(B) \in \mathcal{T}$ and $\mathcal{K}' = [H', Y']_k \in \mathcal{S}$**:** Either $\mathcal{K}_0 = \emptyset$, or $\mathcal{K}_0 = \mathbf{p}(H', B) \in \mathcal{P}$ is a line of $\mathfrak{U}_k$.

$*$ $\mathcal{K} = \mathrm{T}(B) \in \mathcal{T}$ and $\mathcal{K}' = [U, Y']^*_k \in \mathcal{S}^*$**:** As above, if $\mathcal{K}_0 \neq \emptyset$ then $\mathcal{K}_0 = \mathbf{p}^*(U, B) \in \mathcal{P}^*$ is a line of $\mathfrak{U}^\dagger_k$.

$*$ $\mathcal{K} = [H, Y]_k \in \mathcal{S}$ and $\mathcal{K}' = [H', Y']_k \in \mathcal{S}$**:** If $H \neq H'$ then $|\mathcal{K}_0| \leq 1$ (because in the affine Grassmannian we have $|\mathrm{S}(H) \cap \mathrm{S}(H')| \leq 1$ in this case). Assume that $H = H'$. Then $\mathcal{K}_0 = [H, Y \cap Y']_k$ and (cf. [19, Fact 2.5]) $\dim(Y \cap Y')$ may vary from $k-1$ (then $\mathcal{K}_0 = \emptyset$) through k (then $\mathcal{K}_0$ is a point) and $k+1$ (then $\mathcal{K}_0$ is a line of $\mathfrak{U}_k$) to m.

$*$ $\mathcal{K} = [A, Y]^*_k \in \mathcal{S}^*$ and $\mathcal{K}' = [A', Y']^*_k \in \mathcal{S}^*$**:** As previously, to have $|\mathcal{K}_0| \geq 2$ we need (apply properties of projective stars) to know that A and A' determine the same subspace of the horizon H^∞, which means, in fact, that $A \parallel A'$. Assume that $A \parallel A'$, then $\mathcal{K}_0 = [A^\infty, Y \cap Y']^*_k$. Without loss of generality we can assume that $A \subset Y, Y'$ and then the above reasoning can be applied to justify that $\mathcal{K}_0$ may be a point, a line (of $\mathfrak{U}^\dagger_k$, but not of $\mathfrak{U}_k$), and so on.

$*$ $\mathcal{K} = [H, Y]_k \in \mathcal{S}$ and $\mathcal{K}' = [A, Y']^*_k \in \mathcal{S}^*$**:** To have at least a point in $\mathcal{K}_0$ we must have at least one $A' \parallel A$ such that $H \subset A'$; without loss of generality we can simply assume that $H \subset A \subset Y$. One can note that in such a case $\mathcal{K}_0 = \{A\}$ is a single point.

Proposition 3.2. *Assume that $k < m-1$. Let $A_1, A_2 \in \wp_k(\mathfrak{U})$ with $A_1 \sim^+ A_2$ and $A_1 \neq A_2$. Then the set*

$$\bigcap\{\mathcal{K}\colon A_1, A_2 \in \mathcal{K},\ \mathcal{K} \text{ is a maximal } \sim^+\text{-clique}\} \tag{2}$$

is a k-pencil (proper or parallel) of $\mathfrak{U}$ i.e. it is a line of $\mathfrak{U}^\dagger_k$. Consequently, the structure $\mathfrak{U}^\dagger_k$ is definable in terms of the relation $\sim^+$ considered in $\wp_k(\mathfrak{U})$.

Proof. Let $B = A_1 \sqcup A_2$, thus $B \in \wp_k(\mathfrak{U})$. If

1) $A_1 \cap A_2 \neq \emptyset$ we put $L = \mathbf{p}(A_1 \cap A_2, B) \in \mathcal{P}_k(\mathfrak{U})$,

2) otherwise $A_1 \parallel A_2$ and we put $L = \mathbf{p}^*(A_1, B) \in \mathcal{P}^*_k(\mathfrak{U})$.

From 2.3 and 2.4 we know that every maximal $\sim^+$-clique is a subspace of $\mathfrak{U}^\dagger_k$ and thus L is a subset of the set defined by (2). Finally, we extend B to some $Y \in \wp_m(\mathfrak{U})$ and this extension is essential. ($B \subsetneq Y$). We have $L = \mathrm{T}(B) \cap [A_1 \cap A_2, Y]_k$ in case 1), and $L = \mathrm{T}(B) \cap [A_1, Y]^*_k$ in case 2), which closes the proof. $\Diamond$

Proposition 3.3. *Let $k < m-1$ and let $\mathcal{K}$ be a maximal $\sim^+$-clique. The property*

if a maximal $\sim^+$-clique $\mathcal{K}'$ crosses $\mathcal{K}$ in at least two points,
then $\mathcal{K} \cap \mathcal{K}'$ is a line of $\mathfrak{U}_k^\dagger$

characterizes the relation $\mathcal{K} \in \mathcal{T}$ so, it distinguishes the class $\mathcal{T}$ in the class of all maximal $\sim^+$-cliques. Consequently, the classes $\mathcal{S} \cup \mathcal{S}^*$ and $\mathcal{T}$ both are definable in terms of the relation $\sim^+$.

Proof. It suffices to apply the classification of intersections of maximal cliques given above and note, that for every $\mathcal{K} \in \mathcal{S} \cup \mathcal{S}^*$ there is $\mathcal{K}' \in \mathcal{S} \cup \mathcal{S}^*$ such that $\mathcal{K} \cap \mathcal{K}'$ is greater than a line. $\Diamond$

Lemma 3.4. *The classes $\mathcal{S}$ and $\mathcal{S}^*$ are distinguishable in terms of the relation $\sim^+$.*

Proof. From 2.3 we learn that if $\mathcal{K} \in \mathcal{S}$, then the geometry of $\mathfrak{U}_k^\dagger$ restricted to $\mathcal{K}$ is a projective geometry; in particular, it satisfies the Veblen condition. If $\mathcal{K} \in \mathcal{S}^*$, then it carries the geometry of an affine space and therefore it does not satisfy the Veblen condition. This together with 3.2 closes the proof. $\Diamond$

Corollary 3.5. *For $\mathcal{K}, \mathcal{K}' \in \mathcal{S}$ we write $\mathcal{K} \approx_0 \mathcal{K}'$ iff $|\mathcal{K} \cap \mathcal{K}'| \geq 2$; let $\approx$ be the transitive closure of $\approx_0$. If $k < m - 1$, then the two formulas*

$$A_1 \approx A_2 \iff \exists \mathcal{K} \in \mathcal{S}\, [A_1, A_2 \in \mathcal{K}] \tag{3}$$

$$A_1 \sim_- A_2 \iff \exists \mathcal{K}_1, \mathcal{K}_2 \in \mathcal{S}\, [A_1 \in \mathcal{K}_1,\ A_2 \in \mathcal{K}_2,\ \mathcal{K}_1 \approx \mathcal{K}_2] \tag{4}$$

define corresponding adjacencies in terms of $\sim^+$.

