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HORIZONTAL SUMS OF BOUNDED LATTICES

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Abstract: Horizontal sums of bounded chains, respectively Boolean algebras are characterized.

Horizontal sums play an important role in constructing new bounded posets from given ones. Since bounded chains are the simplest bounded lattices and Boolean algebras are the simplest orthomodular lattices it is natural to ask when a bounded lattice is a horizontal sum of bounded chains, respectively when an orthomodular lattice is a horizontal sum of Boolean algebras. The aim of this note is to solve these questions. (In [2] horizontal sums of Boolean rings were characterized.)

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We start with the definition of the horizontal sum of bounded lattices.

Definition 1. An algebra $(L, \vee, \wedge, 0, 1)$ of type $(2, 2, 0, 0)$ is called the *horizontal sum* of its subalgebras $(L_i, \vee, \wedge, 0, 1), i \in I$, if

(i) $L = \bigcup\limits_{i \in I} L_i$,

(ii) $L_i \cap L_j = \{0, 1\}$ for all $i, j \in I$ with $i \neq j$ and

(iii) $a \vee b = 1$ and $a \wedge b = 0$ for all $i, j \in I$ with $i \neq j$ and all $a \in L_i \setminus \{0, 1\}$ and $b \in L_j \setminus \{0, 1\}$.

Now we characterize horizontal sums of bounded chains.

Theorem 2. *A bounded lattice $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ is the horizontal sum of bounded chains if and only if $a \vee b = 1$ and $a \wedge b = 0$ for every pair a, b of incomparable elements of L.*

Proof. First assume $\mathcal{L}$ to be the horizontal sum of the bounded chains $\mathcal{L}_i = (L_i, \vee, \wedge, 0, 1)$, $i \in I$. Let a, b be a pair of incomparable elements of L. According to (i) of Def. 1 there exist $i, j \in I$ with $a \in L_i$ and $b \in L_j$. Since a, b are incomparable, $i \neq j$ and $a, b \neq 0, 1$. Hence $a \vee b = 1$ and $a \wedge b = 0$ according to (iii) of Def. 1.

Conversely, assume that $a \vee b = 1$ and $a \wedge b = 0$ for every pair a, b of incomparable elements of L. Let α denote the binary relation of comparability on $L \setminus \{0, 1\}$. Then α is reflexive and symmetric. Let $a, b, c \in L \setminus \{0, 1\}$ and assume $a \,\alpha\, b \,\alpha\, c$, but not $a \,\alpha\, c$. Then $a \vee c = 1$ and $a \wedge c = 0$. Now $a \leq b \geq c$ would contradict $a \vee c = 1$ and $a \geq b \leq$ $\leq c$ would contradict $a \wedge c = 0$. Hence α is transitive and therefore an equivalence relation on $L \setminus \{0, 1\}$. We claim that $\mathcal{L}$ is the horizontal sum of the bounded chains $M \cup \{0, 1\}, M \in (L \setminus \{0, 1\})/\alpha$. Since any two elements of an equivalence class of α are comparable, the sets mentioned in the previous sentence are in fact base sets of bounded chains the join of which equals L. Let $M_1, M_2 \in (L \setminus \{0, 1\})/\alpha$ with $M_1 \neq M_2$. Then $(M_1 \cup \{0, 1\}) \cap (M_2 \cup \{0, 1\}) = (M_1 \cap M_2) \cup \{0, 1\} = \emptyset \cup \{0, 1\} = \{0, 1\}$. If $a \in M_1$ and $b \in M_2$ then a, b are incomparable and hence $a \vee b = 1$ and $a \wedge b = 0$. $\diamondsuit$

Next we define orthomodular lattices.

Definition 3. An *ortholattice* is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and for all $x, y \in L$ it holds $x \vee x' = 1$, $x \wedge x' = 0$, $(x')' = x$, $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$. An ortholattice $(L, \vee, \wedge, ', 0, 1)$ is called an *orthomodular lattice* if for all $x, y \in L$ it holds $x \vee ((x \vee y) \wedge x') = x \vee y$.

Now we extend the notion of a horizontal sum to algebras of type $(2, 2, 1, 0, 0)$.

Definition 4. An algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ is called the *horizontal sum* of its subalgebras $(L_i, \vee, \wedge, ', 0, 1)$, $i \in I$, if $(L, \vee, \wedge, 0, 1)$ is the horizontal sum of $(L_i, \vee, \wedge, 0, 1)$, $i \in I$.

It should be remarked that certain classes of algebras of type $(2, 2, 0, 0)$, respectively $(2, 2, 1, 0, 0)$ are closed with respect to forming horizontal sums whereas other classes are not.

Theorem 5. *The variety of bounded lattices, ortholattices, respectively orthomodular lattices is closed with respect to forming horizontal sums, whereas the class of bounded chains, respectively the variety of Boolean algebras is not.*

Proof. The proof is immediate. The horizontal sum of two three-element chains is not a chain and the horizontal sum of two four-element Boolean algebras has six elements and therefore cannot be a Boolean algebra. $\diamondsuit$

Before we are able to state and prove our characterization of horizontal sums of Boolean algebras we recall some notions and facts concerning orthomodular lattices. (For the theory of orthomodular lattices see the monographs [6] and [1].)

