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ON ω-ELONGATIONS BY $P^{\omega+N}$-PROJECTIVE P-GROUPS

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Abstract: We prove that any reduced thick abelian p-group which is a strong ω-elongation by a $p^{\omega+n}$-projective p-group is bounded. We also prove that an n-Σ-group for every n is precisely the pillared group. The first claim supplemented our results published in *Commun. Algebra*, 2007, whereas the second one answers a problem of ours posed in *Rend. Ist. Mat. Univ. Trieste*, 2007.

Throughout this brief article, let us assume that all groups into consideration are additively written abelian p-groups, where p is an arbitrary but fixed prime. As usual, for such a group G, $G^1 = \cap_{k<\omega} p^k G$ denotes the first Ulm subgroup of G, where $p^k G = \{p^k g \mid g \in G\}$ is the p^k-th power of G, and $G[p] = \{a \in G \mid pa = 0\}$ denotes the socle of G. By induction, $p^\alpha G = p(p^{\alpha-1}G)$ if $\alpha - 1$ exists or $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$ otherwise; thus $p^\omega G = G^1$.

Following ([BW]) a group G is called *essentially finitely indecomposable* (= efi for facility) if for each direct decomposition $G = A \oplus C$ with C a direct sum of cyclics it follows that C is bounded. Moreover, a group G is said to be *thick* if there is a non-negative integer i such

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that $(p^iG)[p] \subseteq A$ whenever G/A is a direct sum of cyclics. It is readily seen that each thick group is efi. In this aspect Irwin conjectured that these two classes of groups coincide. Unfortunately, this conjecture fails (see [DuIr] and [D7], [D8]). First, recall that a group G is called $p^{\omega+n}$-*projective* (see [N1]) if there exists $P \leq G[p^n]$ such that G/P is a direct sum of cyclics; thus $G^1 \subseteq P$.

In order to show that there exists an efi non thick group, we obtained in [D7] and [D8] the following two assertions, respectively.

Proposition A ([D7]). *Each thick $p^{\omega+n}$-projective group is bounded.*

Note that this is not the case for efi $p^{\omega+n}$-projective groups when $n \geq 2$ (see [CM] and [D2]).

Since each $p^{\omega+n}$-projective group has first Ulm factor which is again $p^{\omega+n}$-projective, the above statement may be naturally extended to

Proposition B ([D8]). *Each thick group with $p^{\omega+n}$-projective first Ulm factor is algebraically compact (= the direct sum of a bounded group and a divisible group).*

We shall show that these two statements are, in fact, equivalent. They are also equivalent to a more generally stated assertion, marked below as Prop. C.

Before stating and proving the equivalence theorem, we need the following technical claim.

Lemma. *For each group G the following relation hold:*

(a) $G^1 = p(\cap_{k<\omega}(p^kG + G[p]))$.

(b) $(G/G^1)[p] = (\cap_{k<\omega}(p^kG + G[p]))/G^1$.

Proof. (a) It is straightforward that the inclusion "$\supseteq$" is true. Reciprocally, to show that "$\subseteq$" is valid, letting $g \in G^1 = \cap_{i<\omega} p^iG$. So, we can write $g = pa_1 = p^2a_2 = \cdots = p^{k+1}a_{k+1} = \cdots$. Furthermore, $p(a_1 - p^ka_{k+1}) = 0$ whence $a_1 - p^ka_{k+1} \in G[p]$ and $a_1 \in p^kG + G[p]$. Since k is an arbitrary integer, it follows that $a_1 \in \cap_{k<\omega}(p^kG + G[p])$ and hence $g = pa_1 \in p(\cap_{k<\omega}(p^kG + G[p]))$ as required.

(b) Choose $x = g + G^1 \in G/G^1$ with $pg \in G^1$. According to point (a), $pg \in p(\cap_{k<\omega}(p^kG + G[p]))$, so that $g \in \cap_{k<\omega}(p^kG + G[p]) + G[p] = \cap_{k<\omega}(p^kG+G[p])$. This insures that the inclusion "$\subseteq$" is fulfilled, that is tantamount to the desired equality because "$\supseteq$" is trivially satisfied. $\Diamond$

We recall that a group G is said to be *a strong ω-elongation by a $p^{\omega+n}$-projective group* if there exists $P \leq G[p^n]$ with $G/(P + G^1)$ a direct sum of cyclics. Since $G/(P + G^1) \cong G/G^1/(P + G^1)/G^1$ and