Proof. The validity of (3) is evident. To justify the validity of (4) it suffices to note that if $\mathcal{K}_i = [H_i, Y_i]_k \in \mathcal{S}$ and $|\mathcal{K}_1 \cap \mathcal{K}_2| \geq 2$, then $H_1 = H_2$ so, $A_i \in \mathcal{K}_i$ gives $A_1, A_2 \in \mathrm{S}(H_1)$. Conversely, assume that $A_1, A_2 \in \mathrm{S}(H)$, where $H \in \wp_{k-1}(\mathfrak{U})$. For any two $Y_1, Y_2 \in \wp_m(\mathfrak{U})$, such that $H \subset A_i \subset Y_i$ for $i = 1, 2$ we consider a sequence of maximal strong subspaces of $\mathfrak{U}$ which joins (via $\approx_0$) $\mathcal{K}_1$ and $\mathcal{K}_2$, where $\mathcal{K}_i = [H, Y_i]_k$. $\Diamond$

3.3. The $\sim_-$-adjacency

In view of 2.3 we assume here that $k < m$. When we deal with the relation $\sim_-$, the situation becomes more complex, as the cliques of this relation need not to be subspaces of $\mathfrak{U}_k^\dagger$.

Let $A_1, A_2, A_3 \in \wp_k(\mathfrak{U})$, assume that $A_i \sim_- A_j$ for $1 \leq i < j \leq 3$. From 2.6, $\boldsymbol{\Delta}^{\sim_-}(A_1, A_2, A_3)$ iff A_1, A_2, A_3 are in some $\mathrm{S}(H)$ with $H \in \wp_{k-1}(\mathfrak{U})$, and they are not in one pencil. In such a case we obtain

$$\mathrm{S}_0(H) = [A_1, A_2, A_3]_{\sim_-} = \big\{A \in \wp_k(\mathfrak{U})\colon A \sim_- A_1, A_2, A_3\big\}.$$

Let us write $\mathcal{S}$ for the family of maximal $\sim_-$-cliques defined in (i) of 2.3; in view of the above the set $\mathcal{S} = \big\{[A_1, A_2, A_3]_{\sim_-} : \mathbf{\Delta}_3^{\sim_-}(A_1, A_2, A_3)\big\}$ is definable in terms of the relation $\sim_-$.

Proposition 3.6. *For pairwise distinct $A_1, A_2, A_3 \in \wp_k(\mathfrak{U})$ with $A_1 \sim_- A_2$ we write*

$$\boldsymbol{L}(A_1, A_2, A_3) \iff \exists \mathcal{K} \in \mathcal{S}\ \big[\, A_1, A_2, A_3 \in \mathcal{K} \wedge A_1, A_2, A_3 \in \mathcal{K}' \\ \text{for some } \sim_- \text{-clique such that } \mathcal{K}' \setminus \mathcal{K} \neq \emptyset \,\big]. \tag{5}$$

Assume that $k < m$. Then we obtain (comp. [22]) that $\boldsymbol{L}(A_1, A_2, A_3)$ iff A_1, A_2, A_3 are in one (proper) k-pencil of $\mathfrak{U}$. Consequently, the structure $\mathfrak{U}_k$ is definable in terms of the relation $\sim_-$ considered in the set $\wp_k(\mathfrak{U})$.

Proof. Indeed, if $A_1, A_2 \in \mathcal{K} \in \mathcal{S}$, then $\mathcal{K} = \mathrm{S}_0(A)$ and $A_1 \cap A_2 = A \in \in \wp_{k-1}(\mathfrak{U})$. Let B be the affine space $A_1 \sqcup A_2$ (clearly, $B \in \wp_{k+1}(\mathfrak{A})$ but we do not claim that $B \in \wp(\mathfrak{U})$). Since $A_3 \in \mathcal{K}$ we get that $A \subset A_3$. Since $A_1, A_2 \in \mathcal{K}'$ and $\mathcal{K}' \neq \mathcal{K}$ we get that $\mathcal{K}' \subset \mathrm{T}(B)$ and thus $A_1, A_2, A_3 \in \in \mathbf{p}(A, B) =: q$. Finally, from 2.2 we get that $q \in \mathcal{P}_k(\mathfrak{U})$.

Conversely, let $A_1, A_2, A_3 \in \mathbf{p}(A, B) \in \mathcal{P}_k(\mathfrak{U})$ be pairwise distinct. Set $\mathcal{K} = \mathrm{S}_0(A)$ so, clearly, $A_1, A_2, A_3 \in \mathcal{K} \in \mathcal{S}$. Next, from 2.6, there is $A_0 \in \mathrm{T}(B) \setminus \mathrm{S}_0(A)$ such that $A_0 \sim_- A_1, A_2, A_3$; for a clique $\mathcal{K}' = \{A_1, A_2, A_3, A_0\}$ we have $A_0 \in \mathcal{K}' \setminus \mathcal{K}$. Finally, we conclude with $\boldsymbol{L}(A_1, A_2, A_3)$. $\Diamond$

As a corollary to 3.6 we obtain also

Proposition 3.7. *Let $A_1 \neq A_2$. We have*

$$A_1 \approx A_2 \iff \exists A_3\, \boldsymbol{L}(A_1, A_2, A_3) \wedge A_3 \neq A_1, A_2$$

so, the relation $\approx$ is definable in terms of $\sim_-$ provided that $k < m$.

