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SOME PROBLEMS IN THE THEORY OF NEARRING MODULES

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Abstract: This paper consists of a survey of some problems involving isomorphism questions about nearring modules and questions about minimal nearring modules.

1. Introduction

This note is an expanded written version of an invited hour lecture by the author at the 20th International Conference on Near-rings and Near-fields 2007 in Linz, Austria, July 23rd to July 27th, 2007 surveying several problems about nearring modules. The first section of this paper will primarily focus on an isomorphism question involving quotient modules of faithful nearring modules. Questions about the isomorphism classes of (to be defined) irreducibly faithful modules and indecomposable modules will also be touched on at the end of this section. The second section will begin with an overview of idempotent quivers (which are directed graphs using primitive nearring idempotents for their vertices) and will end with a discussion of the potential of these quivers indicating placements of type 2 factors within nearring modules. The third and final section will begin with the observation that the integral group ring of a group is a homomorphic image of the free distributively

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generated (d. g.) nearring of the group leading us to problems extending results from group representation theory involving types of minimal and indecomposable modules to the d. g. nearring setting.

2. Isomorphism questions

We begin this section with a problem which this author has been referring to as the isomorphism question for nearrings:

Given an isomorphism φ from a nearring R to a nearring S and two faithful modules G and H of R and S, respectively, are there respective canonical R- and S-ideals of N of G and M of H so that there is a group isomorphism β from G/N to H/M such that

$$((g+N)r)\beta = (g+N)\beta(r\varphi)$$

for all $g \in G$ and all $r \in R$?

Of course, if we identify R and S we can restate this problem in the form:

Given two faithful modules G and H of a nearring R, are there canonical R-ideals N of G and M of H so that G/N and H/M are isomorphic R-modules?

For particular nearrings R natural candidates begin to emerge. For instance if $R = I(G)$ (and $S = I(H)$) it is natural to ask if $G/Z(G) \simeq \simeq H/Z(H)$ ($Z(G)$ being the center of G) because of the elementary result that $Inn(G) \simeq Inn(H)$ ($Inn(G)$ being the inner automorphism group of G) implies $G/Z(G) \simeq H/Z(H)$. However, the answer to this question is no. Indeed in [21] Sergei Syskin has given two centerless nonisomorphic solvable groups G and H for which $I(G) \simeq I(H)$. Gordon Mason and John Meldrum have done work extending Syskin's example in [8]. But something can be salvaged as Syskin went on to show that if G and H are finite groups with no nontrivial factors of normal subgroups that are abelian (called strictly nonabelian groups in [21]), then $I(G) \simeq I(H)$ implies $G \simeq H$. Actually Stuart Scott obtained a more general result in [18] shortly before Syskin: If $R/J_2(R)$ has dccr (descending chain condition on right ideals)and R is ring free (meaning that R contains no proper ideals I with R/I a ring), then any two faithful 2-tame R-modules are isomorphic. In [13] this author showed that if G and H are finite perfect groups with isomorphic compatible automorphism nearrings, then $G/Z(G) \simeq H/Z(H)$. In [14] this was generalized to only requiring $G/Z(G)$ and $H/Z(H)$ to be perfect and R to have dccr.

Scott continued to make progress in [19] showing that if R is a nearring that has no nonzero ideals that are ring modules (Scott calls such a nearring **semiprimary**) and R has dccr, then all faithful compatible R-modules are isomorphic if and only if all faithful compatible R-modules are Z-constrained. (A precise definition of Z-constrained will be given shortly. For the time being, just think of it roughly meaning there are no central factors.) This brings us to Scott's paper in the proceedings of the 2003 Hamburg conference [20], which we will consider in greater detail beginning with a discussion of centralizers and Fitting submodules. Scott has used several types of centralizers of a subset X of an R-module G in his work over the years. Two of these using the terminology and notation of [14] are the **module centralizer** of X,

$$MC_G(X) = \{g \in G | gr + x = x + gr \; \forall x \in X, r \in R\},$$

and the **distributor** $D_G(X)$ of X in G, which consists of the union of all R-subgroups H of G for which