$(P+G^1)/G^1 \cong P/(P\cap G^1)$ is obviously bounded by p^n, it follows that each such ω-elongation G has first Ulm factor G/G^1 which is $p^{\omega+n}$-projective. Thus we can also call these strong ω-elongations as groups with *strongly $p^{\omega+n}$-projective* first Ulm factor. So, if a group G has strongly $p^{\omega+n}$-projective first Ulm factor, then it has $p^{\omega+n}$-projective first Ulm factor as well, while the converse fails when $P \neq 0$. Moreover, it is obvious that any $p^{\omega+n}$-projective group possesses strongly $p^{\omega+n}$-projective first Ulm factor.

Now we are in a position to prove the following.

Proposition C. *Each thick group with strongly $p^{\omega+n}$-projective first Ulm factor is algebraically compact.*

Proof. Suppose that G is such a group. Write $G/(P+G^1)$ is a direct sum of cyclics for some $P \leq G[p^n]$. Since $G/G^1/(P+G^1)/G^1 \cong G/(P+G^1)$ is also a direct sum of cyclics and since G/G^1 is thick by employing either [D1] or [K3], we derive with the help of [BW] that there is $s \in \mathbb{N}$ with $p^s(G/G^1)[p] \subseteq (P+G^1)/G^1$. But since $(p^sG)^1 = G^1$, an appeal to point (b) of the lemma above ensures that $p^s(G/G^1)[p] = (p^sG/G^1)[p] = = \cap_{k<\omega}(p^kG+G[p])/G^1$. Therefore, $\cap_{k<\omega}(p^kG+G[p]) \subseteq P+G^1$. Henceforth, point (a) of the lemma above implies that $p(\cap_{k<\omega}(p^kG+G[p])) = = G^1 \subseteq pP+pG^1$ and thus by taking in both sides the power operation p^n we obtain that $p^nG^1 \subseteq p^{n+1}G^1$.

First, assume that G is reduced, whence so is G^1. Furthermore, $p^nG^1 = 0$ and hence $P+G^1$ is also bounded by p^n. That is why G has to be $p^{\omega+n}$-projective and thereby Prop. A applies to show that G is bounded.

As for the general case, we decompose $G = G_d \oplus G_r$. It is only a technical exercise to check that G_r is thick with strongly $p^{\omega+n}$-projective first Ulm factor being a direct summand of G. Indeed, it is known by [D9] that a direct summand of a thick group is thick as well. Besides, if A is a direct summand of G, say $G = A \oplus B$, we put $K = A \cap (P+G^1)$ to find that $A/K \cong (A+P+G^1)/(P+G^1) \subseteq G/(P+G^1)$. Thus, via the modular law, $K = A\cap(P+G^1) = A\cap(P+A^1+B^1) = A^1+[A\cap(P+B^1)] = A^1+T$ where $T = A\cap(P+B^1)$. Next, because $p^nP = 0$, we observe that $p^nT \subseteq \subseteq p^nA \cap B = 0$ hence $T \leq A[p^n]$, which gives the claim. Consequently, the previous point works to get that G_r is bounded and this obviously assures that G is algebraically compact, as stated. $\Diamond$

We are now ready to demonstrate that the three propositions are deducible one from other, i.e., Props. A, B and C are equivalent, indeed.

Theorem (Equivalence). *The following three claims are equivalently stated:*

(1) *Each thick $p^{\omega+n}$-projective group is bounded.*

(2) *Each thick group with strongly $p^{\omega+n}$-projective first Ulm factor is algebraically compact.*

(3) *Each thick group with $p^{\omega+n}$-projective first Ulm factor is algebraically compact.*

Proof. Since every $p^{\omega+n}$-projective group possesses $p^{\omega+n}$-projective first Ulm factor which is itself strongly $p^{\omega+n}$-projective, it is plainly seen that implications (3)⇒(2)⇒(1) hold. Finally, the implication (1)⇒(3) follows like this: Let G be a thick group such that G/G^1 is $p^{\omega+n}$-projective. Utilizing [D1], G/G^1 is also thick and hence by the assumption in (1) we have that G/G^1 must be bounded. Therefore, p^tG is divisible for some natural number t and so $G \cong p^tG \oplus (G/p^tG)$ is obviously algebraically compact, as asserted. $\diamondsuit$