3.4. The $\approx$-adjacency

In this case the situation needs some other methods. First we assume that $k < m - 1$. Evidently, if $a \in H \subset B \subset Y \in \wp_m(\mathfrak{U})$ and $H \in \wp_{k-1}(\mathfrak{U})$, $B \in \wp_{k+1}(\mathfrak{U})$, then $q = \mathbf{p}(H, B) = [a, B]_k \cap [H, Y]_k$ is the intersection of two maximal cliques of $\approx$ in $\mathfrak{U}$. However, not every nontrivial intersection of two maximal $\approx$-cliques is a line, and if $A_1 \approx A_2$, $A_1 \neq A_2$ then $\bigcap\{\mathcal{K} \colon A_1, A_2 \in \mathcal{K},\ \mathcal{K} \text{ is a maximal } \approx\text{-clique}\} = = \{A_1, A_2\}$. What is more, no triple may satisfy $\mathbf{\Delta}_3^{\approx}$, unless $k = m - 2$. Therefore, the lines of $\mathfrak{U}_k$, even if definable, must be reconstructed with some other methods.

In view of 2.4, maximal $\approx$-cliques fall into two classes. We write $\mathcal{S}$ for the family of maximal cliques of the form 2.3(ii), and $\mathcal{T}$ for the set of maximal cliques defined in 2.3(v). Let $\mathcal{K}$ be a maximal $\approx$-clique in $\mathfrak{U}_k$. Take arbitrary $A \in \mathcal{K}$ and put $\mathcal{K}_0 = \mathcal{K} \setminus \{A\}$. In view of 2.4(iii) two possibilities arise:

$*$ $\mathcal{K} = [H, Y]_k \in \mathcal{S}$**:** Then $[\mathcal{K}_0]_{\approx} = \mathcal{K}$ so, $\mathcal{K}_0$ has the unique extension $\mathcal{K}$ to a maximal clique of $\approx$.

$*$ $\mathcal{T} \ni \mathcal{K} \subset \mathrm{T}(B)$**:** For arbitrary $A' \in \wp_k(B)$ such that $A \parallel A'$ we have $\mathcal{K}_0 \cup \{A'\} \in \mathcal{T}$. Consequently, $\mathcal{K}_0$ has at least two distinct extensions to a maximal $\approx$-clique.

Besides, for any two A', A'' which complete $\mathcal{K}_0$ to a maximal $\approx$-clique we have $A' \sim^{\shortparallel} A''$.

As a consequence of these considerations we get

Corollary 3.8. *If $k < m - 1$, then the two types $\mathcal{S}$ and $\mathcal{T}$ of maximal $\approx$-cliques can be distinguished in terms of the relation $\approx$. Therefore, the families $\mathcal{T}$ and $\mathcal{S}$ are definable in terms of the relation $\approx$ considered in $\wp_k(\mathfrak{U})$. Moreover, the relation $\sim^{\shortparallel}$ is also definable.*

Lemma 3.9. *Let $k = m - 1$. For $A_1, A_2 \in \wp_k(\mathfrak{U})$ we have*

$$A_1 \sim^{\shortparallel} A_2 \iff \exists A', A''\big[A_1, A_2 \approx A', A'' \wedge A' \approx A'' \wedge \neg A_1 \approx A_2\big].$$

Proof. It suffices to note that in this case sets of the form 2.3(ii) are lines of $\mathfrak{U}_k$ and they are not maximal $\approx$-cliques. Therefore, if $A_i \approx A', A''$ and distinct $A', A'' \in \mathbf{p}(H, B)$ then $A_i \in \mathrm{T}(B)$. $\Diamond$

Corollary 3.10. *Let $k < m$. Since, evidently, we have $A_1 \sim^{+} A_2 \iff$ $\iff A_1 \approx A_2 \vee A_1 \sim^{\shortparallel} A_2$, the relation $\sim^{+}$ on $\wp_k(\mathfrak{U})$ is definable in terms of $\approx$.*

Corollary 3.11. *If $k < m - 1$, then in view of 3.8, the formula* (4) *defines the relation $\sim_{-}$ in terms of $\approx$.*

Proposition 3.12. *Assume that $k < m - 1$. Let $A_1 \approx A_2$ and $A_1 \neq A_2$ for some $A_1, A_2 \in \wp_k(\mathfrak{U})$. This is equivalent to say that $H := A_1 \cap A_2 \in$ $\in \wp_{k-1}(\mathfrak{U})$, $B := A_1 \sqcup A_2 \in \wp_{k+1}(\mathfrak{U})$, and $A_1, A_2 \in \mathbf{p}(H, B) \in \mathcal{P}_k(\mathfrak{U})$. Then*

$$\mathbf{p}(H, B) \quad = \quad \bigcap\{\mathcal{K} \colon A_1, A_2 \in \mathcal{K},\, \mathcal{K} \in \mathcal{S}\}.$$

Consequently, the structure $\mathfrak{U}_k$ is definable in terms of the relation $\approx$.

3.5. The $\sim^{\shortparallel}$-adjacency

It is rather clear that the relation $\sim^{\shortparallel}$ cannot be used to express the geometry of $\mathfrak{U}_k$. From 2.3 and 2.4 we learn immediately that the maximal $\sim^{\shortparallel}$-cliques are sets of the form 2.3(iv), though in case $k = m-1$ they are simply lines of $\mathfrak{U}_k^*$; we write $\mathcal{S}^*$ (comp. Subs. 3.2) for the class of such cliques. Consequently, a direct analogue of 3.12 holds:

Remark 3.13. Assume that $k < m$. Let $A_1 \sim^{\shortparallel} A_2$ and $A_1 \neq A_2$ for some $A_1, A_2 \in \wp_k(\mathfrak{U})$. This is equivalent to say that $A_1 \parallel A_2$ and $B := A_1 \sqcup A_2 \in \wp_{k+1}(\mathfrak{U})$, i.e. $A_1, A_2 \in \mathbf{p}^*(A_1, B) \in \mathcal{P}_k^*(\mathfrak{U})$. Then

$$\mathbf{p}^*(A_1, B) \;=\; \bigcap\{\mathcal{K}\colon A_1, A_2 \in \mathcal{K},\, \mathcal{K} \in \mathcal{S}^*\}.$$

Thus the structure $\mathfrak{U}_k^*$ is definable in terms of the relation $\sim^{\shortparallel}$.

4. Reducing dimensions, definability of the affine structure

The idea is simple and standard.