Definition 6. Two elements a, b of an orthomodular lattice $(L, \vee, \wedge, ', 0, 1)$ are said to *commute* with each other, in signs $a \mathrm{C} b$, if $a = (a \wedge b) \vee (a \wedge b')$.

The facts presented in the next lemma are well-known results in the theory of orthomodular lattices. The second one (which is the most non-trivial) heavily uses a famous result by D. J. Foulis and S. S. Holland Jr. ([3] and [5]):

Theorem 7 (Foulis–Holland Theorem)**.** *If three elements of an orthomodular lattice $\mathcal{L}$ have the property that one of them commutes with the other two then the sublattice of $\mathcal{L}$ generated by these three elements is distributive.*

Now we formulate the announced lemma which will be used in the proof of the last theorem.

Lemma 8. *Let $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and $a, b \in L$. Then*

(i) *$a \mathrm{C} b$ if and only if there exists a Boolean subalgebra $(B, \vee, \wedge, ', 0, 1)$ of $\mathcal{L}$ with $a, b \in B$. (This follows from* [6]*, Lemma, pp. 38 and 39; see also the original paper* [4]*.)*

(ii) *Every maximal set of pairwise commuting elements of* L *is the universe of a Boolean subalgebra of* $\mathcal{L}$.

(iii) ([6], *p. 20*) $a \leq b$ *implies* $a \, \mathrm{C} \, b$.

Now we are able to state and prove our characterization of horizontal sums of Boolean algebras.

Theorem 9. *An orthomodular lattice* $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ *is a horizontal sum of Boolean algebras if and only if* $\mathrm{C}|(L \setminus \{0,1\})$ *is transitive.*

Proof. First assume $\mathcal{L}$ to be the horizontal sum of its Boolean subalgebras $\mathcal{B}_i = (B_i, \vee, \wedge, 0, 1)$, $i \in I$. Let $a, b, c \in L \setminus \{0,1\}$ with $a \, \mathrm{C} \, b \, \mathrm{C} \, c$. According to Def. 4 and (i) of Def. 1 there exist $i, j, k \in I$ with $a \in L_i$, $b \in L_j$ and $c \in L_k$. Now $i \neq j$ would imply $a = (a \wedge b) \vee (a \wedge b') = 0 \vee 0 = 0$ according to Def. 4 and (iii) of Def. 1, a contradiction. Hence $i = j$. Analogously, it follows $j = k$. Therefore $i = k$ and hence $a = a \wedge 1 = a \wedge (c \vee c') = (a \wedge c) \vee (a \wedge c')$ proving $a \, \mathrm{C} \, c$.

Conversely, assume $\mathrm{C}|(L \setminus \{0,1\})$ to be transitive. According to (i) of Lemma 8, $\mathrm{C}|(L \setminus \{0,1\})$ is an equivalence relation on $L \setminus \{0,1\}$. We claim that $\mathcal{L}$ is the horizontal sum of its Boolean subalgebras $M \cup \{0,1\}$, $M \in (L \setminus \{0,1\})/(\mathrm{C}|(L \setminus \{0,1\}))$. Since the sets M, $M \in (L \setminus \{0,1\})/(\mathrm{C}|(L \setminus \{0,1\}))$ as equivalence classes of $\mathrm{C}|(L \setminus \{0,1\})$ are maximal sets of pairwise commuting elements of $L \setminus \{0,1\}$, the sets $M \cup \{0,1\}$, $M \in (L \setminus \{0,1\})/(\mathrm{C}|(L \setminus \{0,1\}))$ are maximal sets of pairwise commuting elements of L which implies according to (ii) of Lemma 8 that they are base sets of Boolean subalgebras of $\mathcal{L}$ the union of which equals L. Let $M_1, M_2 \in (L \setminus \{0,1\})/(\mathrm{C}|(L \setminus \{0,1\}))$ with $M_1 \neq M_2$. Then $(M_1 \cup \{0,1\}) \cap (M_2 \cup \{0,1\}) = (M_1 \cap M_2) \cup \{0,1\} = \emptyset \cup \{0,1\} = \{0,1\}$. Now let $a \in M_1$ and $b \in M_2$. Assume $a \vee b < 1$. Since $a \leq a \vee b$ we have $a \, \mathrm{C} \, a \vee b$ according to (iii) of Lemma 8 which together with $a, a \vee b \in L \setminus \{0,1\}$ implies $a \vee b \in M_1$. Analogously, it follows $a \vee b \in M_2$ contradicting $M_1 \cap M_2 = \emptyset$. Hence $a \vee b = 1$. Analogously, it follows $a \wedge b = 0$ completing the proof of the theorem. $\Diamond$

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