$$(h + x)r = hr + xr$$

for all $h \in H$, $x \in X$, and $r \in R$. We have

$$D_G(X) \subseteq MC_G(X)$$

with equality if R is distributively generated by a multiplicative semigroup S, G is an (R, S)-module, and X is an R-subgroup of G by [14, Prop. 2.2(ii)]. Module centralizers and distributors can be extended to factors of G in the obvious ways. Continuing to follow the notation and terminology of [14], we say that an R-module H is **R-nilpotent** if H has a series of R-ideals

$$0 = H_0 < H_1 < \ldots < H_n = H$$

such that $D_H(H_{i+1}/H_i) = H$ for each i and define the **distributive Fitting ideal** of an R-module G, denoted $DF(G)$, to be the sum of the R-nilpotent ideals H of G. If $R = I(G)$, $DF(G)$ is the same as the Fitting subgroup $F(G)$ of G. If G is 3-tame and R satisfies dccr, then

$$DF(G) = \cap D_G(H/K)$$

where the intersection runs over all the type 2 factors of G, which generalizes the well-known group theoretic result that the Fitting subgroup of a finite group is the intersection of the centralizers of the chief factors of the group. The distributive Fitting ideal has emerged as a candidate for the desired canonical R-ideal in the isomorphism problem. Indeed in Cor. 39.4 of [20], Scott obtains the following result where an R-module G is called **Z-constrained** if G contains no R-ideals $K < L$ such that $D_G(L/K) = G$.

Theorem 2.1. *Suppose that G and H are faithful compatible R-modules and R satisfies dccr. If $G/DF(G)$ is Z-constrained, then $G/DF(G)$ and $H/DF(H)$ are isomorphic R-modules.*

In the case of endomorphism nearrings, this author was able to obtain the following stronger result [14, Th. 3.3]:

Theorem 2.2. *Suppose R is a compatible endomorphism nearring of two groups G and H and R satisfies dccr. If $G/DF(G)$ and $H/DF(H)$ are both perfect groups, then $G/DF(G)$ and $H/DF(H)$ are isomorphic R-modules.*

Th. 2.2 generalizes Th. 2.1 in the compatible endomorphism nearring setting since a Z-constrained module is a perfect group in this setting. Moreover, if we define an R-module G to be R-**perfect** if it contains no proper R-ideal I such that $D_G(G/I) = G$, it is easily seen that an R-perfect module is a perfect group in the endomorphism nearring setting and a Z-constrained module is R-perfect. Thus we have the question of whether the endomorphism nearring assumption can be dropped in Th. 2.2 obtaining a stronger version of Th. 2.1 with R-perfect in place of Z-constrained. But this is not only possible generalization. A large part of Scott's Hamburg paper is devoted to proving the following result which plays a crucial role in the proofs of Thms. 2.1 and 2.2.

Theorem 2.3. *Suppose that G and H are faithful 3-tame R-modules and R satisfies dccr. Further, suppose that G and H are both R-perfect. If $K < L$ are R-ideals of G and $M < N$ are R-ideals of H such that L/K and N/M are isomorphic type 2 R-modules, then*

$$G/D_G(L/K) \simeq H/D_H(N/M)$$

as R-modules.

An R-perfect module is a monogenic module and parts of the work involved in proving Th. 2.3 suggest the possibility that it might be obtained by requiring only monogenicity of the module. Could perfect be replaced by monogenic in Th. 2.3? If so, it might open the door to extending Th. 2.1 to the case where $G/DF(G)$ is monogenic. In fact, Syskin constructs his example of two nonisomorphic centerless groups G and H for which $I(G) \simeq I(H)$ by forming the direct product $D_1 \times D_2$ of two copies

$$D_1 = \langle u, a | u^{11} = a^5 = 1, u^a = u^4 \rangle$$

and

$$D_2 = \langle v, b | v^{11} = b^5 = 1, v^b = v^4 \rangle$$

of a metacyclic group of order 55, setting

$$g = ab^{-1} \text{ and } h = ab^2,$$

and then using

$$G = \langle u, v, g\rangle \text{ and } H = \langle u, v, h\rangle.$$

In this example, $F(G) = \langle u, v\rangle = F(H)$, $G/F(G) \simeq H/F(H)$, and $G/F(G)$ is monogenic. In fact, we do not need R-ideals as big as $F(G)$ and $F(H)$ to obtain an isomorphism since $G/\langle v\rangle \simeq H/\langle v\rangle$. Thus we are led to the question: Does a smaller ideal than $DF(G)$ suffice to obtain an isomorphism in general? If not, does a smaller ideal suffice if we work with nearrings distributively generated by groups as is the case with Syskin's example and the automorphism nearring results of this author mentioned earlier?