(I) In terms of $\sim_-$ we can interpret the family $\{\mathrm{S}_0(H)\colon H \in \wp_{k-1}(\mathfrak{U})\}$; via the map $H \longmapsto \mathrm{S}_0(H)$ its elements can be identified with the elements of $\wp_{k-1}(\mathfrak{U})$. Clearly, $\mathrm{S}_0(H_1) \cap \mathrm{S}_0(H_2) \neq \emptyset$ iff $H_1 \sim^+ H_2$ so, our interpretation enables us to define $\langle \wp_{k-1}(\mathfrak{U}), \sim^+ \rangle$ within the structure $\langle \wp_k(\mathfrak{U}), \sim_- \rangle$.

(II) In terms of both $\sim^+$ and of $\approx$ we can interpret the family $\{[H, Y]_k\colon H \in \wp_{k-1}(\mathfrak{U}), Y \in \wp_m(\mathfrak{U}), H \subset Y\}$. Equivalence classes of elements of this set under the relation $\approx$ defined in 3.5 correspond to the elements of $\wp_{k-1}(\mathfrak{U})$: $[H_1, Y_1]_k \approx [H_2, Y_2]_k$ iff $H_1 = H_2$. As above, we obtain an interpretation of $\langle \wp_{k-1}(\mathfrak{U}), \sim^+ \rangle$ in the structure $\langle \wp_k(\mathfrak{U}), \sim^+ \rangle$ (in $\langle \wp_k(\mathfrak{U}), \approx \rangle$ resp.).

Inductively, we come to the relation $\sim_-$ on $\wp_1(\mathfrak{U})$ and, finally, to $\sim^+$ on $\wp_0(\mathfrak{U})$. Finally, 3.1 can be applied.

Summing up we have

Theorem 4.1. *Let $k < m$ and $\sim \in \{\sim^+, \sim_-, \approx\}$. If $\sim = \sim^+, \approx$ we assume, additionally, that $k < m-1$ and if $\sim = \sim_-$ we assume that $k > 0$. Then the affine space $\mathfrak{A}$ and the affine polar space $\mathfrak{U}$ contained in $\mathfrak{A}$ are both definable in $\langle \wp_k(\mathfrak{U}), \sim \rangle$. Consequently, both $\mathfrak{A}$ and $\mathfrak{U}$ are also definable in $\mathfrak{U}_k$ and in $\mathfrak{U}_k^\dagger$ provided that $k < m-1$.*

5. One extraordinary and one special case

Theorem 4.1 does not decide three cases: $\sim_-$ on $\wp_m(\mathfrak{U})$ and $\approx, \sim^+$ on $\wp_{m-1}(\mathfrak{U})$. In this section these cases will be decided.

The family $\wp_k(\mathfrak{U})$ admits one more adjacency, directly inherited from the structure of the Grassmannian over $\mathfrak{A}$. Note that for distinct $A, B \in \wp_k(\mathfrak{U}) \subset \wp_k(\mathfrak{A})$ the relation $A \sim_- B \vee A \parallel B$ holds iff $A \sim^+ B$ holds in $\mathbf{P}^\dagger_k(\mathfrak{A})$; in such a case we write $A \simeq^+ B$.

Fact 5.1. *Let $D \in \wp_{k+1}(\mathfrak{A}) \setminus \wp_{k+1}(\mathfrak{U})$ contain two nonparallel subspaces in $\wp_k(\mathfrak{U})$, let $H \in \wp_{k-1}(\mathfrak{U})$, let $D' \in \wp_{k+1}(\mathfrak{U})$, and let $A \in \wp_k(\mathfrak{U})$.*

(I) *The sets $\mathrm{T}(D')$, $\mathrm{S}_0(H)$, and $[A]_\parallel$ are maximal cliques of the relation $\simeq^+$.*

(II) *Let $\wp_k(\mathfrak{U}) \ni A', A'' \subset D$ and $A' \nparallel A''$. The set*

$$\{A \in [A']_\parallel \cup [A'']_\parallel \colon A \subset D\}$$

is a maximal $\simeq^+$-clique.

Proposition 5.2. *Every maximal $\simeq^+$-clique is one of the above.*

Proof. Let $A' \simeq^+ A''$, then either $A' \parallel A''$ or $A' \sim_- A''$. Let D be the affine subspace spanned by $A' \cup A''$ and, in the second case, we set $H := := A' \cap A''$. Assume that $A' \sim_- A''$ and let $A''' \simeq^+ A', A''$. Then either $A''' \supset H$ or $A''' \subset D$. In the first case if $A_0 \simeq^+ A', A'', A'''$ then $A_0 \in \mathrm{S}_0(H)$. In the second one we have to consider two cases. Either $A''' \nparallel A', A''$ and then $D \in \wp_{k+1}(\mathfrak{U})$ and a clique which contains A', A'', A''' is contained in $\mathrm{T}(D)$. Or there is no such A''', $D \in \wp_{k+1}(\mathfrak{A}) \setminus \wp_{k+1}(\mathfrak{U})$ and then the corresponding clique is contained in the set of the form (II). Analogously we proceed when $A' \parallel A''$. $\Diamond$

Let us write $\mathcal{T}^*$ for the set of cliques of the form (II), $\mathcal{S}^*$ for the class of the affine parallel classes of elements of $\wp_k(\mathfrak{U})$, $\mathcal{T}$ for the class of tops $\mathrm{T}(D)$ with $D \in \wp_{k+1}(\mathfrak{U})$, and $\mathcal{S}$ for the class of stars $\mathrm{S}_0(H)$ with $H \in \wp_{k-1}(\mathfrak{U})$. Let $\mathcal{K}$ be the class of all the maximal $\simeq^+$-cliques.

Lemma 5.3. *Let $K \in \mathcal{K}$. Then*

$$K \in \mathcal{T}^* \iff \exists^2 K' \in \mathcal{K}\big[|K \cap K'| \geq 2\big], \tag{6}$$

so the class $\mathcal{T}^*$ is elementarily distinguishable in terms of $\simeq^+$.

If $K \in \mathcal{T}^*$, $K' \in \mathcal{K}$, and $|K \cap K'| \geq 2$ then $K' \in \mathcal{S}^*$; if $K' \in \mathcal{S}^*$ then there is $K \in \mathcal{T}^*$ such that $|K \cap K'| \geq 2$; this justifies that the class $\mathcal{S}^*$ is also elementarily distinguishable in terms of $\simeq^+$.

We have for distinct $A, B \in \wp_k(\mathfrak{U})$

$$A \parallel B \iff \exists K \in \mathcal{S}^* \, [A, B \in K],$$

$$A \sim_- B \iff A \simeq^+ B \wedge \neg A \parallel B,$$

and thus the relation $\parallel$ *and, consequently,* $\sim_-$ *can be characterized in terms of* $\simeq^+$.