A group G is said to be in [D3] and [D4] an *n-Σ-group* if $G[p^n] = = \cup_{i<\omega} G_i, G_i \subseteq G_{i+1} \leq G[p^n]$ and, for all $i < \omega$, $G_i \cap p^iG = G^1[p^n]$. It was established in [DK] and [D5] via different proofs that G is an n-Σ-group if and only if some (and hence every) $p^{\omega+n}$-high subgroup of G is a direct sum of countable groups. It was also proved in ([D5], Th. 1) that any n-Σ-group is a strong ω-elongation by a $p^{\omega+n}$-projective group if and only if it is a pillared group, i.e., its first Ulm factor is a direct sum of cyclics. So, a question which naturally arises is what is the structure of a group G if it is an n-Σ-group for every positive integer n. It was asked whether it is again a pillared group (see [D5]). Before resolving this question in the affirmative, we need some preliminaries.

Following [M], a group G is called a *C_α-group* whenever $\alpha \leq \omega_1$ is an ordinal if $G/p^\beta G$ is a direct sum of countable groups for each $\beta < \alpha$. Motivated by [K2], a group G is said to be a *K_α-group* whenever $\alpha \leq \omega_1$ is an ordinal if some (and hence every) p^β-high subgroup of G is a direct sum of countable groups for any $\beta < \alpha$.

If α is isolated, only $\alpha = \beta - 1$ is enough to be considered. When α is limit, the following assertion holds:

Proposition ([K1]). *If α is a limit ordinal, a group G is a C_α-group if and only if it is a K_α-group.*

We are now coming to the following pivotal

Proposition. *A group G is pillared if and only if G is a $C_{\omega+1}$-group if and only if G is a $K_{\omega.2}$-group if and only if G is a $K_{\omega+n}$-group for all $n<\omega$.*

Proof. We claim that $G/p^{\omega}G$ is a direct sum of cyclics if and only if $G/p^{\omega+n}G$ is a direct sum of countable groups for all naturals n. In order to show this, observe that

$$G/p^{\omega}G \cong G/p^{\omega+n}G/p^{\omega}G/p^{\omega+n}G = G/p^{\omega}G/p^{\omega}(G/p^{\omega+n}G).$$

Since $p^{\omega}(G/p^{\omega+n}G)$ is bounded by p^n, the claim follows in virtue of [N2]. Thus, the proposition from [K1] listed above allows us to infer that the second equivalence is true. The first and the third equivalencies are valid directly using the definitions alluded to above. $\diamondsuit$

So, we are ready to prove the following.

Theorem. *A group G is an n-Σ-group for every $n<\omega$ if and only if G is a pillared group.*

Proof. As aforementioned, it was showed in [DK] and independently in [D5] that G is an n-Σ-group precisely when G is a $K_{\omega+n}$-group. Furthermore, we apply the previous proposition. $\diamondsuit$

We terminate the paper with some still left-open problems.

Problem 1. Determine under what extra circumstances on the group structure efi Σ-groups are thick.

Notice that for separable groups this holds always, that is, separable efi Σ-groups are bounded and hence thick. By that reason in [CM] and [DuIr] these groups are not Σ-groups since otherwise, being separable, they should also be direct sums of cyclics, whence bounded, which is impossible. In this aspect we note also that in [BW] was showed that pure-completeness plus efi implies thickness; thus in the counterexamples of an efi non thick group constructed in [DuIr] and [CM] the groups are not pure-complete as well, i.e., not every subsocle supports a pure subgroup.

Problem 2. Determine under what additional circumstances efi $p^{\omega+n}$-projective Σ-groups are bounded.

It is worth noting that in the general case there exists an unbounded and inseparable efi $p^{\omega+n}$-projective Σ-group (see, for instance, [D6]), so that in accordance with Theorem (Equivalence) efi Σ-groups are not thick. It is also worthwhile noticing that combining [BW] and the theorem we obtain that pure-complete efi $p^{\omega+n}$-projective groups are bounded. Likewise, the same claim is true by virtue of [IK] since pure-complete $p^{\omega+n}$-projective groups are direct sums of cyclics, thus bounded being efi.

Correction. Although it is clear from the context, we emphasize that in [D2], p. 26, lines 5–6) the word "*essentially indecomposable*" should be read as "*essentially finitely indecomposable*".

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