Another natural isomorphism problem to study involves the isomorphism classes of the types of faithful modules of a nearring R. One particular type arises as follows: Suppose that R is a nearring with dccr and G is a faithful tame R-module. Since

$$\cap_{g\in G}\mathrm{Ann}_R(g) = 0,$$

the descending chain condition on R gives us that

$$\mathrm{Ann}_R(g_1) \cap \cdots \cap \mathrm{Ann}_R(g_n) = 0$$

for some finite subset $\{g_1, \ldots, g_n\}$ of G. As each $g_iR \simeq R/\mathrm{Ann}_R(g_i)$ has both the ascending and descending chain conditions on R-ideals by [17, Th. 5.7], it follows that

$$H = g_1R + \cdots + g_nR$$

also has both chain conditions on R-ideals. Moreover, as H is a faithful R-module, we have:

Theorem 2.4. *If R is a nearring satisfying dccr and G is a faithful tame R-module, then G contains a faithful R-ideal H that satisfies both the ascending and descending chain conditions on its R-ideals.*

Now, let us call an R-module K of a nearing R **irreducibly faithful** if K contains no nontrivial proper R-ideal L such that either L or K/L is faithful. Or, equivalently, we could say that K is irreducibly faithful if it contains no nontrivial proper factor M/N of R-ideals (that is, there are no R-ideals $N < M$ with either $0 < N$ or $M < K$) with M/N a faithful R-module. (This concept of irreducibly faithful modules is a stronger form of minimality than the minimal faithful modules in §19 of [20].) As a consequence of Th. 2.4, we get the following result which includes existence of irreducibly faithful R-modules in the tame-dccr setting.

Theorem 2.5. *Suppose the nearring R satisfies dccr. If G is a faithful tame R-module, then G contains R-ideals $L \leq M$ such that M/L is irreducibly faithful. Also, any irreducibly faithful tame R-module satisfies both the ascending and descending chain conditions on its R-ideals.*

Proof. Let H be a faithful R-ideal of G satisfying both chain conditions on R-ideals guaranteed by Th. 2.4. If H is irreducibly faithful, we are done, so suppose it is not irreducibly faithful. Then H contains a proper nontrivial R-ideal L_1 such that either L_1 or H/L_1 is faithful. Repeating this with either L_1 or H/L_1 in place of H as appropriate, the chain conditions on H force this process to terminate after a finite number of steps in an irreducibly faithful R-module M/L with $0 \leq L \leq M \leq H \leq G$. The last sentence in an immediate consequence of Th. 2.4. ♢

Now that we have their existence, the study of the isomorphism classes irreducibly faithful modules becomes an avenue of investigation. One beginning point is to reexamine Scott's Z-constrained semiprimary result of [19]. Might it be that his Z-constrained assumption can be eliminated if we restrict to irreducibly faithful modules obtaining all irreducibly faithful compatible R-modules are isomorphic when R is semiprimary and satisfies dccr? Do notice too, however, that we cannot expect to have a single isomorphism class in general due to Syskin's example as both of his groups G and H are irreducibly faithful nearring modules for their respective endomorphism nearrings generated by their inner automorphisms.