Let us consider the structure $\langle \wp_{m-1}(\mathfrak{U}), \wp_m(\mathfrak{U}), \subset \rangle$ and write down this almost trivial fact:

Lemma 5.4. *If* D, D_1, $D_2 \in \wp_m(\mathfrak{A})$ *and* $D \sim_- D_1, D_2$, *then* $\dim(D_1 \sqcup D_2) \leq m + 2$.

Proof. It suffices to notice that $\dim D_i \sqcup D = m+1$ for $i = 1, 2$, and that subspaces $D_1 \sqcup D$, $D_2 \sqcup D$ share a hyperplane D, so $\dim(D_1 \sqcup D_2 \sqcup D) \leq$ $\leq m + 2$. $\Diamond$

Now we are able to express the adjacency $\simeq^+$ in terms of $\sim_-$.

Proposition 5.5. *Let* $A_1, A_2 \in \wp_{m-1}(\mathfrak{U})$.

$$A_1 \simeq^+ A_2 \iff$$
$$\iff \forall D_1, D_2 \in \wp_m(\mathfrak{U}) \left[\bigwedge_{i=1,2} (A_i \subset D_i) \Longrightarrow \exists D \in \wp_m(\mathfrak{U}) \left[D \sim_- D_1, D_2 \right] \right].$$

Proof. Assume that $A_1 \simeq^+ A_2$. If $A_1 \sim_- A_2$, then D is any from $\wp_m(\mathfrak{U})$ through $D_1 \cap D_2$. If $A_1 \parallel A_2$, then we can find necessary D through say A_1 so that $D \sim_- D_2$.

In case $A_1 \not\simeq^+ A_2$, that is when $\dim(A_1 \cap A_2) < m - 2$ and $A_1 \not\parallel A_2$, there are $D_1, D_2 \in \wp_m(\mathfrak{U})$ such that $A_i \subset D_i$, $i = 1, 2$ and $m + 2 <$ $\dim(D_1 \sqcup D_2)$ in $\mathfrak{A}$. We are through by 5.4. $\Diamond$

Taking into account the fact that the maximal $\sim_-$-cliques in the family $\wp_m(\mathfrak{U})$ are the stars of the form $\mathrm{S}_0(H)$ with $H \in \wp_{m-1}(\mathfrak{U})$, and the maximal $\sim^+$-cliques in the set $\wp_{m-1}(\mathfrak{U})$ are the tops $\mathrm{T}(D)$ with $D \in \wp_m(\mathfrak{U})$ we see that in terms of any of the two above adjacencies the structure $\langle \wp_{m-1}(\mathfrak{U}), \wp_m(\mathfrak{U}), \subset \rangle$ is definable, and thus the relation $\simeq^+$ on $\wp_{m-1}(\mathfrak{U})$ is definable. Finally, in view of 5.3, the adjacency $\sim_-$ on $\wp_{m-1}(\mathfrak{U})$ is definable as well. This all leads to:

Theorem 5.6. *The affine space* $\mathfrak{A}$ *and affine polar space* $\mathfrak{U}$ *are definable in terms of adjacency* $\sim_-$ *on* $\wp_m(\mathfrak{U})$ *as well as in terms of* $\sim^+$ *and of* $\approx$ *on* $\wp_{m-1}(\mathfrak{U})$.

Let us emphasize one important instance of 5.6: $m = 1$, which corresponds to Minkowskian geometry. Note that 5.6 in the case $m = 1$ is exactly 3.1.

6. From the point of view of foundations

The structures considered as ground structures, affine polar spaces, were defined by us in an analytical manner, originating from vector spaces equipped with symmetric bilinear forms; to put it in more geometrical terms, from metric affine spaces that satisfy the theorem on the three altitudes and admit isotropic lines but have no line orthogonal to all the others (axiomatic characterization to be found e.g. in [25] in the logically equivalent language of equidistance relation, see also [17]). The incidence structures obtained by restricting the class of lines of a metric affine space to its isotropic lines are precisely our affine polar spaces. The collinearity relation of the underlying metric affine space can be expressed in terms of the derived affine polar space.

Even a stronger result is valid (a form of the Alexandrov–Zeeman theorem): the collinearity of the underlying affine space *and* corresponding metric structure (say, orthogonality and/or "equidistance relation") are definable in terms of binary collinearity of the induced affine polar space (cf. [17] and Theorem 3.1). The result does not require that the index m is finite nor that it assumes any specific value > 0. In view of the above, the class of our affine polar spaces is axiomatisable.

Let $\mathfrak{U}$ be an affine polar space (in the above meaning, with the ground metric affine space $\mathfrak{R}$). Clearly, its k-dimensional strong subspaces (i.e. k-dimensional isotropic subspaces of $\mathfrak{R}$) are definable in $\mathfrak{U}$ and thus also the adjacencies $\approx$, $\sim^+$, and $\sim_-$ are definable in $\mathfrak{U}$. That means there are formulas ψ_1^k, ψ_2^k, and ψ_3^k in the language of $\mathfrak{U}$ such that the class $\wp_k(\mathfrak{U})$ of such subspaces is exactly

$$\Big\{\big\{x\colon \mathfrak{U}\models\psi_3^k[a_1,\ldots,a_{k+1},x]\big\}\colon \mathfrak{U}\models\psi_1^k[a_1,\ldots,a_{k+1}]\Big\} \tag{7}$$

and if $\mathfrak{U}\models\psi_1^k\big[a'_1,\ldots,a'_{k+1}\big]$, $\mathfrak{U}\models\psi_1^k\big[a''_1,\ldots,a''_{k+1}\big]$ then

$$\begin{aligned}\big\{x\colon \mathfrak{U}\models\psi_3^k\big[a'_1,\ldots,a'_{k+1},x\big]\big\} = \big\{x\colon \mathfrak{U}\models\psi_3^k\big[a''_1,\ldots,a''_{k+1},x\big]\big\} \iff\\ \iff \mathfrak{U}\models\psi_2^k\big[a'_1,\ldots,a'_{k+1},a''_1,\ldots,a''_{k+1}\big].\end{aligned} \tag{8}$$

Analogously, elementary $(2k+2)$-ary formulas define adjacencies.

What we have proved (cf. 4.1, 5.6) is that (identifying, intuitively, $A\in\wp_{t-1}(\mathfrak{U})$ with the set $\{B\in\wp_t(\mathfrak{U})\colon A\subset B\}$ for $t=k,k-1,\ldots,1$) one can define the underlying structure $\mathfrak{U}$ in terms of the considered adjacencies on $\wp_k(\mathfrak{U})$.

The general schema of such an interpretation of a structure in terms of relations on its subspaces is well presented in [16] and, in a manner even more suitable for this exposition, in [21]. To make presentation of our results simpler we have presented most of them in terms of maximal cliques of the relation $\sim = \sim^+, \sim_-, \approx$ and thus in a nonelementary language. It is relatively easy to rephrase them in an elementary language, though. Generally, to this aim it suffices to note that for a particular type $\mathcal{J}$ of maximal $\sim$-cliques there is a natural number s such that

$$\mathcal{J} = \big\{ [A_1, \ldots, A_s]_\sim \colon A_1, \ldots, A_s \in \wp_k(\mathfrak{U}),\ \mathbf{\Delta}^\sim_s(A_1, \ldots, A_s) \big\}. \tag{9}$$

However, we must be cautious, as s may depend on the index; e.g., (9) is valid for $\sim = \sim^+$, $\mathcal{J} = \mathcal{S}$, and $s = m - k + 1$. Elementarizing formulas of Subs. 3.2 entirely that way we get the results simply but for particular values of k only. That is why below we follow a bit more complicated way.

Note that (9) is valid for $\sim = \sim^+$ and $s = 3$ when $\mathcal{J} = \mathcal{T}$ and $k \neq m-2$, and when $\mathcal{J} = \mathcal{T} \cup \mathcal{S} \cup \mathcal{S}^*$ and $k = m-2$. If $k = m-2$ all the reasonings of Subs. 3.2 are directly repeated. Assume that $k \neq m-2$. Let us write $\boldsymbol{L}^\dagger(A_1, A_2, A_3)$ iff A_1, A_2, A_3 are on a line of $\mathfrak{U}^\dagger_k$. Let $A_1 \sim^+ A_2$. Then

$$\boldsymbol{L}^\dagger(A_1, A_2, A_3) \iff \forall A_0 \,\big[\, A_0 \sim^+ A_1, A_2 \Longrightarrow A_0 \sim^+ A_3 \,\big]. \tag{10}$$

For distinct $A_1 \sim^+ A_2$ let $\overline{A_1, A_2}$ be the line of $\mathfrak{U}^\dagger_k$ that contains them. Note that if A_1, A_2, A_3 are pairwise $\sim^+$-adjacent then the plane $\Pi(A_1, A_2, A_3)$ that contains them is definable in terms of the relation $\boldsymbol{L}^\dagger$. In the class of all planes one can elementarily distinguish the class $\mathcal{E}$ of projective planes: these are subspaces that satisfy the projective Veblen condition. Let distinct A_1, A_2 satisfy $A_1 \sim^+ A_2$.

$$\begin{aligned} &\overline{A_1, A_2} \text{ is a proper pencil (i.e. } A_1 \approx A_2) \iff \\ &\quad \iff \forall A_3 \,\big[\, \neg \boldsymbol{L}^\dagger(A_1, A_2, A_3) \wedge \neg \mathbf{\Delta}^{\sim^+}(A_1, A_2, A_3) \wedge \\ &\qquad\qquad \wedge\, A_3 \sim^+ A_1, A_2 \Longrightarrow \Pi(A_1, A_2, A_3) \in \mathcal{E} \,\big]. \end{aligned} \tag{11}$$

Let $\mathcal{E}'$ be the class of star-planes i.e. the class of planes $\Pi(A_1, A_2, A_3)$ in $\mathcal{E}$ such that $A_1 \approx A_2$ and $\neg \mathbf{\Delta}^{\sim^+}(A_1, A_2, A_3)$. Its elements have form $[H, D]_k$ with $H \in \wp_{k-1}(\mathfrak{U})$, $D \in \wp_{k+2}(\mathfrak{U})$.

$$\begin{aligned} A_1 \sim_- A_2 \iff \exists \Pi_0, \Pi_1, \Pi_2, \Pi_3 \in \mathcal{E}' \,\big[\, & A_1 \in \Pi_0 \wedge A_2 \in \Pi_3 \wedge \\ & \wedge\, |\Pi_{i-1} \cap \Pi_i| \geq 2 \text{ for } i = 1, 2, 3 \,\big]. \end{aligned} \tag{12}$$

When $\sim = \sim_-$ then (9) remains valid with $s = 3$ and $\mathcal{J} = \mathcal{S}$. As previously, we write $\boldsymbol{L}(A_1, A_2, A_3)$ when A_1, A_2, A_3 are on a line of $\mathfrak{U}_k$. Then for pairwise distinct A_1, A_2, A_3 we have

$$\begin{aligned} &\boldsymbol{L}(A_1, A_2, A_3) \iff \exists A_0', A_0'' \Big[\boldsymbol{\Delta}^{\sim_-}(A_1, A_2, A_0') \wedge \\ &\qquad \wedge A_3 \sim_- A_1, A_2, A_0', A_0'' \wedge A_0'' \sim_- A_1, A_2 \wedge \\ &\wedge \neg\exists A_1''', A_2''', A_3''' \big[\boldsymbol{\Delta}^{\sim_-}(A_1''', A_2''', A_3''') \wedge A_1, A_2, A_0'' \sim_- A_1''', A_2''', A_3''' \big]\Big]. \end{aligned} \tag{13}$$

For $\sim = \approx$, (9) does not hold for any family of maximal $\sim$-cliques, as determined in Subs. 3.4. Nevertheless, when $A_1 \approx A_2$ we have

$$\begin{aligned} &\boldsymbol{L}(A_1, A_2, A_0) \iff A_0 \approx A_1, A_2 \wedge \forall A_3 \Big[A_3 \approx A_1, A_2 \wedge \\ &\qquad \wedge \exists A_1', A_2', A_3' \big[A_1', A_2', A_3' \approx A_1, A_2, A_3 \wedge \\ &\qquad \wedge \neg A_1' \approx A_2' \wedge \neg A_2' \approx A_3' \wedge A_1' \approx A_3' \big] \Longrightarrow A_0 \approx A_3 \Big]. \end{aligned} \tag{14}$$