Since the Krull–Schmidt Theorem holds for nearring modules satisfying both the ascending and descending chain conditions [10], we can express an irreducibly faithful tame module K of a nearring R with dccr uniquely as

$$K = K_1 \oplus \cdots \oplus K_n$$

where the summands are indecomposable and unique up to order and isomorphism of the direct summands. Thus in the study of isomorphism classes of irreducible modules we may have to consider another problem that has received considerable attention within the subject of representation theory of groups and artinean algebras: What are the isomorphism classes of indecomposable modules? Directed graphs called **quivers** have played an important role in studying this problem in the artinean algebra setting. An excellent introduction to this can be found in [16] and additional material can be found in [2] and [3]. An approach to forming quivers in the nearring setting was developed in [15] that we shall consider in the next section. It is not clear whether these quivers will have any impact on the determination of the isomorphism classes of indecomposable nearring modules, but, as we shall see, they do have other relationships to the structure of nearring modules. The study of the isomorphism classes of indecomposable nearring modules is one that has received little attention among nearringers to this author's knowledge. We will return to it in the last section of this note.

3. Idempotent quivers

We begin by reviewing some of the details found in [15] for constructing the idempotent quiver of a nearring. We start with a 0-symmetric nearring with identity R that has the descending chain condition on right R-subgroups and has nilpotent J_2-radical. Our quiver will be formed using selected primitive idempotents for its vertices where (following Lausch's definition [6]) an idempotent e of R is **primitive** if there does not exist an idempotent $f \in R$ such that $ef = f$ and $fe \neq e$. The next theorem gives us equivalent characterizations on primitivity of idempotents. In its second part, the terminology of saying that a right R-subgroup M of R is **self-monogenic** means that $mM = M$ for some $m \in M$.

Theorem 3.1. *For e an idempotent of R, the following are equivalent:*

1. *e is primitive.*
2. *eR is a minimal self-monogenic right R-subgroup of R.*
3. *eR is a minimal nonnilpotent right R-subgroup of R.*

The first step in forming the set of primitive idempotents for the vertex set of our quiver is to form a **principal set of primitive orthogonal idempotents** (**PPO-set**) by which we mean a set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ of R so that modulo $J_2(R)$,

$$\overline{r} = \overline{e_1 r} + \cdots + \overline{e_n r}$$

for all $r \in R$ where bars denote images in $R/J_2(R)$. The existence of PPO-sets can be seen by using [11, Cor. 8] to lift the idempotents from a Wedderburn decomposition of $R/J_2(R)$ into minimal right ideals to obtain the desired PPO-set. Indeed it can be shown that all PPO-sets correspond to such a Wedderburn decomposition of $R/J_2(R)$. An alternative way to obtain a PPO-set that (in conjunction with Lausch's definition of primitivity) can be implemented on software packages such as SONATA [1] involves adjoining an additional primitive idempotent e_{k+1} orthogonal to a set of orthogonal primitive idempotents $\{e_1, \ldots, e_k\}$ until $\mathrm{Ann}_R(e_1, \ldots, e_n)$ becomes nilpotent.

Next, we define two primitive idempotents e and f to be **linked** if there exist primitive idempotents

$$e = e_1, e_2, \ldots, e_n = f$$

such that e_iR and $e_{i+1}R$ have isomorphic R-factors for each i. An alternative equivalent definition is to say that e and f are linked if there exist primitive idempotents

$$e = e_1, e_2, \ldots, e_n = f$$

such that $e_iRe_{i+1} \neq 0$ or $e_{i+1}Re_i \neq 0$ for each i. Linkage of primitive idempotents is an equivalence relation on the set of primitive idempotents which, if we fix a PPO-set $W = \{e_1, \ldots, e_n\}$ of R, partitions W into equivalence classes $W_1, \ldots, W_r$ of linked idempotents. There is a relationship between these equivalence classes and direct sum decompositions, at least in the case of tame nearrings, that we next discuss.

Our nearring R is uniquely expressible as

$$R = B_1 \oplus \cdots \oplus B_t$$

where each B_i is an indecomposable ideal of R. The ideals B_i are called the **blocks** of R. If R is a tame nearring, the number r of equivalence classes W_i of linked primitive idempotents of a PPO-set W is the same as the number of blocks t of R. In fact, the blocks of R are the same as the ideals of R generated by the W_i. For notational purposes, let us relabel if necessary so that B_i is the ideal generated by W_i. The unique decomposition of R into blocks induces a similar unique decomposition of any faithful tame R-module G into the form

$$G = G_1 \oplus \cdots \oplus G_t$$

with $G_i = GB_i$. The G_i are called the **blocks** of G and are characterized by the fact that G_i and G_j have no common isomorphic factors for $i \neq j$ and each G_i cannot be further decomposed into a nontrivial direct sum of R-ideals without the summands having common isomorphic factors. It is an open question as to whether these unique block decompositions in the tame case extend to other nearrings and associated faithful modules.

Now let us develop how we construct the idempotent quiver of R. We start by choosing a PPO-set W of R. Next define an equivalence relation $\sim$ on W by having $e_i \sim e_j$ if $\overline{e_i}\overline{R} \simeq \overline{e_j}\overline{R}$ where $\overline{R} = R/J_2(R)$. In fact, the equivalence classes of W under $\sim$ correspond to the isomorphism classes of type 2 R-modules. There are several other equivalent ways of defining $\sim$ that can be found in [15]. One that is useful for computer calculation is to define $e_i \sim e_j$ if e_iRe_jR contains a primitive idempotent as this can be checked by using only the multiplication of the nearring. (A by-product of this is that it gives us a way based solely on the nearring multiplication of determining the number of isomorphism classes of type 2 modules of R. Often this has been done by resorting to determining the Wedderburn decomposition of $R/J_2(R)$ or, in the case of a compatible endomorphism nearring, by finding the isomorphism classes of the type 2 summands of socle series factors of the group.) To form the vertex set V of our quiver, we choose a set of representatives

$$V = \{e_1, \ldots, e_m\}$$

(relabeling if necessary) of the equivalence classes of W under $\sim$. The directed edges of our quiver are formed by having and arrow from e_i to e_j if $e_iRe_j \neq 0$. It can be shown that different choices of a vertex set from W as well as different choices of PPO-set W result in isomorphic quivers allowing us to speak of the quiver of R, which we shall denote by $\Gamma(R)$.

There are some connections between $\Gamma(R)$ and the structure of R and its modules. One is the following result relating the connected components of $\Gamma(R)$ and blocks in the tame case.

Theorem 3.2. *If R has a faithful tame R-module G, then the connected components of $\Gamma(R)$ are in one-to-one correspondence with the blocks of R and G.*

Some other relationships between the direction of the arrows of $\Gamma(R)$ and the arrangement of type 2 factors of an associated faithful tame R-module G are emerging. For statements of results known at this point the reader is referred to [15], but let us give some examples to illustrate connections between $\Gamma(R)$ and type 2 factors. The quiver of $E(G)$ where $G = D_{24}$ is the dihedral group of order 24 is:

$$\begin{tikzcd} & e_1 \arrow[dl] \arrow[dr] & \\ e_2 & & e_3 \end{tikzcd}$$

Notice the resemblance of this to the socle series of G viewed as an $E(G)$-module which is

$$\begin{array}{c} D_{24} \\ | \\ C_2 \oplus C_3 \end{array}$$

where C_n denotes the cyclic group of order n. Here $\mathrm{Soc}_1(G) = C_2 \oplus C_3$, $\mathrm{Soc}_2(G) = G$, and $G/\mathrm{Soc}_1(G)$ is the Kline 4-group, V_4. As another example, the quiver of $I(G)$ where $G = S_4$ is:

$$\begin{tikzcd} & e_1 \arrow[dl] \arrow[ddr] & \\ e_2 \arrow[drr] & & \\ & & e_3 \end{tikzcd}$$

Here too notice the similarity between this quiver and the socle series for S_4 which is

$$\begin{array}{c} S_4 \\ | \\ A_4 \\ | \\ V_4 \end{array}$$

with $\mathrm{Soc}_1(G) = V_4$, $\mathrm{Soc}_2(G) = A_4$, and $\mathrm{Soc}_3(G) = G$. One obvious question these examples raise is whether we can use the quiver $\Gamma(R)$ to read off (up to multiplicity) direct summands of (using the terminology of Lyons and Meldrum in [7]) radical series of a faithful tame R-module G. And there are many other questions that are natural to ask. For instance, the Dynkin diagrams from the classification of simple Lie algebras play an important role in the study of algebras of finite representation type. Are there any such diagrams that play an essential role here?