Now again the class of planes of $\mathfrak{U}^\dagger$ can be defined; note that from the point of view of $\mathfrak{U}$ they can be projective planes, affine planes, and slit planes as well. We have

$$\begin{aligned} &A_1 \sim^{\shortparallel} A_2 \iff \neg A_1 \approx A_2 \wedge \\ &\qquad \wedge \exists A_1', A_2', A_3' \Big[\neg \boldsymbol{L}(A_1', A_2', A_3') \wedge \approx (A_1', A_2', A_3') \wedge \\ &\qquad \wedge A_1, A_2 \approx A_1', A_2', A_3' \wedge \\ &\qquad \wedge \forall A_3 \big[A_3 \approx A_1', A_2', A_3' \wedge \neg A_3 \approx A_1 \Longrightarrow \neg A_3 \approx A_2 \big]\Big]. \end{aligned} \tag{15}$$

Well, finally we have rewritten definitions from Subs. 3.2–3.4 in an elementary language of the relation $\sim$ on $\wp_k(\mathfrak{U})$. In an analogous way we can elementarize definitions of Sec. 5.

Our main result yields now

Theorem. *Let $\sim = \sim_-$, $0 < k \leq m$ or $\sim = \sim^+, \approx$, $k < m$. The class of the graphs of adjacency $\sim$ on k-subspaces associated with affine polar spaces is axiomatizable* and thus *the geometry of our affine polar spaces can be elementarily characterized in terms of adjacency $\sim$ on k-subspaces.*

A bijection of the class of k-dimensional isotropic subspaces of an at least 3 dimensional metric affine space $\mathfrak{R}$ with index m which preserves (in both directions) a respective adjacency is determined by an automorphism of $\mathfrak{R}$.

As we already noted in the introduction, the term "affine polar space" refers in the literature to a wider class of structures; namely to structures obtained from a polar space $\mathfrak{Q}$ by deleting a hyperplane H. An axiom system for such structures (except those derived from $\mathfrak{Q}$ of index 1) is presented in [5]; from the point of view of polar geometry, $\mathfrak{Q}$ with index 1 is simply a generalized quadrangle, but its suitable derived affine polar space corresponds to an important geometry: to Minkowskian geometry. Thus affine polar spaces in the sense of [5] contain all of our affine polar spaces except those associated with Minkowskian spaces. Note that each of our spaces is determined by a symmetric bilinear form and thus the corresponding affine space must be Pappian; from the point of view of polar geometry it is derived from $\mathfrak{Q}$ determined by a polarity associated with a symmetric bilinear form, where H consists of all the points collinear with a fixed point of $\mathfrak{Q}$. If a polar space $\mathfrak{Q}$ is determined in a Pappian projective space by a polarity associated with a sesquilinear form η then η is bilinear iff the following *Net Axiom* is satisfied on $\mathfrak{Q}$:

(Net) If points a_1, a_2, a_3, a_4 yield a quadrangle without diagonal lines, K, L are lines such that K crosses $\overline{a_1, a_2}, \overline{a_3, a_4}$ and L crosses $\overline{a_1, a_4}, \overline{a_2, a_3}$, then K, L have a common point.

Consequently, our non-Minkowskian affine polar spaces can be characterized as those models of [5] which satisfy, additionally:

– all their planes are Pappian with characteristic $\neq 2$;

– there are lines L, M such that $L \equiv M$ and $L \not\parallel M$ (in the notation of [5]);

– suitable affine variant of (Net) is satisfied.

Our affine polar spaces are line reducts of affine spaces; they arise by restricting the line set of a (metric-) affine space to its subset. Note that the class of affine polar spaces in the sense of [5] contains another subclass of line reducts of affine spaces. Namely, if a polar space $\mathfrak{Q}$ is associated with a symplectic polarity in a projective space $\mathfrak{P}$ and H is its hyperplane then $H = [a]_\sim$ ($H = a^\perp$, in the terminology of [5]) for some point a of $\mathfrak{P}$ and H is a hyperplane of $\mathfrak{P}$ as well. In that case deleting H we arrive to an affine space $\mathfrak{A}$ and corresponding associated affine polar space $\mathfrak{U}$ is a line reduct of $\mathfrak{A}$. Let $\mathcal{G}$ be the class of lines of $\mathfrak{U}$. One can note that the formula

$$\forall y\,[y \sim a, b \Longrightarrow y \sim x] \tag{16}$$

defines the ternary collinearity of $\mathfrak{A}$ in terms of the binary collinearity $\sim$ of $\mathfrak{U}$ and thus $\mathfrak{A}$ is definable in $\mathfrak{U}$. In this case, however, the class $\mathcal{G}$ is not closed under the parallelism of $\mathfrak{A}$, which was crucial in investigations on our affine polar spaces. On the other hand, in this case the parallelism $\widetilde{\parallel}$ on $\mathcal{G}$ inherited from $\mathfrak{A}$ coincides with the parallelism $\parallel$ defined in [5], while for our affine polar spaces we have $\parallel \subsetneq \widetilde{\parallel}$.

A. Final remarks

An affine polar space can be also represented in terms of the derived affine (more precisely: metric-affine) space of a quadric. Let $\mathfrak{P}$ be the projective space $\mathbf{P}_1(\mathbb{V})$. For given $W \in \mathrm{Sub}(\mathbb{V})$ we write $\Theta(\mathbb{V}, W)$ for the set of all linear complements of W in $\mathbb{V}$. Assume that $\mathrm{codim}(W) = k$, then $\Theta(\mathbb{V}, W) \subset \mathrm{Sub}_k(\mathbb{V})$. We write then $\mathbf{A}_k(\mathbb{V}, W)$ for the *structure of linear complements of W*, i.e. for the spine space $\mathbf{A}_{k,0}(\mathbb{V}, W)$ in the terminology of [23] (a suitable restriction of $\mathbf{P}_k(\mathbb{V})$). Recall that $\mathbf{A}_1(\mathbb{V}, W) \cong \cong \mathbf{A}(W)$ and, generally, $\mathbf{A}_k(\mathbb{V}, W)$ is representable as a substructure of the affine space $\mathbf{A}(\mathrm{Hom}(\mathbb{V}/W, W))$ (see [24]).