4. Free d. g. nearrings

In this last section, we look at a type of nearring that may well deserve more attention than it has up to this point. It is the free d. g. nearring generated by a finite group G which we shall denote by $F(G)$. The reader desiring a reference on free d. g. nearrings may consult Ch. 12 of [9] where a more general form of a free d. g. nearring is constructed within a given variety V of groups. We will stick to the most basic case with the free d. g. nearring of a group (i.e., V will be the variety of all groups), but some of the ideas mentioned here may well have corresponding versions of interest for other varieties V as well as in the case where G is replaced by a semigroup.

A convenient way to view $F(G)$ following from [9, Th. 12.12(i)] is to note that $(F(G),+)$ is the free group on G and then realize that the multiplication on $F(G)$ is determined by first applying the left-hand distributive law and then distributing the elements of G on the right; that is, elements of $F(G)$ are multiplied as

$$\begin{aligned}&(\varepsilon_1 g_1 + \cdots + \varepsilon_m g_m)(\delta_1 h_1 + \cdots + \delta_n h_n) =\\ &= \delta_1(\varepsilon_1 g_1 + \cdots + \varepsilon_m g_m)h_1 + \cdots + \delta_n(\varepsilon_1 g_1 + \cdots + \varepsilon_m g_m)h_n =\\ &= \delta_1(\varepsilon_1 g_1 h_1 + \cdots + \varepsilon_m g_m h_1) + \cdots + \delta_n(\varepsilon_1 g_1 h_n + \cdots + \varepsilon_m g_m h_n)\end{aligned}$$

where each ε_i and δ_i is ± 1 and each g_i and h_i is an element of G. The reader familiar with group rings will notice that this is the same rule of multiplication followed for the group ring $R(G)$ of a commutative ring R (group rings over commutative rings are the primary focus in the study of group rings due to their role in group representation theory) where the ε_i and δ_i are elements of R and $(R(G),+)$ is the free R-module on G. Another connection with group rings is that $F(G)$ is related to the integral group ring $\mathbb{Z}(G)$ in a natural way: Letting $F(G)'$ denote the commutator subgroup of $(F(G),+)$, which is an ideal of $F(G)$, the nearring $F(G)/F(G)'$ is $\mathbb{Z}(G)$.

Observe that studying the category of groups M on which a homomorphic image of G acts as group of automorphisms of M is equivalent to studying the category of $(F(G),G)$-modules. In particular, the study of an automorphism nearring of a group generated by a finite group of automorphisms of that group falls within this realm. If M is an abelian group, then $F(G)' \leq \mathrm{Ann}_{F(G)}(M)$ and hence M is a $\mathbb{Z}(G)$-module—not only in the nearring sense, but as a ring module. To put it another way, studying the abelian $(F(G),G)$-modules is the same studying the $\mathbb{Z}(G)$-modules. This puts us into the group representation setting where we have a wealth of results at our disposal that we can use in the study of $(F(G),G)$-modules. For instance, suppose that one is interested in studying either simple, type 0, or type 2 $(F(G),G)$-modules. In all three situations the same reasoning used to obtain [12, Prop. 1.1] gives us that if M is any one of these three types of $(F(G),G)$-modules, then M is either (1) an elementary abelian p-group, (2) a direct sum of copies of the additive group of the rational numbers, or (3) a perfect group. In the abelian cases (1) and (2), simple, type 0, and type 2 are all the same. In particular, M is then monogenic, say, $M = mF(G), m \in M$. But now case (2) cannot occur because M is finitely generated by the elements

$mg, g \in G$, as a $\mathbb{Z}$-module. Thus if M is abelian it is a simple $\mathbb{Z}_p(G)$-module in which case a theorem of S. D. Berman [5, VII,3.11] gives us information on the number of isomorphism classes of such modules M. What about in the perfect situation? Can we get any results about the number of perfect simple, type 0, or type 2 $(F(G), G)$-modules?

Another place a theorem from group representation theory has an impact is in the study of when the number of indecomposable $(F(G), G)$-modules is finite. Here a remarkable result of A. Jones [4, Th. 81.18] gives us that we may restrict out attention to those groups G for which each Sylow p-subgroup of G is cyclic of order at most p^2 in the study of this question.

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