Now assume that $W \in \mathrm{Q}_k(\xi)$ and let $\mathrm{Q}_k^W = \mathrm{Q}_k(\xi) \cap \Theta(\mathbb{V}, W^\perp)$. The restriction of the structure $\mathbf{A}_k(\mathbb{V}, W)$ to the set Q_k^W will be called *the derived space of* $\mathfrak{Q}_k$ *at* W, where $\mathfrak{Q}_k$ is the geometry of pencils on the quadric Q_k (a Grassmann space of the polar space $\langle \mathrm{Q}_1, \mathrm{Q}_2 \rangle$) (cf. [19]). This restriction will be denoted by $\mathbf{Q}_k(\mathbb{V}, W)$. The following is known:

Fact A.1 ([15]). *Let* $\langle w \rangle = W \in \mathrm{Q}_1(\xi)$, *where* ξ *is a symmetric bilinear form on a vector space* $\mathbb{V}$. *There is a vector space* $\mathbb{V}^\circ$ *and a nondegenerate symmetric bilinear form* ξ° *on* $\mathbb{V}^\circ$ *such that* $\mathbf{Q}_1(\mathbb{V}, W) \cong \mathbf{U}_1(\mathbb{V}^\circ, \xi^\circ)$. *Every space* $\mathbf{U}_1(\mathbb{V}^\circ, \xi^\circ)$ *can be represented in this way.*

Problem A.2. Can the above be generalized to arbitrary $k \le \mathrm{ind}(\xi)$? And for symplectic form ξ? If so, is it possible to generalize our main results concerning adjacencies for $\mathbf{Q}_k(\mathbb{V}, W)$?

Acknowledgements. We wish to thank the referee for his valuable comments which made our presentation more precise as it comes to bibliographical references and richer in the area of logic and foundations of geometry.

References

[1] ALEXANDROV, A. D.: Mappings of spaces with families of cones and space-time-transformations, *Annali di Mat.* **103** (1975), 229–256.

[2] BENZ, W.: Classical geometries in modern contexts, Birkhäuser Verlag, Basel, 2005.

[3] BUEKENHOUT, F. and SHULT, E.: On the foundations of polar geometry, *Geometriae Dedicata* **3** (1974), 155–170.

[4] CHOW, W.-L.: On the geometry of algebraic homogeneous spaces, *Ann. of Math.* **50** (1949), 32–67.

[5] COHEN, A. M. and SHULT, E. E.: Affine polar spaces, *Geom. Dedicata* **35** (1990), 43–76.

[6] COHEN, M. A.: Point-line spaces related to buildings, in: Handbook of incidence geometry, F. Buekenhout, Ed. North-Holland, Amsterdam, 1995, pp. 647–737.

[7] DIEUDONNÉ, J.: La géométrie des groupes classiques, Springer-Verlag, Berlin, 1971.

[8] GOLONKO, I., PRAŻMOWSKA, M. and PRAŻMOWSKI, K.: Adjacency in generalized projective Veronese spaces, *Abh. Math. Sem. Univ. Hamb.* **76** (2006), 99–114.

[9] HAVLICEK, H.: Chow's theorem for linear spaces, *Discrete Mathematics* **208/209** (1999), 319–324.

[10] HUANG, W.-L.: Adjacency preserving transformations of Grassmann spaces, *Abh. Math. Sem. Univ. Hamb.* **68** (1998), 65–77.

[11] HUANG, W.-L.: Adjacency preserving mappings of invariant subspaces of a null system, *Proc. Amer. Math. Soc.* **128**, 8 (1999), 2451–2455.

[12] KREUZER, A.: A remark on polar geometry, *Abh. Math. Sem. Univ. Hamb.* **61** (1991), 213–215.

[13] LESTER, J. A.: Distance preserving transformations, in: Handbook of incidence geometry, F. Buekenhout, Ed. North-Holland, Amsterdam, 1995, pp. 921–944.

[14] LINGENBERG, R.: Metric planes and metric vector spaces, John Wiley & Sons, New York, 1979.

[15] ORYSZCZYSZYN, H. and PRAŻMOWSKI, K.: Inversive closure of metric affine spaces and its automorphisms, *Demonstratio Math.* **XXXII**, 1 (1999), 151–155.

[16] PAMBUCCIAN, V.: Elementary axiomatizations of projective space and of its associated grassmann space, *Note di Matematica* **24**, 1 (2005), 129–141.

[17] PAMBUCCIAN, V.: Alexandrov–Zeemann type theorems expressed in terms of definability, *Aequationes Math.* **74** (2007), 249–261.

[18] PANKOV, M., PRAŻMOWSKI, K. and ŻYNEL, M.: Transformations preserving adjacency and base subsets of spine spaces, *Abh. Math. Sem. Univ. Hamb.* **75** (2005), 21–50.

[19] PANKOV, M., PRAŻMOWSKI, K. and ŻYNEL, M.: Geometry of polar Grassmann spaces, *Demonstratio Math.* **39**, 3 (2006), 625–637.

[20] PORTEOUS, I. R.: Topological geometry, Cambridge University Press, Cambridge, 1981.

[21] PRAŻMOWSKA, M., PRAŻMOWSKI, K. and ŻYNEL, M.: Euclidean geometry of orthogonality of subspaces, *Aequationes Math.* **76**, 1–2 (2008), 151–167.

[22] PRAŻMOWSKI, K. and PRAŻMOWSKA, M.: Grassmann spaces over hyperbolic and quasi hyperbolic spaces, *Math. Pannonica* **17**, 2 (2006), 195–220.

[23] PRAŻMOWSKI, K., and ŻYNEL, M.: Automorphisms of spine spaces, *Abh. Math. Sem. Univ. Hamb.* **72** (2002), 59–77.

[24] PRAŻMOWSKI, K. and ŻYNEL, M.: Geometry of the structure of linear complements, *J. Geom.* **79** (2004), 177–189.

[25] SCHRÖDER, E. M.: Zur Kennzeichnung Fanoscher Affin-Metrischer Geometrien, *J. Geom.* **16**, 1 (1981), 56–62.

[26] TITS, J.: Buildings of spherical type and finite BN-pairs, Lecture Notes in Mathematics, vol. 386, Springer, Berlin, 1974.

[27] VELDKAMP, F. D.: Polar geometry I–IV, *Indag. Math.* **21** (1959), 512–551.

[28] ZEEMAN, E. C.: Causality implies the Lorentz group, *J. Mathemathical Physics* **5** (1964), 490–493.

[29] ŻYNEL, M.: Finite grassmannian geometries, *Demonstratio Math.* **XXXIV**, 1 (2001), 145–160.